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UPPER MIDDLE ANNIHILATORS

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Each ring contains a unique smallest ideal which when factored out yields a ring with zero middle annihilator. Various results concerning this ideal are obtained including theorems about how it behaves in connection with normalising extensions and smash products.

1. INTRODUCTION

Sands [12] has introduced the upper middle annihilator $\overline{M}(A)$ of a ring A, and de la Rosa [5] the quasi-radical of A. We observe that these concepts coincide and study properties of the ideal $\overline{M}(A)$. This notion itself does not seem to be useful for rings in general, so the ideal we actually study is $\overline{M}(P(A))$, which we denote by $\Delta(A)$, where $P(A)$ is the prime radical of A.

The next section contains definitions and various preliminary results. In Section 3 we show that in several well-known situations where A and S are rings with $A \subseteq S$ and S a free A-module, $\Delta(S) = \Delta(A)S$. Section 4 concerns the question of when the middle annihilator of $P(A)$ is essential in $P(A)$, and it contains a generalisation of a theorem of Shock. In the final section we show that a result of Pascaud on T-nilpotence and fixed rings cannot be extended to the M-nilpotent case.

Throughout this paper rings are associative but, at least at the beginning, need not have an identity. The notation $I \lhd R$ means that I is a (two-sided) ideal of R.

2. DEFINITIONS AND PRELIMINARY RESULTS

The *middle annihilator* of a ring A is $M(A) = \{a \in A \mid AaA = 0\}$. In [12] Sands defines the *upper middle annihilator* of a ring A inductively: $M_0(A) = 0$, if α is an ordinal and $M_\alpha(A)$ has been defined then $M_{\alpha+1}(A)$ is defined by the equation

$$M(A/M_\alpha(A)) = M_{\alpha+1}(A)/M_\alpha(A),$$

if β is a limit ordinal then $M_\beta(A) = \cup\{M_\alpha(A) \mid \alpha < \beta\}$; finally, the upper middle annihilator of A is $\overline{M}(A) = \cup\{M_\alpha(A) \mid \alpha \text{ is an ordinal}\}$.

The *quasi-radical* of a ring A was defined and studied by de la Rosa [5]. He calls an ideal I of A *quasi-semiprime* if $M(A/I) = 0$. The *quasi-radical* $q(A)$, is then defined as the intersection of all the quasi-semiprime ideals of A.

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PROPOSITION 1. *For any ring* A, $\overline{M}(A) = q(A)$, *so* $\overline{M}(A)$ *is the unique minimal quasi-semiprime ideal of* A.

PROOF: A straightforward transfinite induction shows that $\overline{M}(A) \subseteq Q$ for each quasi-semiprime ideal Q of A, and since $\overline{M}(A)$ is quasi-semiprime the result follows. ∎

In [**13**] Sands gave a characterisation of rings A such that $M(A) = 0$ (equivalently, $\overline{M}(A) = 0$). We include a proof of this result which is more straightforward than the original.

PROPOSITION 2. *(Sands). Let* A *be a ring. The following are equivalent:*

1. $M(A) = 0$,
2. *if* $R \lhd S \lhd T$ *and* $S/R \cong A$, *then* $R \lhd T$.

PROOF: First assume that $M(A) = 0$ and let R^* be the ideal of T generated by R where $R \lhd S \lhd T$.
Then $SR^*S = S(R + RT + TR + TRT)S \subseteq SRS \subseteq R$ and so $M(S/R) = 0$ implies that $R^* = R$.

Conversely, if $M(A) \neq 0$, then A has either a nonzero left annihilator or a nonzero right annihilator. Without loss of generality we may assume that A has a nonzero ideal I such that $AI = 0$. Let

$$R = \begin{bmatrix} A & I \\ A & 0 \end{bmatrix}, \quad S = \begin{bmatrix} A & I \\ A & A \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} A & I \\ A^1 & A \end{bmatrix}$$

where A^1 is the ring A with an identity adjoined in the usual way. It is straightforward to check that $R \lhd S \lhd T$, $S/R \cong A$ but R is not an ideal of T. ∎

The next result characterises rings A such that $M(\overline{A}) = 0$ for all homomorphic images $\overline{A}$ of A.

PROPOSITION 3. *Let* A *be a ring. The following are equivalent:*

1. $M(\overline{A}) = 0$ *for all homomorphic images* $\overline{A}$ *of* A,
2. *for every* $a \in A$, $a \in AaA$,
3. *if* n *is a positive integer and* T *is an ideal of the* $n \times n$ *matrix ring* A_n, *then* $T = B_n$ *for some ideal* B *of* A.

PROOF: Clearly 1 and 2 are equivalent, and Jacobson [**7**, p. 40, Proposition 1] shows that 2 implies 3. Sands [**11**, p. 50] observes that 3 implies 2. The equivalence of 1 and 3 is also given in de la Rosa [**4**, Theorem 10]. ∎

We shall denote the *prime radical* of a ring A; that is, the intersection of all the prime ideals of A, by $P(A)$. If $P(A) \neq A$, the upper middle annihilator may not be particularly useful in studying A. In particular, $\overline{M}(A) = 0$ if A has an identity. Because of this we shall consider not $\overline{M}(A)$ but $\overline{M}(P(A))$. It follows from Proposition 2 that $\overline{M}(P(A)) \lhd A$ and we shall denote this ideal by $\Delta(A)$.

If A is a ring without identity and A^1 is the usual unital extension of A, then $P(A) = P(A^1)$ and so $\Delta(A) = \Delta(A^1)$. In view of this we shall henceforth assume, unless the contrary is stated explicitly, that all rings have identity.

An ideal I of a ring A is *left T-nilpotent* if for any sequence of elements $a_1, a_2, \ldots, a_n \ldots$ in I there is a positive integer k such that $a_1, a_2 \ldots a_k = 0$. *Right T-nilpotence* is defined in a similar way. In [**12**] Sands calls an ideal I *M-nilpotent* if for any doubly infinite sequence of elements $\ldots, a_{-n}, \ldots, a_0, \ldots, a_n, \ldots$ there is a positive integer k such that $a_{-k} \ldots a_0 \ldots a_k = 0$. He then establishes the following result:

THEOREM 3. (Sands). *For any ring A, $\Delta(A)$ is M-nilpotent and $\Delta(A) = P(A)$ if and only if $P(A)$ is M-nilpotent.*

PROPOSITION 4. *Let A be a ring and suppose that $B \lhd A$. Then:*

1. $M(A/\Delta(A)) = 0$ *and so* $\Delta(A/\Delta(A)) = 0$,
2. $\Delta(B) \lhd A$,
3. $\Delta(\Delta(A)) = \Delta(A)$,
4. $\Delta(A) = \overline{M}(\Delta(A))$,
5. *if* $B \subseteq P(A)$, *then* $(\Delta(A) + B)/B \subseteq \Delta(A/B)$.

PROOF: First observe that 1 is true because

$$M(P(A/\Delta(A))) = M(P(A)/\Delta(A)) = 0.$$

Now 2 follows from 1 and Proposition 2. Also, 3 is an immediate consequence of Theorem 3, and 4 is merely a restatement of 3. In view of Proposition 1, 5 will follow if we show that $\Delta \cap P(A)$ is a quasi-semiprime ideal of $P(A)$ where $\Delta(A/B) = \Delta/B$. Suppose that $x \in P(A)$ and $P(A) \subset P(A) \subseteq \Delta \cap P(A)$. Since $B \subseteq P(A)$, $P(A/B) = P(A)/B$ and so $P(A/B)(x + B)P(A/B) \subseteq \Delta/B$. Thus $x \in \Delta$ and the proof is complete. ∎

Concerning 5 in the Proposition we note that both Sands [**12**, Theorem 2] and de la Rosa [**5**, Lemma 4.5] observe that the class of M-nilpotent rings (quasi-radical rings in the terminology of [**5**]) is homomorphically closed. Also, the assumption that $B \subseteq P(A)$ can not be omitted as the following example shows.

Let F be a field and let R be the polynomial ring over F with commuting indeterminates $\{X_\lambda \mid \lambda \in \mathbf{R}, 0 < \lambda < 1\}$. Let I be the ideal of $\mathbf{R}$ generated by $(X_{.5})^2$ and let J be the ideal of $\mathbf{R}$ generated by $\{X_\lambda X_\alpha - X_{\lambda+\alpha} \mid 0 < \lambda + \alpha < 1\}$ and

$\{X_\lambda X_\alpha \mid 0 < \lambda, \alpha < 1, \lambda + \alpha \geqslant 1\}$. Finally, let $A = R/I$ and $B = J/I$. We see that $\Delta(A) = P(A) =$ the ideal generated by $X_{.5} + I$, $\Delta(A) \nsubseteq B$ and A/B is the Zassenhaus algebra with $\Delta(A/B) = 0$.

3. Free extensions

A ring A is a *free normalising extension* of a subring S if S has the same identity as A and A contains a subset X such that A is a free left and right S-module with basis X and $xS = Sx$ for all $x \in X$.

THEOREM 5. . *If A is a free normalising extension of S and $P(A) = P(S)A$, then $\Delta(A) = \Delta(S)A$.*

PROOF: Each $x \in X$ determines an automorphism $\varphi = \varphi(x)$ of S defined by $sx = xs^\varphi$ for all $s \in S$ (here s^φ is the image of s under the automorphism φ). Since $P(S)$ is invariant under automorphisms of S, $P(S)A = AP(S)$. Now, to see that $\Delta(S) \subseteq \Delta(A)$ if suffices to show that $\Delta(A) \cap P(S)$ is a quasi-semiprime ideal of $P(S)$. Suppose that $P(S)tP(S) \subseteq \Delta(A)$ where $t \in P(S)$. Then $AP(S)tP(S)A \subseteq A\Delta(A)A \subseteq \Delta(A)$ and hence $P(A)tP(A) \subseteq \Delta(A)$. Since $\Delta(A)$ is a quasi-semiprime ideal of A, $t \in \Delta(A)$. Thus $\Delta(A) \cap P(S)$ is quasi-semiprime as required.

If θ is an automorphism of $P(S)$, then $\Delta(S)^\theta$, the image of $\Delta(S)$ under θ, is clearly a quasi-semiprime ideal of $P(S)$ and so $\Delta(S) \subseteq \Delta(S)^\theta$. Since this applies equally well to the automorphism θ^{-1}, $\Delta(S)^\theta = \Delta(S)$. Now, the automorphisms $\varphi(x)$, $x \in X$, restrict to automorphisms of $P(S)$ and so $x\Delta(S) = \Delta(S)x$ for all $x \in X$. Thus $\Delta(S)A \lhd A$ and the proof will be complete if we can show that $\Delta(S)A$ is a quasi-semiprime ideal of $P(A)$. Suppose that $P(A)aP(A) \subseteq \Delta(S)A$ where

$$a = \sum\{t_i x_i \colon t_i \in P(S), x_i \in X, i = 1, \ldots, n\} \in P(A).$$

Then $P(S)aP(S) \subseteq \Delta(S)A$. Since X is a free basis and $x_i P(S) = P(S)x_i$ for all $i = 1, \ldots, n$, $P(S)t_iP(S) \subseteq \Delta(S)$ for all $i = 1, \ldots, n$. Thus $t_i \in \Delta(S)$ for all $i = 1, \ldots, n$ and hence $a \in \Delta(S)A$ as required. ∎

COROLLARY 6. $\Delta(A[x]) = \Delta(A)[x]$.

PROOF: Amitsur [1] has shown that $P(A[x]) = P(A)[x]$. ∎

A free normalising extension A of S is a (right) *excellent extension* if (i) the free left and right basis X is finite with $1 \in X$ and (ii) if whenever M is a right A-module with A-submodule N which is a direct summand of M as an S-module, N is also a direct summand as an A-module. Examples include matrix rings $A = S_n$, group rings $A = SG$ where $|G|$ is finite and $|G|^{-1} \in S$ and, more generally, crossed products $A = S * G$ where $|G|$ is finite and $|G|^{-1} \in S$.

COROLLARY 7. *If A is an excellent extension of S, then $\Delta(A) = \Delta(S)A$.*

PROOF: The Fisher-Montgomery theorem asserts that $P(A) = P(S)A$, see [8] for details. ∎

If A is graded by a group G, then the smash product $A\#G^*$ is the free unital left A-module with basis $\{p_g \mid g \in G\}$ and multiplication defined by $ap_g bp_h = ab_{gh^{-1}}p_h$ where a, $b \in A$, g, $h \in G$ and $b_{gh^{-1}}$ is the gh^{-1} component of b.

THEOREM 8. *Let A be a G-graded ring such that $P(A)$ is a graded ideal and $P(A\#G^*) = P(A)\#G^*$. Then $\Delta(A)$ is a graded ideal and $\Delta(A\#G^*) = \Delta(A)\#G^*$.*

PROOF: If I is an ideal of A we shall denote the ideal $\{a \in I \mid a_g \in I \text{ for all } g \in G\}$ by I_G. Suppose that $P(A)aP(A) \subseteq (\Delta(A))_G$ where $a \in P(A)$ and the homogeneous components of a are $a_1, \ldots, a_n$. If $x, y \in P(A)$ are homogeneous, $xay = \sum\{xa_iy \mid i = 1, \ldots, n\}$ and $xa_1y, \ldots, xa_ny$ are the homogeneous components of xay. Thus $xa_iy \in \Delta(A)$ for all $i = 1, \ldots, n$ and so $P(A)a_iP(A) \subseteq \Delta(A)$ for all $i = 1, \ldots, n$ forcing $a_i \in \Delta(A)$ for all i. It follows that $(\Delta(A))_G$ is quasi-semiprime and hence $\Delta(A) = (\Delta(A))_G$ is a graded ideal.

Let $T = \{a \in A\colon ap_g \in \Delta(A\#G^*) \text{ for all } g \in G\}$. It is straightforward to check that T is a graded ideal of A. Suppose that $P(A)bP(A) \subseteq T$ where $b \in P(A)$. We wish to show that $b \in T$, and since T is graded we may assume that b is homogeneous. For each $g \in G$,

$$\begin{aligned} P(A\#G^*)bp_gP(A\#G^*) &= (P(A)\#G^*)bp_g(P(A)\#G^*) \\ &= (P(A)bp_g)(P(A)\#G^*) \\ &\subseteq (P(A)bP(A))\#G^* \\ &\subseteq T\#G^* \subseteq \Delta(A\#G^*). \end{aligned}$$

Also, $bp_g \in P(A)\#G^* = P(A\#G^*)$ and thus $bp_g \in \Delta(A\#G^*)$ for all $g \in G$. It follows that T is quasi-semiprime and so $\Delta(A) \subseteq T$ and hence $\Delta(A)\#G^* \subseteq \Delta(A\#G^*)$.

For the other containment it is enough to show that $\Delta(A)\#G^*$ is a quasi-semiprime ideal of $P(A)\#G^*$. Suppose that $u + \Delta(A)\#G^* \in M(P(A)\#G^*/\Delta(A)\#G^*)$. We wish to show that $u \in \Delta(A)\#G^*$ and it is sufficient to consider the case when u is of the form bp_g where $b \in P(A)$ and $g \in G$. For each $h \in G$ the function θ_h defined by $\theta_h(ap_k) = ap_{kh}$ induces an automorphism of $A\#G^*$. This automorphism restricts to an automorphism of $P(A\#G^*)$ under which $\Delta(A\#G^*)$ is invariant (as we saw in the proof of Theorem 5), and so it lifts to an automorphism of $P(A)\#G^*/\Delta(A)\#G^*$. Now since middle annihilators are clearly invariant under automorphisms, $(\forall m \in G)(bp_m + \Delta(A)\#G^* \in M(P(A)\#G^*/\Delta(A)\#G^*))$. If $x \in P(A)$ and $y \in P(A)$ is homogeneous of grade g, then $xp_hbp_eyp_{g^{-1}} = xb_hyp_{g^{-1}}$ is in

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$\Delta(A)\#G^*$ and hence $xb_hy \in \Delta(A)$. It follows that $P(A)b_hP(A) \subseteq \Delta(A)$ for all homogeneous components b_h of b. Hence all these homogeneous components, and so b too, are in $\Delta(A)$. Consequently, $\Delta(A)\#G^*$ is quasi-semiprime and the proof is complete. ∎

COROLLARY 9. *If A is graded by a finite group G and A has no $|G|$-torsion, then* $\Delta(A\#G^*) = \Delta(A)\#G^*$.

PROOF: Cohen and Montgomery [**3**, Theorem 5.3 and Corollary 5.5] have shown that the hypotheses of the theorem are satisfied in this case. ∎

COROLLARY 10. *If A is a prime radical ring (without 1 of course) graded by a group G, then* $\Delta(A\#G^*) = \Delta(A)\#G^*$.

PROOF: Let $A^1 = \{(a, n) \mid a \in A, n \in \mathbb{Z}\}$ be the usual unital extension of A. Let $(A^1)_e = \{(a, n) \mid a \in A_e, n \in \mathbb{Z}\}$ and $(A^1)_g = \{(a, 0) \mid a \in A_g\}$ if $e \neq g \in G$. Then A^1 is G-graded and $P(A^1) = \{(a, 0) \mid a \in A\}$ is a graded ideal which, as is usual, we will identify with A.

Since $(A^1\#G^*/P(A^1)\#G^*) \cong (A^1/P(A^1))\#G^* \cong \mathbb{Z}\#G^*$ is just a direct sum of $|G|$ copies of $\mathbb{Z}$, $P(A^1\#G^*) \subseteq P(A^1)\#G^*$.

Let A_{fin} be the ring of $|G| \times |G|$ matrices with only a finite number of nonzero entries. Since $P(A) = A$, $P(A_{fin}) = A_{fin}$ and since $A\#G^*$ embeds as a subring in A_{fin} (see [2] and/or [**10**]) $P(A\#G^*) = A\#G^*$. It follows that $P(A^1\#G^*) = P(A^1)\#G^*$ and so the theorem applies. ∎

4. ESSENTIAL MIDDLE ANNIHILATORS

PROPOSITION 11. *The ideal $M(P(A))$ is essential as a two-sided ideal of $\Delta(A)$.*

PROOF: Let $0 \neq F$ be a finite subset of $\Delta(A)$. We will show that there are $a, b \in (\Delta(A))^1$ such that $0 \neq aFb \subseteq M(P(A))$, thus establishing somewhat more than is required.

Since $\Delta(A) = \overline{M}(P(A))$ we may choose an ordinal γ minimal with respect to the property that $0 \neq aFb \subseteq M_\gamma(P(A))$ for some $a, b \in (\Delta(A))^1$. Since F is finite, γ is not a limit ordinal. Let $\gamma = \alpha + 1$. Then $P(A)aFbP(A) \subseteq M_\alpha(P(A))$ and so $P(A)aFbP(A) = 0$. Thus $aFb \subseteq M(P(A))$ and the proof is complete. ∎

The following example, due to Sasiada [**6**], shows that $M(A)$ may not be essential as a right ideal.

Let k be a field and let I be the ideal of the polynomial ring $k[X_1, X_2, \ldots]$ in noncommuting indeterminates $X_1, X_2, \ldots$ which is generated by X_iX_j, $i \geqslant j$. Let $A = k[X_1, X_2, \ldots]/I$ and denote $X_i + I$ by x_i. Now $P(A)$ is the ideal generated by $x_1, x_2, \ldots$ and $P(A)$ is right T-nilpotent, so $\Delta(A) = P(A)$. The middle annihilator ideal $M(P(A))$ is generated by x_1 and it has zero intersection with the nonzero right ideals $x_k\Delta(A)$, $k \geqslant 2$.

Also, we note that in general the ideal $M(P(A))$ need not be essential in $P(A)$. For example, if $P(A)$ is a direct sum $T_1 \oplus T_2$ where $M(T_1) = 0$ and $M(T_2) \neq 0$, then $M(P(A)) \cap T_1 = 0$.

THEOREM 12. *If $P(A)$ has the ascending chain condition on left annihilators of the form $ann_l(P(A)xP(A))$, $x \in P(A)$, then $M(P(A))$ is essential as a left ideal of $P(A)$.*

PROOF: Let $0 \neq L$ be a left ideal of $P = P(A)$. Choose z such that $ann_l(PzP)$ is maximal among annihilators of the form $ann_l(PxP)$, $0 \neq x \in L$.

Suppose that $I \lhd P$ and $I^2 \subseteq ann_l(PzP)$. If $IPz = 0$, then $I \subseteq ann_l(PzP)$. Otherwise, let $0 \neq y \in IPz$. Clearly we have $ann_l(PzP) \subseteq ann_l(PyP)$, so the maximality of $ann_l(PzP)$ forces $ann_l(PzP) = ann_l(PyP)$. Now, $IPyP \subseteq I^2PzP = 0$ and so $I \subseteq ann_l(PyP)$. So in any case $I \subseteq ann_l(PzP)$.

Since we have shown that $ann_l(PzP)$ is semiprime, $ann_l(PzP) = P$. Thus $P(Pz)P = 0$ and hence $L \cap M(P) \neq 0$. ∎

This generalises a result of Shock [**14**, Corollary 3.4] which asserts that if A satisfies the maximum condition on left annihilators, then $P(A)$ contains a nilpotent ideal which is essential as a left ideal. In general, middle annihilators are smaller than nilpotent ideals. For instance, $P(A)$, where A is the Sasiada ring discussed before the theorem, has the ascending chain condition on left annihilators, a rather small middle annihilator but is the sum of its nilpotent ideals.

5. FIXED RINGS

Pascaud [**9**] has shown that if A is a ring (without identity) and G is a group of automorphisms of A such that the fixed ring A^G is left T-nilpotent, then A is left T-nilpotent. An example of Sands [**12**, Example 2] can be used to show that the analogous result for M-nilpotence does not hold. We will give a variation of this example below, but first we require the following lemma.

Let A and B be algebras over a field F such that the right annihilator of A is zero and the left annihilator of B is zero. If A has an identity, let $A^1 = A$; otherwise let $A^1 = \{(a, \alpha) \mid a \in A, \alpha \in F\}$ with the usual ring operations and identify A and $\{(a, 0) \mid a \in A\}$ as is customary. Define B^1 similarily and note that the right annihilator of A in A^1 is zero and so is the left annihilator of B in B^1.

LEMMA 13. *With the notation established in the preceding paragraph and $M = A^1 \otimes_F B^1$ we have:*

1. $AaM = 0$, $a \in A$, *implies* $a = 0$,

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2. $MbB = 0$, $b \in B$, *implies* $b = 0$,
3. $AmB = 0$, $m \in M$, *implies* $m = 0$.

PROOF: If $AaM = 0$ where $a \in A$, then $Aa(1 \otimes 1) = 0$ and so $Aa = 0$. Thus $a = 0$ because A has zero right annihilator. This establishes 1 and 2 is similar.

If $0 \neq x \otimes y \in M$, there is an $a \in A$ and an element $b \in B$ such that $ax \neq 0$ and $yb \neq 0$. Thus $ax \otimes yb \neq 0$ and so $A(x \otimes y)B \neq 0$. Now let k be an integer, $k \geqslant 2$, and suppose that if $AmB = 0$ where m is a sum of fewer than k tensors, then $m = 0$. Assume that $AmB = 0$ where $m = x_1 \otimes y_1 + \ldots + x_k \otimes y_k \neq 0$. From our induction hypothesis we see that $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are both linearly independent over F.

Suppose that $b \in B$ and $y_k b = 0$. Then $AmbB = 0$ and so $mb = 0$ by the induction hypothesis. Since $\{x_1, \ldots, x_k\}$ is linearly dependent, $y_i b = 0$ for all $i = 1, \ldots, k$. Similarly, if $a \in A$ is such that $ax_1 = 0$, then $ax_i = 0$ for all $i = 1, \ldots, n$.

Let $a \in A$ be such that $ax_1 \neq 0$. Since $AamB = 0$ and $am \neq 0$ (because $\{y_1, \ldots, y_k\}$ is linearly independent), the induction hypothesis implies that $\{ax_1, \ldots, ax_k\}$ is linearly independent. Thus, if $b \in B$ is such that $y_k b \neq 0$, then $amb \neq 0$. This contradiction establishes the lemma. ∎

Let A be a left T-nilpotent algebra over a field F with zero right annihilator (for instance, the opposite ring of the prime radical of the Sasiada example discussed earlier or the ring of those $\aleph_0 \times \aleph_0$ matrices in F_{fin} which are strictly lower triangular). Let B be the right T-nilpotent algebra over F with zero left annihilator (for instance, A^{op}). If $M = A^1 \otimes_F B^1$ is as in the lemma, then

$$R = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$$

is a ring such that $P(R) = R$ and the lemma guarantees that $M(R) = 0$. The group $G = \{e, \theta\}$ of two elements acts on R via

$$\theta\left(\begin{bmatrix} a & m \\ o & b \end{bmatrix}\right) = \begin{bmatrix} a & -m \\ o & b \end{bmatrix}$$

and the fixed ring is $R^G \cong A \oplus B$. The fixed ring is M-nilpotent (in fact, a direct sum of a right T-nilpotent ring and a left T-nilpotent ring), so $\Delta(R^G) = R^G$. This shows that the Pascaud result does not extend to M-nilpotence; in fact, for this example R^G is M-nilpotent and R has zero middle annihilator.

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