J. Austral. Math. Soc. (*Series A*) **35** (1983), 308–326

THE ENDOMORPHISM RING OF A LOCALLY FREE MODULE

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(Received 10 March 1982; revised 9 August 1982)

Communicated by R. Lidl

Abstract

We identify a large class of rings over which locally free modules are determined by their endomorphism rings. We characterize these endomorphism rings and consider under what circumstances the conditions on the locally free modules can be relaxed, for example by requiring that only one of the rings need be in the special class, or by replacing "free' by "projective".

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 16 A 65.

1. Introduction

The ring $E_D(V)$ of linear operators on a finite-dimensional vector space V over a division ring D can be characterized in purely ring-theoretical terms by the Wedderburn-Artin Structure Theorem [8, page 39]: a ring E is isomorphic to $E_D(V)$ for some D and V if and only if E is a simple ring with descending chain condition on right ideals. The parameters D and V can be recovered from E as follows: $D \cong eEe$ for any primitive idempotent e of E, and $V \cong \oplus_n D$, where n is the length of any maximal chain of right ideals of E. A consequence of this result is that if $E_D(V) \cong E_{D'}(V')$, then $D \cong D'$ and $V \cong V'$. Moreover, as Baer showed in [1, page 183], every ring isomorphism of $E_D(V)$ onto $E_{D'}(V')$ is induced by a semi-linear isomorphism of V_D onto $V'_{D'}$.

There have been several extensions of these results from finite-dimensional vector spaces to wider classes of modules. It is convenient to classify these

https://doi.org/10.1017/S1446788700027002 Published online by Cambridge University Press

theorems into two types, corresponding respectively to the Wedderburn-Artin and the Baer Theorems.

Let $\mathfrak{R}$ be a class of rings, and $\mathfrak{M}$ a class of modules over rings in $\mathfrak{R}$.

(1) Characterization Problem: describe in ring-theoretic terms $E_R(M)$ for all $R \in \mathfrak{R}$ and $M_R \in \mathfrak{M}$.

(2) Uniqueness Problem: prove or disprove that any ring isomorphism of $E_R(M)$ onto $E_S(N)$ is induced by a semi-linear isomorphism of M_R onto N_S for all $R, S \in \mathfrak{R}$ and $M_R, N_S \in \mathfrak{M}$. These problems are very difficult if $\mathfrak{M}$ is allowed to be at all general, even for very restricted classes $\mathfrak{R}$ [2, 6, 7, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24]. In this paper, we restrict $\mathfrak{M}$ to consist only of free or "almost free" modules, but we allow as much freedom as possible in the choice of $\mathfrak{R}$.

The first result of this nature was obtained by Baer's student Wolfson, who in his 1953 doctoral dissertation [25] solved both the Characterization and the Uniqueness Problems for vector spaces of arbitrary dimension over division rings. In 1962 Wolfson [26] also solved the Uniqueness Problem for free modules over complete discrete valuation rings, and in the same year [27] for free modules over not necessarily commutative principal left ideal domains. In 1963 [28], he extended the same results to locally free modules.

In 1973, Hauptfleisch [7], as part of a more general study, solved the Uniqueness Problem for locally free modules over subrings of the rationals, and in 1975 Metelli and Salce [21] completely solved the Characterization Problem for this class of modules. Important advances were made by Liebert, who in 1974 [18] solved the Characterization Problem for free modules over principal ideal domains and in doing so, introduced a topological method that has proved exceedingly fruitful, not only in solving problems of this type, but also in characterizing endomorphism rings for classes of modules which are not locally free.

In Sections 3 and 4 of this paper we solve both problems for locally free modules over the largest class of rings for which the uniqueness property holds and which contains all the rings described above. We describe this class in Section 2, the simplest description being that it is the class of all rings R for which a summand of a free module is indecomposable if and only if it is isomorphic to R_R.

In Section 5 we extend the uniqueness results in several ways. For example, we consider what happens if only one of R, S is required to be in this class, or if only one of M_R, N_S is required to be free or locally free. We also consider whether "free" can be replaced by "projective".

Finally, in Section 6 we examine what happens if the Uniqueness Condition

> "every isomorphism of $E_R(M)$ onto $E_S(N)$ is induced by a semilinear isomorphism of M_R onto N_S"

is replaced by

> "if $E_R(M)$ is isomorphic to $E_S(N)$ then there exists a semilinear isomorphism of M_R onto N_S".

We show that the second conditon is stronger than the first if and only if $E_R(M)$ admits an automorphism which is not inner. We then exhibit examples of free modules with this property.

We use the standard notation of ring and module theory, as found for example in [4, 5, 9, 10]. In particular, scalars act on the right and homomorphisms on the left; all rings have an identity and all modules are unital; some important concepts mentioned above which may not be familiar are:

(1) A module is *locally free* if each finite subset is contained in a free direct summand.

(2) If M_R is an R-module and N_S an S-module, a *semi-linear homomorphism* $\alpha: M_R \to N_S$ is a pair (ϕ, θ) in which $\phi: R \to S$ is a ring isomorphism, and $\theta: M \to N$ an additive homomorphism such that for all $r \in R$ and $m \in M$, $\theta(mr) = \theta(m)\phi(r)$.

(3) If $\alpha: M_R \to N_S$ is a semi-linear isomorphism, α *induces* a ring isomorphism $\phi: E_R(M) \to E_S(N)$ if for all $f \in E_R(M)$, $\phi(f) = \alpha f \alpha^{-1}$.

2. The class of IF-rings

In his paper [18] characterizng the endomorphism ring of a free module over a not necessarily commutative principal ideal domain, Liebert pointed out that this ring property, like those used by Wolfson in his papers [27, 28] of 1962 and 1963 could be replaced by the weaker conditions:

PF : All projective modules are free; and

IBN : R has Invariant Basis Number.

It was proved by Leavitt [11] that for any ring R, if $R^m \cong R^n$ for some cardinals $m \neq n$, then m and n are both finite. Furthermore, for all $0 < m < n$, there exist rings R for which $R^h \cong R^k$ with $h < k$ if and only if $m \leqslant h$ and $k \equiv h \pmod{n - m}$. Thus the following condition is strictly weaker than IBN:

WIBN : (Weak invariant basis number) If $R \cong R^n$, then $n = 1$. Franzsen, in his 1981 Honours Dissertation proved that if R has PF and WIBN, then free R-modules satisfy the Uniqueness Problem.

However, an analysis of the proofs of Wolfson, Liebert and Franzsen show that these conditions are only used to establish:

IF : A non-zero summand of a free R-module is indecomposable if and only if it is isomorphic to R_R.

This ring condition seems to be the minimum required to solve both the Characterization and Uniqueness Problems for free modules of arbitrary cardinality. Certainly all the methods which have been discovered so far require it.

Because of its importance for algebraic geometry, much research has been devoted to the condition PF for commutative rings; in the non-commutative case, the subject has hardly been touched [23]. The condition has a useful counterpart for locally free modules.

SLF: (Strongly locally free) All summands of any locally free module are locally free.

Similarly, the condition IF has an analogue for locally free modules:

LIF: A non-zero summand of a locally free R-module is indecomposable if and only if it is isomorphic to R_R.

It is known (see for example [8, Theorem 17] and [28, Lemma 1.2]) that all five properties hold for principal ideal domains.

A property strictly weaker than PF which has proved useful in several contexts is:

FGPF: All finitely generated projective modules are free. The relationship between these six properties is described by the following:

PROPOSITION 2.1. *For any ring R,*
(1) *$SLF \Rightarrow FGPF \Rightarrow WIBN$;*
(2) *$SLF \Rightarrow LIF \Rightarrow IF \Rightarrow WIBN$;*
(3) *$PF \Rightarrow IF$.*

PROOF. (1) *$SLF \Rightarrow FGPF$*: Let M be a finitely generated projective module. Then M is a summand of a free, so locally free module. Hence M is locally free. But any finitely generated locally free module is free.

FGPF $\Rightarrow$ WIBN: Suppose R has FGPF but not WIBN, say $R_R \cong R_R^n$ for some n. Then $R \cong E_R(M) \cong E_R(R^n) \cong M_n(R)$ as rings, and $B = e_{11}M_n(R)$ is a projective $M_n(R)$ module, where e_{11} is the matrix unit having 1 in the (1, 1) position, 0 elsewhere. By FGPF, B is free, and there is a projection of B onto R ((1, 1) component) with kernel $K = [{}^{0\ RR\cdots R}_{\ \ \ 0}]$.

Thus $B \cong R \oplus K$, so K is projective and therefore free. Let $\alpha: K \to M_n(R)$ be a projection, and let $\alpha(x) = 1$ where $x = (x_{ij}) \in K$. Then $xe_{11} = 0$, and $0 = \alpha(xe_{11}) = \alpha(x)e_{11} = e_{11}$, a contradiction.

Hence R has WIBN as required.

(2) SLF $\Rightarrow$ LIF: Let M be an indecomposable summand of a locally free module. By SLF, M is indecomposable and locally free so $M \cong R_R$.

Conversely, let $M \cong R_R$ be a summand of a locally free module and suppose $M = E \oplus F$. Then E and F, being homomorphic images of R_R are finitely generated summands of a locally free module, so by SLF they are free. By (1), SLF implies R has WIBN so one of E or F is zero.

LIF $\Rightarrow$ IF: Let M be a non-zero summand of a free R-module F. Then F is locally free, so M is indecomposable if and only if it is isomorphic to R_R.

IF $\Rightarrow$ WIBN: Suppose $R \cong R^n$; by IF, R^n is indecomposable, so $n = 1$.

(3) PF $\Rightarrow$ IF: Let M be an indecomposable summand of a free module, so M is free and hence $M \cong R_R$. Conversely, let $M \cong R_R$ be a non-zero summand of a free module, and suppose $M = E \oplus F$. Then each of E and F is free. Since PF $\Rightarrow$ FGPF $\Rightarrow$ WIBN by (1), one of E or F is zero, so M is indecomposable.

3. The characterization theorem

In his characterization of the endomorphism ring of a free module over a principal ideal domain [18], Liebert used a combination of algebraic and topological conditions which involve an intrinsic characterization of the finite topology [5]. These conditions can be described as follows:

A *finite idempotent* in a ring E is an idempotent which is a sum of finitely many orthogonal primitive idempotents. For any finite idempotent e, let $e_L = \{x \in E : xe = 0\}$ be the left annihilator of e. The set of left annihilators of finite idempotents is a neighborhood basis of 0 for a linear topology on E if and only if for all finite idempotents e and f there is a finite idempotent u such that $u_L \subseteq e_L \cap f_L$. If this condition on E is satisfied, we call the resultant topology the *L-topology*.

A module is called *separable* if every finite subset is contained in a direct summand which is a direct sum of indecomposable submodules. It is well known (see for example [5, Section 107]) that for any ring R and R-module M, the L-topology exists on $E_R(M)$ and is equivalent to the finite topology if and only if M is separable. In this case $E_R(M)$ is a complete Hausdorff topological ring in which the left ideal generated by the finite idempotents is dense.

Using our notation and the results above, we can paraphrase Liebert's Theorem [18] as follows:

Let E be a unital ring. There is an IF-ring R and a free R-module F such that $E \cong E_R(F)$ if and only if E satisfies the following four conditions:

(1) *For all primitive idempotents e, eEe is an IF-ring.*

(2) *For all primitive idempotents e and f, $Ee \cong Ef$ as left ideals.*

(3) *E is Hausdorff and complete in the L-topology.*

(4) *E contains an orthogonal set I of finite idempotents such that the left ideal of E generated by the finite idempotents is $\bigoplus_{e \in I} eE$.*

It is clear that the characterization depends on the fact that E has an ample supply of idempotents. We shall now establish a similar theorem for locally free modules which also have enough idempotents, and for this we need the following lemma also due to Liebert:

LEMMA 3.1. *Let e, f be idempotents in a ring E. IF $Ee \cong Ef$, then fEe is a free right eEe module generated by f, and $EfEe = Ee$.*

PROOF. [18, Lemma 2.3].

The proof of the following theorem uses also techniques due to Metelli and Salce [21] and to Fuchs and Schultz [6].

THEOREM 3.2. *Let E be a unital ring. There is an LIF-ring R and a locally free R-module M such that $E \cong E_R(M)$ if and only if E satisfies the following five conditions*:

(1) *For all primitive idempotents e, eEe is an LIF-ring.*

(2) *For all primitive idempotents e, Ee is a locally free eEe-module.*

(3) *For all primitive idempotents e and f, $Ee \cong Ef$ as left ideals.*

(4) *E is Hausdorff and complete in the L-topology.*

(5) *The left ideal of E generated by the finite idempotents is dense in E in the L-topology.*

PROOF. A. The five conditions are necessary:

Let R be an LIF-ring. M a locally free R-module and $E = E_R(M)$. If e is a primitive idempotent in E, then $e(M)$ is an indecomposable summand of M, so isomorphic to R_R by the LIF condition. Say $e(M) = aR$, with $e(a) = a$.

Define a mapping $\phi : eEe \to R$ as follows: for $\eta \in E$, let $e\eta(a) = ar_\eta$ for some $r_\eta \in R$. Then $\phi : e\eta e \mapsto r_\eta$. It is routine to check that ϕ is a ring isomorphism, so (1) is satisfied.

Now define $\theta : Ee_{eEe} \to M_R$ by $\theta : \eta e \mapsto \eta(a)$. Since $e(a) = a$, $\eta(a) = 0$ implies $\eta e(M) = 0$, so θ is an additive monomorphism. Let $y \in M$; since

$$M = e(M) \oplus (1 - e)(M) = aR \oplus (1 - e)(M),$$

there exists an R-homomorphism μ such that $\mu(a) = y$ and $\mu(1 - e)(M) = 0$. Hence $\theta(\mu e) = y$ so θ is surjective. Next, for any $\eta, \mu \in E$,

$$\theta(\mu e)\theta(e\eta e) = \mu(a)r_\eta = \mu(ar_\eta) = \theta(\mu e\eta e),$$

so (ϕ, θ) is a semi-linear isomorphism. Consequently Ee is a locally free eEe-module, as required.

Now Ee is also a left E-module, and for all $\eta, \mu \in E$, $\theta(\mu\eta e) = \mu(\eta(a)) = \mu\theta(\eta e)$, so θ is also a left E-homomorphism. Hence if f is any other primitive idempotent in E,

$$_E Ef \cong {}_E M \cong {}_E Ee \quad \text{as left } E\text{-modules.}$$

Every finite subset of M is contained in a finitely generated free direct summand. But by the LIF property, R_R is indecomposable, so M is separable. Hence E admits the L-topology and is Hausdorff and complete by [5, Section 107]. By the same result, the left ideal generated by the finite idempotents is dense.

B. The five conditions are sufficient:

Let E be a ring admitting the L-topology and satisfying the five conditions. In particular, E has primitive idempotents. Let I be the set of primitive idempotents and E_0 the left ideal generated by I.

Let $e \in I$, $R = eEe$ and $M = Ee$; define $\phi: E \to E_R(M)$ by $\phi(\eta)(\mu e) = \eta\mu e$, so ϕ is a ring homomorphism. Suppose $\phi(\eta) = 0$; then η is in the left annihilator of Ee. By Lemma 3.1, for any finite idempotent f in E, $Ef = EeEf$, so $\eta f = 0$. Hence η is in the left annihilator of every finite idempotent in E. Since E is Hausdorff in the L-topology, $\eta = 0$, so ϕ is injective.

Now $M = Ee$ is a left ideal in E; we now want to show that $\phi(M)$ is a left ideal in $E_R(M)$. Let $f \in E_R(M)$ and suppose $f(e) = \alpha e$ for some $\alpha \in E$. Then for all $\beta \in E$,

$$\begin{aligned}\phi(f(e))(\beta e) &= f(e)\cdot\beta e = \alpha e\beta e = \alpha e \cdot e\beta e = f(e)\cdot e\beta e \\ &= f(e\beta e),\end{aligned}$$

since M is an eEe-module and f is an eEe-homomorphism. Thus

$$\phi(f(e))(\beta e) = (f \circ \phi(e))(\beta e) \quad \text{for all } \beta \in E, \text{ so}$$
$$\phi(f(e)) = f \circ \phi(e).$$

Hence for all $f \in E_R(M)$ and $\beta \in E$,

$$f \circ \phi(\beta e) = f \circ \phi(\beta) \circ \phi(e) = \phi(f \circ \phi(\beta))(e) = \phi(f(\beta e)) \in \phi(M),$$

so $E_R(M)\phi(M) \subseteq \phi(M)$ as required.

From now on we shall consider ϕ as an embedding of E in $E_R(M)$, so $M = Ee \subseteq E_R(M)e$. We have just shown that conversely $E_R(M)e \subseteq E_R(M)M \subseteq Ee$.

Now let $\Psi: Ef \to Ee$ be an isomorphism postulated by condition 3, and suppose $\Psi(f) = \eta(f, e)e$ for some $\eta(f, e) \in E$, and $\Psi^{-1}(e) = \eta(e, f)f$ for some $\eta(e, f) \in E$. Since Ψ and Ψ^{-1} are E-homomorphisms, $f = \Psi^{-1}\Psi(f) = \eta(f, e)\eta(e, f)f$. Define $\overline{\Psi}: E_R(M)f \to E_R(M)e$ by $\overline{\Psi}(gf) = g\eta(f, e)e$ for all $g \in E_R(M)$. This definition makes sense since $E_R(M)e = Ee$. Clearly $\overline{\Psi}$ is an

https://doi.org/10.1017/S1446788700027002 Published online by Cambridge University Press

$E_R(M)$-isomorphism with inverse $\overline{\Psi}^{-1}: he \mapsto h\eta(e, f)f$ for all $h \in E_R(M)$. It follows that for all $g \in E_R(M)$,

$$gf = \overline{\Psi}^{-1}(g\eta(f, e)e) = \overline{\Psi}^{-1}(\alpha e) \quad \text{for some } \alpha \in E$$
$$= \alpha\eta(e, f)f \in Ef.$$

Hence for all $f \in I$, $E_R(M)f \subseteq Ef \subseteq E_R(M)f$, so $E_R(M)E_0 = E_0$.

Next we show that $M = \langle fM : f \in I \rangle$ as right R-modules. Now $M = eM \oplus (1 - e)M$ as R-modules, and eM is rank 1; say, $eM = aR$. Let $z \in M$, say $z = ar + g$ where $r \in R$, $g \in (1 - e)M$. Then $M = (a + g)R \oplus (1 - e)M$, since both a and $(1 - e)M$ are contained. In this submodule, and the sum is direct for if $as + gs \in (1 - e)M$, then $as = e(as + gs) = 0$, so $s = 0$. Let $h \in E_R(M)$ be the projection onto $(a + g)R$ along $(1 - e)M$, and let $f = he$. Since $E_R(M)e = Ee$, $f \in E$. Furthermore

$$f(1 - e)M = he(1 - e)M = 0 \quad \text{and}$$
$$f(a + g) = he(a + g) = h(a) = h(a + g - g) = a + g,$$

so $f \in I$.

Hence $z = ar + g = a(r - 1) + a + g) = e(a(r - 1)) + f(a + g) \in eM + fM$, so $M = \langle fM : f \in I \rangle$ as required.

Now we can show that the L-topology on $E_R(M)$ restricted to E is the L-topology on E. One way is trivial: if α is a finite idempotent in E, the left annihilator of α in E is the intersection of E with the left annihilator of α in $E_R(M)$. Conversely, let f be a primitive idempotent in $E_R(M)$, and suppose $f(M) = yR$. By the previous paragraph,

$$y = e_1g_1 + \cdots + e_ng_n$$

for $e_1, \ldots, e_n \in I$ and $g_1, \ldots, g_n \in M$. Hence the annihilator of f in E is the annihilator of y in E, which contains the intersection of the annihilators of the e_i in E. Now let β be any finite idempotent in $E_R(M)$, say $\beta = f_1 + f_2 + \cdots + f_k$, where the f_k are primitive idempotents in $E_R(M)$. Then $\beta_L \cap E$ contains the intersection of the annihilators of the f_i in E, and hence the intersection of basic neighbourhoods of the L-topology on E.

Thus E is closed in the L-topology on $E_R(M)$. By condition 5, E_0 is dense in E, so to complete the proof that $e = E_R(M)$ it remains to show that E_0 is dense in $E_R(M)$. Let $\mu \in E_R(M)$ and let α_L be a basic neighbourhood of 0 in $E_R(M)$. Since E_0 is dense in E, there is an $\eta \in E_0$ such that $1 - \eta \in \alpha_L \cap E$. But α_L is a left ideal in $E_R(M)$, so $\mu - \mu\eta = \mu(1 - \eta) \in \alpha_L$. Hence $\mu\eta \in \mu + \alpha_L$, an open neighbourhood of μ. Since $E_R(M)E_0 = E_0$, $\mu\eta \in E_0$, so E_0 is dense in $E_R(M)$ and the proof is complete.

4. Uniqueness

In common with all similar results, the characterization theorem just proved does not contain even an implicit uniqueness statement; indeed the minimal idempotent e was chosen arbitrarily (condition 3 of this theorem only requires that Ee and Ef be isomorphic as ideals not as right eEe and fEf modules). In this section two uniqueness results will be proved which will show that the characterizations given in the previous section are essentially unique (that is, unique up to semilinear isomorphism). First of all uniqueness for Theorem 3.2 will be proved, and then the changes necessary to prove the corresponding result for Liebert's characterization will be noted.

THEOREM 4.1. *Let R and S be any two LIF-rings; then the pair (R, S) satisfes:*
UL: If M_R and N_S are any two nontrivial locally free modules such that $E_R(M) \cong E_S(N)$ then any isomorphism $\alpha : E_R(M) \to E_S(N)$ is induced by a semilinear isomorphism of M_R onto N_S.

PROOF. (i) Let M_R, N_S and α be given, say $\alpha : \eta \mapsto \eta^*$. Since $M \neq 0$, $M = mR \oplus M'$ for some $m \neq 0$. Then α induces an isomorphism $\alpha' = \alpha|_{E_R(mR)}$, $\alpha' : E_R(mR) \to E_S(N_0)$ where $N_0 = \varepsilon^*(N)$, ε being the projection onto mR. R is an IF-ring since LIF $\Rightarrow$ IF so $(mR)_R$ is indecomposable. Hence N_{0_S} is indecomposable, and thus $N_{0_S} \cong S_S$, as S is an LIF-ring. Say $N_0 = nS$, $n \neq 0$.

(ii) $(mR)_R \cong R_R$ and $(nS)_S \cong S_S$ so $E_R(mR) \cong E_R(R) \cong R$ and similarly $E_S(nS) \cong S$. So, if $\eta \in E_R(mR)$ then η has the form $\varepsilon\lambda\varepsilon$ for some $\lambda \in R$. Note that as $mR\lambda \subseteq mR$, $\varepsilon\lambda\varepsilon = \lambda\varepsilon$ and similarly for $E_S(nS)$. Let $\lambda \in R$; then $\varepsilon\lambda\varepsilon \in E_R(mR)$ so $\varepsilon^*\lambda^*\varepsilon^* \in E_S(nS)$, so there is an $\mu \in S$ such that $\varepsilon^*\lambda^*\varepsilon^* = \varepsilon^*\mu\varepsilon^*$. Denoting μ by $\phi''(\lambda)$, we have defined a map $\phi'' : R \to S$. Suppose $\varepsilon^*\mu'\varepsilon^* = \varepsilon^*\mu\varepsilon^*$; then $\varepsilon^*(\mu' - \mu)\varepsilon^* = 0$, that is, $(\mu - \mu')\varepsilon^* = 0$. Thus $\mu - \mu' = 0$ so $\mu = \mu'$ and ϕ'' is well-defined. Clearly ϕ'' is onto. For $\lambda \in R$, $\varepsilon^*\lambda^*\varepsilon^* = 0$ if and only if $\lambda = 0$, so ϕ'' is one-one. Let $\lambda, \pi \in R$, say $\mu = \phi''(\lambda)$, $\eta = \phi''(\pi)$. Then $\varepsilon^*(\lambda\pi)^*\varepsilon^* = \varepsilon^*\lambda^*\pi^*\varepsilon^*$; but $\varepsilon\pi\varepsilon = \pi\varepsilon$ so $\varepsilon^*\pi^*\varepsilon^* = \pi^*\varepsilon^*$; thus $\varepsilon^*(\lambda\pi)^*\varepsilon^* = (\varepsilon^*\lambda^*\varepsilon^*)(\varepsilon^*\pi^*\varepsilon^*)$ (ε^* is idempotent). Hence $\varepsilon^*(\lambda\pi)^*\varepsilon^* = \varepsilon^*\mu\varepsilon^*\eta\varepsilon^* = \varepsilon^*\mu\eta\varepsilon^*$, that is, $\phi''(\lambda\rho) = \phi''(\lambda)\phi''(\pi)$. Similarly $\phi''(\lambda + \pi) = \phi''(\lambda) + \phi''(\pi)$. Hence ϕ'' is a ring isomorphism.

For $a \in M$, there is an $\eta \in E_R(M)$ such that $a = \eta(m)$ (for example take η mapping mr to ar and annihilating M'). Define $\phi' : M \to N$ by $\phi'(a) = \eta^*(n)$. If $a = \eta'(m)$, then $(\eta' - \eta)(m) = 0$ so $(\eta' - \eta)\varepsilon = 0$ and $(\eta'^* - \eta^*)\varepsilon^* = 0$. Hence $\eta^*(n) = \eta'^*(n)$ and ϕ' is well-defined. Similarly $\phi'(a) = 0$ if and only if $a = 0$ so ϕ' is one-one. The steps in the definition of ϕ' can be easily reversed to show that ϕ' is onto. Let $a, b \in M$, say $a = \eta(m)$, $b = \mu(m)$; then $(\eta + \mu)(m) = a + b$, so

$\phi'(a+b) = (\eta+\mu)^*(n) = (\eta^*+\mu^*)n = \eta^*(n)+\mu^*(n) = \phi'(a)+\phi'(b)$. Hence ϕ' is a group isomorphism.

It remains to show that $\phi = (\phi', \phi'')$ is a semilinear isomorphism which induces α. Let $a \in M$, $\lambda \in R$, $a = \eta(m)$ where $\eta \in E_R(M)$; then $a\lambda = \eta(m)\lambda = \eta(m\lambda) = (\eta\lambda)(m) = (\eta\lambda\varepsilon)(m) = (\eta(\varepsilon\lambda\varepsilon))(m)$. So $\phi'(m\lambda) = (\eta\varepsilon\lambda\varepsilon)^*(n) = (\eta^*\phi''(\lambda)\varepsilon^*)(n) = \eta^*(n\phi''(\lambda)) = \eta^*(n)\phi''(\lambda) = \phi'(a)\phi''(\lambda)$ so ϕ is a semilinear isomorphism. Furthermore, if $\xi \in E_R(M)$, $c \in N$ say $c = \eta^*(n)$ with $\eta \in E_R(M)$, then $\xi^*(c) = \xi^*\eta^*(n) = \phi'(\xi\eta(m)) = \phi'\xi(\eta(m)) = \phi'\xi\phi^{-1}(c)$. Thus $\xi^* = \phi'\xi\phi'^{-1}$ and hence ϕ' induces α.

REMARKS. (1) The proof has been split into two parts because part (ii) can be used in every situation where $M \cong R_R \oplus M'$ and N_{0_S} can be shown to be isomorphic to S_S. So from now on whenever this can be established we will refer back to part (ii) above for the remainder of the proof.

(2) It can be seen from the proof that it is sufficient to assume R and S are IF-rings and at least one of them is an LIF-ring so given E in Theorem 3.2, once M_R is constructed, there is no other locally free module N_S over an IF ring with $E_S(N) \cong E$.

(3) Lemmas 1.1 and 1.2 of Wolfson [28, page 591] show that any left pid is an LIF-ring, so Wolfson's Theorem A of the same paper is a corollary to the above.

THEOREM 4.2. *If R and S are any two IF rings, then (R, S) satisfies:*
UF: as for UL with "free" replacing "locally free".

PROOF. Let M_R, N_S and α be given, then $M = mR \oplus M'$ for some $m \neq 0$ and $N = N_0 \oplus N'$ with $E_R(mR) \cong E_S(N_0)$. R is an IF ring so $(mR)_R$ is indecomposable, and hence N_{0_S} is indecomposable. Thus $N_0 \cong S_S$ as S is an IF-ring. The rest follows as above.

REMARK. It is sufficient to assume that S is a PF-ring.

5. Extensions of the uniqueness results

In the last section it was proved that if M_R is a locally free module over an LIF-ring then there is essentially no other such module whose endomorphism ring is isomorphic to $E_R(M)$. But the question still remains, is M_R the only locally free module with $E_R(M)$ as its endomorphism ring? In general the answer is no. Showing exactly when M_R is unique is the first aim of this section. We will then consider the property UF from the point of view of finding conditions on (R, S)

forcing it to satisfy UF. Some comments about various generalizations of UF will then be made.

PF-rings will play a major role in this section, so let $\mathcal{P}$ denote the class of all such rings (that is, $R \in \mathcal{P}$ if and only if R is a PF-ring).

THEOREM 5.1. *Let K be a ring satisfying the conditions in Theorem 3.2, say M_R is a locally free module over an LIF-ring, such that $E_R(M) \cong K$. Then up to semilinear isomorphism M_R is the only locally free module (over any ring) with endomorphism ring isomorphic to K if and only if K is an LIF-ring.*

PROOF. ($\Rightarrow$) If K is not an LIF-ring, then $K \not\cong R$, so there is no semilinear isomorphism of K_K onto M_R; but $E_K(K) \cong K$.

($\Leftarrow$) Let S be any ring with 1 and N_S any locally free S-module such that $E_S(N) \cong K$. Now $E_K(K) \cong K$ and K is an LIF-ring so K_K is indecomposable. Since $E_S(N) \cong K \cong E_K(K)$, N_S is indecomposable. Let $0 \neq n \in N$; N is locally free so there is a nonzero free module F such that $n \in F$ and $N = F \oplus N'$. But N is indecomposable and $F \neq 0$, so $N' = 0$; that is, N_S is free and thus has rank 1. Thus $N_S \cong S_S$, so $E_S(N) \cong S$ implies $K \cong S$. There is a semilinear isomorphism from K_K to S_S induced by this ring isomorphism so there is a semilinear isomorphism from K_K to N_S. In particular there is one from K_K to M_R and hence from M_R to N_S.

COROLLARY 5.2. *The locally free module M_R constructed in Theorem 3.2 is not in general unique up to semilinear isomorphisms.*

PROOF. Theorem 5.1 shows it to be unique only in the case where M_R is free of rank 1.

Before continuing we will need the following results.

LEMMA 5.3. *If R is any ring with 1 and M_R is any free R-module with an infinite basis, then M_R has a well-defined rank.*

PROOF. See Cohn [4, volume 2, page 102].

LEMMA 5.4. *Let M_R and N_R be nontrivial modules such that at least one is free. Then there is a semilinear isomorphism of M_R onto N_R if and only if there is an isomorphism of M_R onto N_R.*

PROOF. ($\Leftarrow$) any isomorphism is semilinear.

($\Rightarrow$) We may assume $M_R = \oplus_\gamma R_R$ is free. Let

(a) $\phi = (\phi', \phi'') : N_R \to \oplus_\gamma R_R$ be the semilinear isomorphism.

(b) $j_i : R_R \to \oplus_\gamma R_R$ be the ith injection.

For all i let $N_i = \phi'^{-1} j_i(R_R)$. It is easily seen that $N_R = \oplus_\gamma N_{i_R}$. So it is sufficient to show $N_{i_R} \cong R_R$ for all i. Let $n_i = \phi'^{-1} j_i(1_R)$ and suppose $x \in N_i$, then $\phi'(x) = j_i(r) \in j_i(R)$. So $x = \phi'^{-1}(j_i(r)) = \phi'^{-1}(j_i(1_R)r) = \phi'^{-1}(j_i(1_R))\phi''^{-1}(r) = n_i\phi''^{-1}(r) \in n_iR$. Thus $N_i \subseteq n_iR \subseteq N_i$ so $N_i = n_iR$ and it is easily seen that $n_iR \cong R_R$. Hence $N_R \cong \oplus_\gamma R_R$.

Thus:

Corollary 5.5. *If M_R is free of infinite rank then $\forall n \in \mathbf{Z}^+$ there is no semilinear isomorphism of M_R onto $\oplus_1^n R_R$.*

It can be seen as a corollary to Theorem 5.1 that even for locally free modules, $E_R(M) \cong E_S(N)$ does not necessarily imply that there is a semilinear isomorphism of M_R onto N_S. So it is not the case that every pair of rings (R, S) has UL or even UF.

Definition. For any infinite cardinal γ, $M_\gamma^c(R)$ is the ring of column finite $\gamma \times \gamma$ matrices with entries from R.

Lemma 5.6. *If (R, S) has UF then (R, S) satisfies*
W: (a) If $R \cong M_\gamma^c(S)$ then $\gamma < \infty$, and
(b) If $S \cong M_\eta^c(R)$ then $\eta < \infty$.

Proof. (a) Suppose $R \cong M_\gamma^c(S)$, then $E_R(R) \cong E_S(\oplus_\gamma S)$ but:

(i) if $R \not\cong S$ then there is no semilinear isomorphism of R_R onto $\oplus_\gamma S_S$ which is a contradiction as (R, S) has UF, so we can assume

(ii) $R \cong S$. Then there is a semilinear isomorphism of S_S on R_R so there is a semilinear isomorphism $S_S \to R_R \to \oplus_\gamma S_S$. Thus by Corollary 5.5 $\nu < \infty$ and similarly for (b).

Hence:

Corollary 5.7. *If R is any ring with 1 then there is a ring S with 1 such that (R, S) does not have UF.*

Proof. Let γ be infinite and $S = M_\gamma^c(R)$. Then by Lemma 5.6 (R, S) cannot have UF.

There are numerous other examples of such pairs of rings. One of the more important examples was given by Cohn [3, page 255] where he constructs nonisomorphic rings R and S with $M_n(R) \cong M_n(S)$ for some $n \in \mathbf{Z}^+$.

As seen earlier R and S being IF-rings is a sufficient condition for (R, S) to have UF. But this condition is not necessary. Clearly if there are no free modules M_R and N_S with $E_R(M) \cong E_S(N)$ then (R, S) has UF but there is no reason why R or S should have IF. Excluding this case, however, what can be said is the following:

LEMMA 5.8. *If (R, S) has UF and there exist free modules with $E_R(M) \cong E_S(N)$ then R and S have WIBN.*

PROOF. Suppose $E_R(M) \cong E_S(N)$. Then there is a semilinear isomorphism of M_R onto N_S as (R, S) has UF. In particular $R \cong S$. Suppose R did not have WIBN. Then Example 2 of Section 6 shows that there is a free module M_R and an automorphism of $E_R(M)$ which is not induced by a semilinear "automorphism". This contradicts UF, so R has WIBN.

Unfortunately this procedure does not show IF to be necessary in this case. But for R and S in $\mathcal{P}$ it is possible to give necessary and sufficient conditions of the type mentioned in Lemma 5.6.

THEOREM 5.9. *If $R, S \in \mathcal{P}$, then (R, S) has UF if and only if $R \cong M^c_\gamma(S)$ implies $\gamma = 1$, and $S \cong M^c_\gamma(R)$ implies $\gamma = 1$.*

PROOF. ($\Rightarrow$) Suppose $R \cong M^c_\gamma(S)$. Then

$$E_R(R) \cong R \cong M^c_\gamma(S) \cong E_S\big(\oplus_\gamma S\big),$$

so by UF, R_R is decomposable into γ summands. Since $R \in \mathcal{P}$, its summands are free, so γ is finite. But by Proposition 2.1, R has WIBN, so $\gamma = 1$.

A similar argument works for S.

($\Leftarrow$) Suppose M_R, N_S and an isomorphism $\alpha : E_R(M) \to E_S(N)$ are given. Then as in Theorem 4.1, $M = mR \oplus M'$ and $N = N_0 \oplus N'$ with $R \cong E_R(mR) \cong E_S(N_0)$. But since $S \in \mathcal{P}$, N_0 is free, so $R \cong M^c_\gamma(S)$ for some γ.

Since $\gamma = 1$, $N_0 \cong S_S$, and the rest follows as in part (ii) of the proof of Theorem 4.1.

If R and S are no longer assumed to be in $\mathcal{P}$ then the situation becomes less clear, but much can still be said. In this case it is natural to widen the class of modules involved. Consider the following property.

UFP: as for UF with "projective modules with at least one free" replacing "free modules".

THEOREM 5.10. *(R, S) has UFP if and only if (R, S) has UF and satisfies*:
(P): *If P_R (Q_S) is projective but not free then $E_R(P) \not\cong S$ ($E_S(Q) \not\cong R$).*

PROOF. ($\Rightarrow$) Clear.
($\Leftarrow$) (i) Let M_R, N_S and α be given. Without loss of generality we may assume M_R is free, so $M_R = mR \oplus M'$ and $N = N_0 \oplus N'$ as before. N_0 is projective and (R, S) has P so $E_S(N_0) \cong R$ shows that N_0 is free. The rest follows as in the proof of Theorem 5.9.

REMARKS. (1) UFP implies UF but the converse is not true in general, for example let $R = \mathbf{Z}_2$, $S = \mathbf{Z}_2 \oplus \mathbf{Z}_3$.
(2) It is expected that relaxing the requirement in UFP that one of M_R and N_S be free will lead to a considerably more difficult problem, that is, when does (R, S) satisfy:
UP: as for UF with "projective" replacing "free".
What can be said is that UP implies UFP but again the converse is not necessarily true (for example $R = \mathbf{Z}_2 \oplus \mathbf{Z}_3$, $S = \mathbf{Z}_2 \oplus \mathbf{Z}_5$), and the following:

PROPOSITION 5.11. *If (R, S) has UFP and one of R or S satisfies*:
P_1: Every projective has a free summand, then (R, S) has UP.

(It is not known whether P_1 is necessary for UP or whether the class of rings with P_1 is disctinct from $\mathcal{P}$.)

6. Semilinear isomorphisms

At first glance, the statement of the Uniqueness Problem seems somewhat unnatural. Why insist on each ring isomorphism being induced (by a semilinear isomorphism) when the results might be found more easily by just proving the existence of some semilinear isomorphism? In this section we will first look at whether dropping this restriction can lead to any great improvement in the results. Then we shall look for modules M_R and N_S which are such that there is an isomorphism $\alpha: E_R(M) \to E_S(N)$, which is not induced, even though there are semilinear isomorphisms of M_R onto N_S. Consider the following properties.
WUF: as for UF but α need not be induced;
WUFP: as for UFP but α need not be induced.

The following lemma shows that no gain at all is made by considering WUFP instead of UFP.

LEMMA 6.1. *(R, S) has WUFP if and only if (R, S) has UFP.*

PROOF. $(\Leftarrow)$ clear.

$(\Rightarrow)$ Suppose (R, S) has WUFP. If there are no projectives M_R, N_S with $E_R(M) \cong E_S(N)$ then (R, S) has UFP. So suppose there are M_R, N_S with $\alpha : E_R(M) \to E_S(N)$ an isomorphism. We may assume M_R is free. So $M = mR \oplus M'$, and $N = N_0 \oplus N'$ with $E_S(N_0) \cong E_R(mR)$. (R, S) has WUFP, N_0 is projective and mR is free so there is a semilinear isomorphism of $mR_R \to N_{0_S}$. In particular $R \cong S$, thus there is a semilinear isomorphism $S_S \to R_R \to N_{0_S}$, that is, $N_{0_S} \cong S_S$.

Part (ii) of 4.1 can now be used to construct the required semilinear isomorphism.

The case including UF and WUF is not quite as general, however.

THEOREM 6.2. *If (R, S) has P (see Theorem 5.10), then (R, S) has UF if and only if (R, S) has WUF.*

PROOF. $(\Rightarrow)$ clear.

$(\Leftarrow)$ Follow the steps for $(\Rightarrow)$ in 6.1 but notice that N_0 is free as (R, S) has P.

We will now search for free modules M_R, N_S with the properties:

(1) there is a semilinear isomorphism of M_R onto N_S;

(2) there is an isomorphism $\alpha : E_R(M) \to E_S(N)$ which is not induced by any semilinear isomorphism.

By considering the proof of Theorem 4.1 we are lead to the following:

LEMMA 6.3. *Suppose M_R, N_S are nontrivial free modules. Let $\alpha : E_R(M) \to E_S(N)$ be a ring isomorphism. Say $M = mR \oplus M'$, $\varepsilon : M \to mR$ a projection, so $N = N_0 \oplus N'$ where $N_0 = \alpha(\varepsilon)(N)$. Then*

(i) α induces a ring isomorphism $\alpha' : E_R(mR) \to E_S(N_0)$ and

(ii) α is induced by a semilinear isomorphism of M_R onto N_S if and only if α' is induced by a semilinear isomorphism of mR onto N_{0_S}.

PROOF. (i) Let $\alpha' = \alpha|_{\varepsilon E_R(M)}$; then α' is the required isomorphism.

(ii) $(\Leftarrow)$ Similar to (ii) in the proof of 4.1.

$(\Rightarrow)$ Suppose α is induced by $\phi = (\phi', \phi'') : M_R \to N_S$;

that is, $\alpha(\eta) = \eta^* = \phi'\eta\phi^{-1}$ $\forall \eta \in E_R(M)$ so $\eta^*\phi' = \phi'\eta$. Clearly $\phi'(mR) = nS$ where $n = \phi'(m)$. As $m = \varepsilon(m)$, $\phi'(m) = \phi'\varepsilon(m)$ but $\varepsilon^*\phi' = \phi'\varepsilon$ so $n = \phi'(m) = \varepsilon^*\phi'(m) = \varepsilon^*(n) \in N_0$. Thus $\phi'(mR) \subseteq N_0$. Similarly $N_0 \subseteq \phi'(mR)$, tht is, $\phi'(mR) = N_0$. So $\phi_0 = (\phi'|_{mR}, \phi'')\colon mR_R \to N_{0_S}$ is a semilinear isomorphism. Let $\mu' \in E_R(mR)$ then $\mu' = \varepsilon\mu\varepsilon$ for some $\mu \in E_R(M)$, and it is easily checked that $\alpha(\mu') = (\phi'|_{mR})\mu'(\phi'|_{mR})^{-1}$, that is, ϕ_0 induces α'.

Remark. This is in fact true for all idempotents ε not just projections onto free summands.

There is the following immediate simplification.

Lemma 6.4. *There are modules M_R and N_S with the above properties if and only if there is a module P_T with an automorphism of $E_T(P)$ not induced by a semilinear automorphism.*

Proof. ($\Leftarrow$) Take $M_R = N_S = P_T$.

($\Rightarrow$) Suppose there is a semilinear isomorphism $\theta\colon M_R \to N_S$ and an isomorphism $\alpha\colon E_R(M) \to E_S(N)$ which is not induced. Then θ^{-1} induces an isomorphism $\beta\colon E_S(N) \to E_R(M)$. So $\gamma = \beta\alpha\colon E_R(M) \to E_R(M)$ is an automorphism of $E_R(M)$. Suppose γ is induced by $\phi = (\phi', \phi'')$. Then $\forall \eta \in E_R(M)$, $\gamma(\eta) = \phi\eta\phi^{-1}$. But this implies that $\alpha(\eta) = (\theta\phi)\eta(\theta\phi)^{-1}$, so that α is induced by the semilinear isomorphism $\theta\phi$ which is a contradiction.

From now on, by an inner (outer) automorphism will be meant an automorphism induced (not induced) by a semilinear automorphism. So Lemma 6.4 shows that we need only look for an outer automorphism.

Suppose we had an outer automorphism. Let ε be a projection onto a rank 1 free summand mR; and $M_0 = \alpha(\varepsilon)(M)$. By Lemma 6.3, as α is not induced, $\alpha' = \alpha|_{E_S(mR)}$ cannot be induced by a semilinear isomorphism of mR_R onto M_{0_R}. There are two possibilities.

(1) $M_{0_R} \cong R_R$ (and hence α' defines an outer automorphism of $E_R(R)$), or

(2) $M_{0_R} \not\cong R_R$.

Only case (2) can occur:

Lemma 6.5. *Let R be any ring with* 1, *then any automorphism of $E_R(R)$ is inner.*

Proof. Let $\alpha\colon E_R(R) \to E_R(R)$ be an automorphism of $E_R(R)$. $E_R(R) \cong R$ under the isomorphism $\zeta\colon \eta \to \eta(1_R)$. Define $\phi = (\phi', \phi'')\colon R_R \to R_R$ by $\phi' = \zeta\alpha\zeta^{-1}$, $\phi'' = \zeta\alpha\zeta^{-1}$. Clearly ϕ is a semilinear automorphism of R_R. Let $\eta \in E_R(R)$,

$r \in R_R$; then $\phi'\eta\phi'^{-1}(r) = r(\alpha(\eta)(1_R))$, but $r(\alpha(\eta)(1_R)) = \alpha(\eta)(r)$, thus $\alpha(\eta) = \phi'\gamma\phi'^{-1}$ and so α is inner.

Let us restrict our attention to those ε whose image is isomorphic to R_R. If we are to find an outer automorphism then for all such idempotents $\varepsilon \in E_R(M)$, $\varepsilon(M) \not\cong \alpha(\varepsilon)(M)$. There are 3 possibilities for $\alpha(\varepsilon)(M)$:

(1) $\alpha(\varepsilon)(M) \cong \oplus_1^n R_R$ for some $n \in \mathbf{Z}^+ \setminus \{1\}$;

(2) $\alpha(\varepsilon)(M) \cong \oplus_\gamma R_R$ for some infinite γ; or

(3) $\alpha(\varepsilon)(M)$ is projective but not free.

As $E_R(\varepsilon(M)) \cong E_R(\alpha(\varepsilon)(M))$ the following conditions on R are easily seen to be necessary for each of the above to occur:

(1′) there is a positive integer n with $R \cong M_n(R)$, but $R_R \not\cong \oplus_1^n R_R$;

(2′) there is an infinite γ with $R \cong M_\gamma^c(R)$;

(3′) there is a non-free projective P_R with $R \cong E_R(P)$.

So if R is a ring for which none of these hold then for a free R-module M_R all automorphisms of $E_R(M)$ are inner, for example if R has P and WIBN. Even if one of these occurs for a given ring it is not immediately apparent that an example of an outer automorphism can be constructed. However, the following two examples show that for any ring over which 1′ or 2′ occurs there is a free module with an outer automorphism. Whether this is so in general for 3′ is not known, but the special case where $R = M_n(R)$ for some $n \in \mathbf{Z}^+$ will be treated. Note that this special case includes all rings satisfying 1′ or without WIBN (so the proof of Lemma 5.8 will be complete.)

EXAMPLE 1. Let $R = E_{\mathbf{Z}}(\oplus_\omega \mathbf{Z})$; then $R \cong M_\omega^c(R)$ so let $\alpha : R \to M_\omega^c(R)$ by any isomorphism. Then $\alpha' : M_\omega^c(R) \to M_\omega^c(M_\omega^c(R))$ given by $\alpha' : (a_{ij}) \mapsto (\alpha(a_{ij}))$ is a ring isomorphism ($M_\omega^c(S)$ is the ring of all $\omega \times \omega$ column convergent matrices, that is, all the column finite matrices, and (a_{ij}) is column finite if and only if $(\alpha(a_{ij}))$ is).

For $n \in \mathbf{Z}^+$, let X_n be the set of positive integers divisible by exactly n distinct primes (with the understanding that $1 \in X_1$). Then $\mathbf{Z}^+ = \cup_\omega X_n$ and $|X_i| = \omega$. Thus $\oplus_\omega R_R \cong \oplus_\omega(\oplus_{X_n} R)_R$ under the map $\beta : \eta \to \eta^*$ where $\eta^*(i, j) = \eta(k)$ with k the jth element (ordered by magnitude) of X_i. Then β induces an isomorphism $\beta' : M_\omega^c(R) \to M_\omega^c(M_\omega^c(R))$.

Now $\gamma = \beta'^{-1}\alpha' : M_\omega^c(R) \to M_\omega^c(R)$ defines an automorphism γ' of $E_R(\oplus_\omega R)$.

Claim. γ' is an outer automorphism of $E_R(\oplus_\omega R)$. Now $\alpha : 1R \mapsto I_\omega$ as α is a ring isomorphism, so $\alpha' : e_{11} \to E_{11}$. Then we have $\gamma(e_{11}) = \Sigma_{i \in X_1} e_{ii}$ but $e_{11}(\oplus_\omega R) \cong R$ and $\gamma(e_{11})(\oplus_\omega R) = \oplus_{X_1} R \cong \oplus_\omega R$, which implies $e_{11}(\oplus_\omega R) \not\cong \gamma(e_{11})(\oplus_\omega R)$. Thus by 6.3 γ' is not inner.

REMARK. This holds for any ring $R \cong M_\gamma^c(R)$, γ infinite.

To give an example for the cases 1′ and 3′ we use the following fact.

COROLLARY 6.6 (to Proposition 2.1). *Suppose* $R = M_n(R)$ *for some* $n \in \mathbf{Z}^+$. *Then* $R_R = \oplus_1^n P_i$ *with* P_i *projective but not free and* $\mathrm{Hom}_R(P_i, P_j) \cong R$.

PROOF. The proof of Proposition 2.1 showed that $M_n(R)_{M_n(R)} \cong \oplus_1^n e_{ii} M_n(R)$, and $\mathrm{Hom}_{M(R)}(e_{ii}M_n(R), e_{jj}M_n(R)) \cong e_{ii}M_n(R)e_{jj} \cong R$. The result follows from the isomorphism $R \cong M_n(R)$.

Thus $E_R(\oplus_1^n P_i) \cong M_n(R)$, with e_{11} corresponding to the projection onto P_1.

EXAMPLE 2. Let $R \cong M_n(R)$, $n \in \mathbf{Z}^+$ and put $M_R = \oplus_\omega R_R$. Then $E_R(M) = M_\omega^c(R)$. Define $\alpha : M_\omega^c(R) \to M_\omega^c(M_n(R))$ by inserting brackets appropriately. But $M_\omega^c(M_n(R)) \cong E_R(\oplus_\omega(\oplus_1^n P_i))$, so $R_R \cong \oplus_1^n P_{i_R}$ implies that α induces an automorphism α' of $E_R(M)$. Clearly $\alpha'(e_{11}^\omega) = E_{11}$ where e_{11}^ω is the matrix with 1 in the (1, 1)th place and zeros elsewhere and E_{11} has e_{11} in the (1, 1)th place and zeros elsewhere. Thus $e_{11}(M_R) \cong R \not\cong P_1 \cong E_{11}(M) = \alpha'(e_{11})(M)$ so α' is outer.

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