# ORTHOMODULAR POSETS FROM SESQUILINEAR FORMS

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(Received 28th April 1971)

Communicated by P. D. Finch

In this paper, we show how to generate orthomodular posets from sesquilinear forms on a vector space.

Let $E$ be a vector space over the division ring $k$. A binary relation $\perp$ on $E$ is called a *linear orthogonality relation* provided

(1) $x \perp y$ iff $y \perp x$, and

(2) for each $x$ in $E$, $\{x\}^{\perp} = \{y \mid y \perp x\}$ is a linear subspace of $E$.

For a subset $M$ of $E$ we define the *orthogonal of* $M$ by

$$M^{\perp} = \{y \mid y \perp m \text{ for all } m \text{ in } M\}.$$

Also we let $[M]$ denote the linear span of $M$ in $E$.

The first lemma is trivial.

LEMMA 1. *For $M$, $M_i$, and $N$ subsets of $E$, we have*

(1) $M \subseteq M^{\perp\perp}$

(2) $M \subseteq N$ *implies* $N^{\perp} \subseteq M^{\perp}$

(3) $M^{\perp} = M^{\perp\perp\perp}$

(4) $(\cup M_i)^{\perp} = \cap M_i^{\perp}$

(5) $M^{\perp}$ is a subspace of $E$

(6) $M^{\perp} = [M]^{\perp}$ *so in particular if* $x \perp y$ *and* $x \perp z$ then $x \perp y + z$ and $x \perp \alpha y$ *for all* $\alpha$ *in* $k$.

(7) $(0)^{\perp} = E$ *and* $E = E^{\perp\perp}$

Note that $M \mapsto M^{\perp\perp}$ is a closure operator on the lattice of all subspaces of $E$.

Let $\perp$ be a linear orthogonality relation on $E$. We say $\perp$ is *nondegenerate* when $E^{\perp} = (0)$. In this case we call $(E, \perp)$ a *linear orthogonality space.*

Call a subspace $M$ of $E$ *orthogonally closed* or $\perp$*-closed* if $M = M^{\perp\perp}$. Let $P_c(E, \perp)$ denote the set of all $\perp$-closed subspaces of $E$ ordered by inclusion. Using well known generalities on closure operators we see that $P_c(E, \perp)$ is a complete involution lattice with zero (0) and unit $E$. Also for $M_i$ in $P_c(E, \perp)$ we have



https://doi.org/10.1017/S1446788700013161 Published online by Cambridge University Press

$$\inf(M_i) = \cap M_i \quad \text{and} \quad \sup(M_i) = [\cup M_i]^{\perp\perp}.$$

Call a vector $x$ in $E$ *isotropic* if $x \perp x$ and *anisotropic* otherwise. For a subspace $F$ of $E$, define the *radical* of $F$ by $\mathrm{rad}(F) = F \cap F^{\perp}$. Say that $F$ is *semisimple* provided $\mathrm{rad}(F) = (0)$. Let $P_{ss}(E, \perp)$ denote the set of all semisimple subspaces of $E$ ordered by inclusion.

It can be shown that the orthogonal of a semisimple subspace need not be semisimple. However,

$$P_{ss}(E, \perp) \cap P_c(E, \perp)$$

is easily seen to be an orthocomplemented poset under the natural involution $F \mapsto F^{\perp}$. It need not though be orthomodular.

LEMMA 2. *Let $\{F_i\}$ be an orthogonal family of linear subspaces of $E$ (i.e. $F_i \subseteq F_j^{\perp}$ for $i \neq j$). Let $F$ be the smallest subspace of $E$ containing all the $F_i$. We write $F = \sum F_i$. Then* $\mathrm{rad}(F) = \sum \mathrm{rad}(F_i)$.

PROOF. First $\mathrm{rad}(F) = F \cap F^{\perp} = F \cap (\cap F_i^{\perp}) = \cap(F \cap F_i^{\perp})$. For each fixed $j$ and for any $i$ we have

$$\mathrm{rad}(F_j) = F_j \cap F_j^{\perp} \subseteq F \cap F_i^{\perp}.$$

Hence for each $j$, $\mathrm{rad}(F_j) \subseteq \mathrm{rad}(F)$ so $\sum \mathrm{rad}(F_i) \subseteq \mathrm{rad}(F)$.

Conversely, suppose $x$ is in $\mathrm{rad}(F)$. Since $x$ is in $F$ we can write $x$ as a finitely nonzero sum $x = \sum x_i$ with $x_i$ in $F_i$. For each $j$,

$$x_j = x - \sum_{i \neq j} x_i.$$

Since $x$ is in $\mathrm{rad}(F)$ then $x$ is in $F_j^{\perp}$. Since the family $\{F_i\}$ is orthogonal, each $x_i$ with $i \neq j$ also belongs to $F_j^{\perp}$. Thus $x_j$ is in $F_j^{\perp}$. It follows each $x_j$ is in $\mathrm{rad}(F_j)$. Thus $x$ is in $\sum \mathrm{rad}(F_i)$.

COROLLARY 3. *If $\{F_i\}$ is an orthogonal family of semisimple subspaces of $E$, the join exists in $P_{ss}(E, \perp)$ and in fact the join is the orthogonal direct sum of the $F_i$.*

Next we have a technical lemma.

LEMMA 4. *Let $F$ and $G$ be linear subspaces of $E$. Suppose $F \subseteq G$ and $G \subseteq F + F^{\perp}$. Let $G$ be semisimple. Then $G \cap F^{\perp}$ is semisimple.*

PROOF. If $G \cap F^{\perp}$ were not semisimple, we would have a vector $w$ different from zero with $w$ belonging to

$$\mathrm{rad}(G \cap F^{\perp}) = (G \cap F^{\perp}) \cap (G \cap F^{\perp})^{\perp}.$$

Since $G$ is semisimple and $w$ is in $G$ we cannot have $w$ in $G^{\perp}$. Thus there is a vector $y$ in $G$ such that $y$ fails to be orthogonal to $w$. Since $G$ is contained in in $F + F^{\perp}$ and $F$ is contained in $G$ we see

$$G = F + (F^{\perp} \cap G) .$$

Hence we can write $y = u + x$ where $u$ belongs to $F$ and $x$ is in $F^{\perp} \cap G$ Since $w$ is in $F^{\perp}$ and $w$ is in $(F^{\perp} \cap G)^{\perp}$, then $w$ is orthogonal to $y$, a contradiction.

A subspace $F$ of the linear orthogonality space $(E, \perp)$ is called *splitting* if $E = F + F^{\perp}$. Let $P_s(E, \perp)$ denote the set of all splitting subspaces of $E$ again ordered by inclusion.

The next lemma is straightforward and we omit the proof.

LEMMA 5. (1) (0) *and $E$ are splitting subspaces*
(2) *if $F$ is in $P_s(E, \perp)$ then so is $F^{\perp}$*
(3) *every splitting subspace is closed and semisimple*
(4) *$P_s(E, \perp)$ is an orthocomplemented poset under the inovolution $F \to F^{\perp}$.*

The next lemma establishes the first crucial property of an orthomodular poset.

LEMMA 6. *Finite orthogonal joins exist in $P_s(E, \perp)$.*

PROOF. Let $e$ be any vector in $E$. Let $F$ and $G$ be in $P_s(E, \perp)$ with $F \subseteq G^{\perp}$. We claim

$$F + G = F \oplus G$$

is in $P_s(E, \perp)$. First $e = w + w_1$ with $w$ in $G$ and $w_1$ in $G^{\perp}$ and $e = v + v_1$ with $v$ in $F$ and $v_1$ in $F^{\perp}$. Clearly

$$e = (v + w) + x$$

where $x = e - v - w, v_1 - w = w_1 - v$. Since $v_1$ is in $F^{\perp}$ and $w$ is in $G$ then $x$ is in $F^{\perp}$. Similarly, $x$ is in $G^{\perp}$. Thus $e$ is in

$$(F + G) + (F^{\perp} \cap G^{\perp}) = (F + G) + (F + G)^{\perp}.$$

Hence $E = (F + G) + (F + G)^{\perp}$

We now come to the main result.

THEOREM 7. *Let $(E, \perp)$ be a linear orthogonality space. Then $P_s(E, \perp)$ is an orthomodular poset.*

PROOF. We have already that $P_s(E, \perp)$ is an orthocomplemented poset with zero (0) and unit $E$ under the involution $F \mapsto F^{\perp}$. Orthogonal joins are just orthogonal direct sums. It suffices then to show the orthomodular identity. Let $F$ and $G$ be spliting subspaces with $F \subseteq G$. Then $G^{\perp} \leqq F^{\perp}$ so $F$ is orthogonal to $G^{\perp}$ so $F \vee G^{\perp} = F + G^{\perp}$. Thus

$$(F \vee G^{\perp})^{\perp} = (F + G^{\perp})^{\perp} = F^{\perp} \cap G.$$

Now $G = G \cap E = G \cap (F + F^{\perp}) = F + (G \cap F^{\perp}) = F \vee (F \vee G^{\perp})^{\perp}$ which completes the proof.

Note if $E$ is finite dimensional, $P_s(E, \perp)$ is necessarily an atomic orthomodular poset. Also note that linear orthogonality relations exist in great abundance. Let $E$ be any vector space. Let $\Phi$ be a $\theta$-sesquilinear nondegenrate orthosymmetric from on $E$. For $x$ and $y$ in $E$, define $x \perp y$ by $\Phi(x,y) = 0$. Then $(E, \perp) = (E, \Phi)$ is a linear orthogonality space. Call such a *quadratic space*. We have characterized which linear orthogonality spaces are quadratic spaces elsewhere. For a quadratic space $(E, \Phi)$ it can also be shown that $P_s(E, \Phi)$ is an ample atomic orthomodular poset with the ortho-covering and ortho-exchange properties.

A crucial problem is to determine when $P_s(E, \Phi)$ is a lattice. The next theorem provides and important partial answer. We are indebted to H. R. Fischer for the proof.

THEOREM 8. *Let $(E, \Phi)$ be a quadratic space of dimension at least 4 over a field of characteristic different from two. Suppose not every vector of E is isotropic. If $P_s(E, \Phi)$ is a lattice then $\Phi$ admits no non-zero isotropic vectors.*

PROOF.

Suppose on the contrary that $\Phi$ admits a nonzero isotropic vector. Since every nondegenerate space of dimension at least 4 contains a four dimensional semisimple subspace, it suffices to consider the case where the dimension of $E$ equals 4 and show that $E$ contains two distinct three dimensional semisimple subspaces whose intersection is a degenerate plane, but not totally isotropic (i.e. with radical properly contained in this plane, thus of dimension one).

The proof proceeds as follows: we shall construct in $E$ a plane $[x, y]$ such that $x \perp x$, $y \not\perp y$, and $x \perp y$. Then we shall find two distinct three dimensional semisimple spaces $F$ and $G$ in $E$ such that $F \cap G = [x, y]$. Once this is done, it is clear that $F$ and $G$ do not possess any infimum in $P_s(E, \Phi)$; $[y]$ and $[x + y]$ are distinct noncomparable lower bounds of $F$ and $G$ in $P_s(E, \Phi)$.

The construction is as follows. Choose any nonzero $x$ in $E$ such that $x \perp x$. Then $[x]^{\perp}$ is a subspace of dimension three. Therefore it cannot be totally isotropic. Now choose anisotropic $y$ in $[x]^{\perp}$. Then $[x, y]$ is the required plane. It is degenerate with $\text{rad}([x, y]) = [x]$.

Next $[y]^{\perp}$ is three dimensional and semisimple. Since $x$ is in $[y]^{\perp}$ there exists $a$ in $[y]$ such that $x \not\perp a$. If $a$ is anisotropic, let $z = a$. If $a$ is isotropic, let $z = a + x$. This will be anisotropic and still not orthogonal to $x$. In either case we have an anisotropic $z$ in $[y]^{\perp}$ such that $x \not\perp z$. From this it follows that $[x, y, z]$ is semisimple; its radical is properly contained in $[x]$ whence is (0).

The three dimensional space $F = [x, y, z]$ is also spanned by $x$, $x + y$, and $z$.

Since $x+y$ is anisotropic $[x+y]^{\perp}$ is semisimple. Hence, $x$ being in $[x+y]$ there is an anisotropic $u$ in $[x+y]^{\perp}$ such that $x \not\perp u$. Then $[x, x+y, u] = [x, y, u]$ is again semisimple and of dimension three.

We now have two cases:

*case a*: $u$ is not in $[x, y, z]$. Then we put $G = [x, y, u]$ and get $F \cap G = [x, y]$.

*case b*: $u$ is in $[x, y, z]$. By construction, $\{x, x+y, u\}$ is a linearly independent subset of $F$ and hence is a basis of $F$. In particular, $[x, x+y, u]^{\perp} = [u]^{\perp} \cap [x]^{\perp} = [x+y]^{\perp}$ is one dimensional semisimple, i.e. is spanned by an anisotropic vector $a$. Now $u$ is in $[x+y]^{\perp} = [u] \oplus M$, $M$ of dimension two and semisimple. Note that $M = [u]^{\perp} \cap [x+y]^{\perp}$. Also $[a] = [u]^{\perp} \cap [x]^{\perp} \cap [x+y]^{\perp} = [x]^{\perp} \cap M$. Thus there exists an anisotropic vector $w$ in $M$ such that $a \perp w$. Clearly then $w$ is not in $[x]^{\perp}$ but $w$ is in $[x+y]^{\perp}$. In this case we put $G = [x, x+y, w] = [x, y, w]$. Again we have $F \cap G = [x, y]$ both $F$ and $G$ semisimple of dimension three. This completes the proof.

We remark that if the dimension of $E$ does not exceed 3, then $P_s(E, \Phi)$ is a lattice simply because there is not enough height for things to go wrong. If in the above theorem, the dimension of $E$ is finite and $\Phi$ admits no nonzero isotropic vectors, then $P_s(E, \Phi) = P_{ss}(E, \Phi)$ is the lattice of all subspaces of $E$. Also in particular we note that if $\Phi$ is the Minkowski metric fo space-time, then $P_s(\mathbb{R}^4, \Phi)$ is an orthomodular poset that is not an orthomodular lattice.

We close with some open questions:

QUESTION 1. *What about the converse of Theorem* 8?

QUESTION 2. *What is the cut completion of* $P_s(E, \perp)$?

**Acknowledgement**

The author wishes to express his thanks to D. J. Foulis for his help.

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