THE MORSE INEQUALITIES FOR LINE BUNDLES

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In place of a real valued differentiable (C^2) function on a closed n-dimensional differentiable manifold M, we may more generally consider a differentiable section s in any line bundle L on M, assumed to have structural group Z_2, the group of integers modulo two. Since the usual definitions of a critical point and of a non-degenerate critical point are local in nature, and since composing a differentiable real valued function with the function $t \to -t$ does not change its set of critical points or its set of non-degenerate critical points, it is clear that we may speak of critical points and non-degenerate critical points of the section s. Unless the bundle has a fixed trivialization however, the index of a non-degenerate critical point must be thought of as a set of two numbers $\{k, n-k\}$, corresponding to the two indices arising from the two trivializations possible for L restricted to a small enough neighborhood of the point, i.e. corresponding to the two possible ways of reading the index. With this understanding we extend the usual definitions, and call a differentiable (C^2) section s of L a Morse section if each of its critical points is non-degenerate. Then non-degenerate critical points are isolated, and one again has:

THEOREM 1. *The set of Morse sections of L forms a dense open subset of the set of sections in the C^2 topology.*

PROOF. The lemmas of the proof in [4] for the case in which L is trivial are all local in nature, so that using a non-trivial L introduces no further complications.

Theorem 1 has an application which is reminiscent of how the projective plane is constructed from the 2-dimensional disk. In order to give it, we will introduce some notation which will be used in all that follows: Let L be the line bundle with structural group Z_2 determined by a non-zero element w of $H^1(M; Z_2)$, and let s be a Morse section of L whose existence is assured by Theorem 1. Further, let W be the two-fold covering of M determined by w and having projection map $p : W \to M$. Lifting L to a bundle on W through p produces a trivial bundle $W \times R$. Now lifting s and

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regarding the resulting section as a real valued function produces a Morse function $f: W \to R$.

COROLLARY. *Suppose that M is connected and that $H^1(M; Z_2) \neq 0$. Then there exists a connected manifold-with-boundary N whose boundary ∂N is connected, non-empty, and has a differentiable involution T with the following property: Points of ∂N corresponding under T to each other may be identified in such a way that the resulting manifold is diffeomorphic to M.*

PROOF. Let $w \in H^1(M; Z_2)$ be non-zero, let L be the line bundle determined by it, and assume that W, p and f are as above. Now by means of an alteration procedure of a local nature, see for example the proof of Lemma 2.8 of [4], it may be supposed that s and hence f do not assume zero as a critical value. To complete the proof, take $N = f^{-1}\{t \in R | t \leqq 0\}$, and $\partial N = f^{-1}\{0\}$. The last assumption ensures that $f^{-1}\{0\}$ is a manifold, while showing the connectedness of N is easy. The involution T interchanges every two points of ∂N mapping under p into the same point of M.

What we plan to do next is to combine the standard Morse relations for f on W with Smith theory to study the homology of M. This way one gets some 'Morse relations' even though s does not give a cell decomposition of M (the way f gives one for W).

For this purpose fix a field F. Let Z_2 consist of e and T with the relation $T^2 = e$, and let $\Lambda = F(Z_2)$ be the group ring of Z_2 with coefficients from the field F. Let F_- denote F considered as a Λ-module through the action $Tx = -x$, and use F also to denote F as a Λ-module in a trivial way. The fact that W is a regular two-fold covering of M implies that Z_2 acts properly on W, so that — see [1] or [2] — the F-chain complex $C_*(W; F)$ of W may be regarded as a free Λ-module with the further property that $C_*(M; F)$ is isomorphic to $C_*(W; F) \otimes_\Lambda F$. Likewise,

$$H_*^w(M) = H_*(C_*(W) \otimes_\Lambda F_-),$$

is the twisted homology of M determined by the element $w \in H^1(M; Z_2)$ defining W and L.

We shall also need the following facts from the Smith theory of covering spaces.

One has two exact sequences of Λ-modules:

$$0 \to F \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} F_- \to 0$$

and

$$0 \to F_- \xrightarrow{\gamma} \Lambda \xrightarrow{\delta} F \to 0$$

where the maps are determined by:

$$\alpha(1) = 1+T, \quad \beta(1) = 1, \quad \beta(T) = -1,$$
$$\gamma(1) = 1-T, \quad \delta(1) = 1, \quad \delta(T) = 1,$$

After tensoring these exact sequences by the Λ-free chain complex $C_*(W;F)$, we may proceed in a standard way to obtain the two long exact sequences:

$$\cdots \to H_*(M) \xrightarrow{\alpha} H_*(W) \xrightarrow{\beta} H^w_*(M) \xrightarrow{\partial} H_*(M) \to \cdots$$

and

$$\cdots \to H^w_*(M) \xrightarrow{\gamma} H_*(W) \xrightarrow{\delta} H_*(M) \xrightarrow{\partial'} H^w_*(M) \to \cdots$$

with $\delta \circ \alpha = 2$, $\beta \circ \gamma = 2$, $\gamma \circ \beta = 1-T$ and $\alpha \circ \delta = 1+T$. Here for instance ∂ is a collection of maps $\partial_n : H^w_n(M) \to H_{n-1}(M)$, while the degrees of α, β, γ and δ are zero.

In order to state the theorem, let $C_{\{\lambda, n-\lambda\}}$ equal the number of critical points of s of index $\{\lambda, n-\lambda\}$, and set

$$C_\lambda = C_{n-\lambda} = C_{\{\lambda, n-\lambda\}} \quad \text{if} \quad \lambda \neq n-\lambda,$$

and

$$C_\lambda = 2C_{\{\lambda, \lambda\}} \quad \text{if} \quad \lambda = n-\lambda.$$

Note that C_λ is also equal to the number of critical points of f having index λ. Finally, set $R_i(X)$ and $R^w_i(X)$ equal respectively to the dimensions as vector spaces over F of $H_i(X; F)$ and $H^w_i(X; F)$.

Then we have:

THEOREM 2. *Let L be a line bundle on M determined by a nonzero element w of $H^1(M; Z_2)$, and suppose that s is a Morse section of L. Then regarding the critical points of s the following relations hold:*

(a) *If the characteristic of F is not two, then*

$$(-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i [R_i(M) + R^w_i(M)] \leqq (-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i C_i.$$

(b) *If the characteristic of F is not two and if w is the Stiefel class $w_1(M)$ so that W is the two-fold orientable covering of the non-orientable manifold M, then*

$$(-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i [R_i(M) + R_{n-i}(M)] \leqq (-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i C_i.$$

(c) *If F is of characteristic two, then*

$$R_\lambda(M) + (-1)^\lambda \sum_{i=0}^{\lambda-1} 2(-1)^i R_i(M) \leqq (-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i C_i.$$

In any case, $\sum_{i=0}^{n} (-1)^i C_i$ is equal to twice the Euler characteristic of M.

PROOF. Since f can be viewed as an equivariant function on W, it is easy to see that to each critical point x of s of index $\{\lambda, n-\lambda\}$ there will correspond two critical points of f, namely those in $p^{-1}(x)$, one of which has index λ and the other has index $n-\lambda$. Thus the usual Morse inequalitie. for f, see [3] or [5], imply that

$$(-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i R_i(W) \leqq (-1)^\lambda \sum_{i=0}^{\lambda} (-1)^i C_i.$$

To arrive at the form of the theorem consider first the case in which the characteristic of F is not two. Then the formula $\delta \circ \alpha = 2$ shows that α is an injection. Thus, considered as a sequence of vector spaces over F, the long exact sequence displayed preceding the theorem may be written as a short exact sequence

$$0 \to H_*(M) \to H_*(W) \to H_*^w(M) \to 0.$$

Hence $R_n(W) = R_n(M)+R_n^w(M)$. Then part (a) follows upon making this substitution in the above inequality. Part (b) follows upon noting that W is the orientation covering of M, whence by Poincaré duality we have $H_i^w(M) \cong H_{n-i}(M)$.

Next assume that the characteristic of F is two. Then $F = F_-$, and $H_*^w(M) = H_*(M)$, so that from the long exact sequence we have dimension kernel ∂_m-dimension cokernel $\partial_m = R_m(M)-R_{m-1}(M)$ which we shall write as:

$$\ker \partial_m - \operatorname{coker} \partial_m = R_m(M)-R_{m-1}(M).$$

Also we have

$$H_m(W) \cong \text{kernel } \partial_m \oplus \text{cokernel } \partial_{m+1}$$

which upon taking dimensions over F we shall write as:

$$R_m(W) = \ker \partial_m + \operatorname{coker} \partial_{m+1}.$$

Using these relations, the rest of the proof runs as follows:

$$\begin{aligned}
R_m(W)-R_{m-1}(W)+R_{m-2}(W)-R_{m-3}(W)+\cdots \\
&= \ker \partial_m + \operatorname{coker} \partial_{m+1} - \ker \partial_{m-1} - \operatorname{coker} \partial_m \\
&\quad + \ker \partial_{m-2} + \operatorname{coker} \partial_{m-1} - \cdots \\
&= \operatorname{coker} \partial_{m+1} + (\ker \partial_m - \operatorname{coker} \partial_m) - (\ker \partial_{m-1} - \operatorname{coker} \partial_{m-1}) \\
&\quad + (\ker \partial_{m-2} - \operatorname{coker} \partial_{m-2}) - \cdots \\
&= \operatorname{coker} \partial_{m+1} + (R_m(M)-R_{m-1}(M)) - (R_{m-1}(M)-R_{m-2}(M)) \\
&\quad + (R_{m-2}(M)-R_{m-3}(M)) - \cdots \\
&= \operatorname{coker} \partial_{m+1} + R_m(M) - 2R_{m-1}(M) + 2R_{m-2}(M) - 2R_{m-3}(M) + \cdots.
\end{aligned}$$

The equation consisting of the end terms of this chain of equalities implies part (c) upon using the above inequality.

The last statement in the theorem follows from part (b) upon examining the inequalities for $\lambda = n$ and $\lambda = n+1$, keeping in mind that C_{n+1} equals

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zero. It may also be deduced from the corresponding equality for f upon making use of the fact that the Euler characteristic of W is twice that of M. Note that for $w = 0$, the set of inequalities in part (a) includes the usual ones. This concludes the proof.

We have stated Theorem 2 for the case when the coefficient domain is a field. As is well known, see [5], the corresponding results when the coefficient domain is the group of integers may be deduced from the universal coefficient theorem and the result for a field. Another comment is of interest. It is of course important to know how good are the estimates of Theorem 2. The following example, suggested by the referee, is one for which the inequalities yield the best possible estimate. It is not hard to see that the twisted line bundle over projective n-space has a Morse section with only one critical point of index $\{0, n\}$. That there must be at least one follows upon setting $\lambda = 0$ in any part of the theorem. We also leave it to the interested reader to formulate relative versions of the above theorems, and remark only that this can be done following the above lines.

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