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Characterizing slopes for the $(-2, 3, 7)$-pretzel knot

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Abstract. In this note, we exhibit concrete examples of characterizing slopes for the knot $12n242$, also known as the $(-2, 3, 7)$-pretzel knot. Although it was shown by Lackenby that every knot admits infinitely many characterizing slopes, the nonconstructive nature of the proof means that there are very few hyperbolic knots for which explicit examples of characterizing slopes are known.

1 Introduction

Given a knot $K \subseteq S^3$, we say that $p/q \in \mathbb{Q}$ is a *characterizing slope* for K if the oriented homeomorphism type of the manifold obtained by p/q-surgery on K determines the isotopy type of K uniquely. That is, p/q is a characterizing slope for K if there does not exist any knot $K' \neq K$ such that $S^3_{p/q}(K) \cong S^3_{p/q}(K')$. It was shown by Lackenby that every knot admits infinitely many characterizing slopes and for a hyperbolic knot any slope p/q with q sufficiently large is characterizing [Lac19]. Although these results show the existence of characterizing slopes, the proofs are nonconstructive and so there are very few hyperbolic knots for which explicit examples of characterizing slopes are known. Ozsváth and Szabó have shown that every slope is characterizing for the figure-eight knot 4_1 [OS19], and recent work of Baldwin and Sivek implies that every noninteger slope is characterizing for 5_2 [BS22]. The aim of this article is to exhibit explicit examples of characterizing slopes for the knot $12n242$, also known as the $(-2, 3, 7)$-pretzel knot (see Figure 1). Since $12n242$ is a hyperbolic L-space knot—Fintushel and Stern showed that it admits two lens space surgeries [FS80]—it has only finitely many noncharacterizing slopes that are not negative integers [McC19]. The following theorem is a quantitative version of this fact. As far as the author is aware, these are the first known explicit examples of characterizing slopes on a hyperbolic knot with genus greater than one.

Theorem 1.1 *Any slope p/q satisfying at least one of the following conditions is a characterizing slope for* $12n242$*:*

(i) $q \geq 49$*;*

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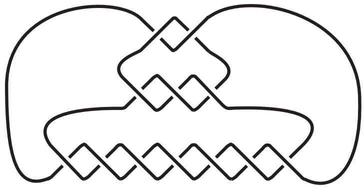

Figure 1: The main protagonist: $12n242$.

(ii) $p \geq \max\{24q, 441\}$; *or*
(iii) $q \geq 2$ *and* $p \leq -\max\{12 + 4q^2 - 2q, 441\}$.

Note that Theorem 1.1 also yields information about the characterizing slopes of the mirror of $12n242$: a slope p/q is characterizing for a knot K if and only if $-p/q$ is characterizing for the mirror mK. The key input allowing us to prove Theorem 1.1 is the fact that $12n242$ is one of the knots with smallest volume (up to reflection it one of only three hyperbolic knots with volume smaller than 3.07) [GHM21]. A result of Futer, Kalfagianni, and Purcell on the change in volume of a hyperbolic manifold under Dehn filling [FKP08] can then be used to restrict potential noncharacterizing slopes coming from surgeries on hyperbolic knots with large volume (and satellites thereof). This leaves only the possibility of noncharacterizing slopes coming from surgeries on the other knots of small volume, explicitly $4_1, 5_2, m5_2$ and $m12n242$, or satellites of these knots. Alongside tools from hyperbolic geometry, we use the Casson–Walker invariant and the ν^+ invariant from Heegaard Floer homology to rule out such surgeries.

In principle, one could use a similar approach to derive information about the characterizing slopes of the other small volume knots: 4_1 and 5_2. However, much better results have already been obtained by other means for both of these knots [BS22, OS19], so we restrict our analysis to $12n242$.

We note that Theorem 1.1 says nothing about negative integer characterizing slope. Although there are knots which possess infinitely many integer noncharacterizing slopes [BM18], all known examples admit infinitely many noncharacterizing slopes of both sign. This suggests that $12n242$ (and L-space knots more generally) should admit only finitely many integer noncharacterizing slopes. However, establishing such a result remains an interesting and challenging problem.

1.1 Noncharacterizing slopes

Lackenby has shown for a hyperbolic knot K any slope p/q with q sufficiently large is characterizing for K [Lac19]. For example, Theorem 1.1 shows that $q \geq 49$ is sufficiently large for $12n242$. However, the "sufficiently large" here is inherently dependant on the specific knot in question. To illustrate this dependence, we exhibit a family of hyperbolic two-bridge knots $\{K_q\}_{q\geq 1}$ such that for each q, the slope $\frac{1}{q}$ is noncharacterizing for K_q. This family is shown in Figure 2. The construction given is essentially the same as the one used by Brakes to find examples of surgeries on distinct satellite knots yielding the same manifold [Bra80].

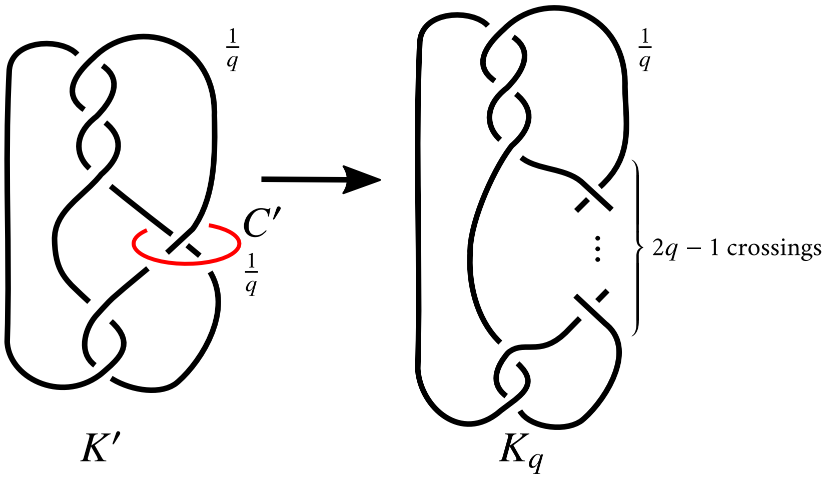

Figure 2: A link $K' \cup C'$, such that twisting along C' yields the two-bridge link K_q. Proposition 4.1 implies that K_q has $\frac{1}{q}$ as a noncharacterizing slope.

1.2 Conventions

The following notational conventions will be in force throughout the paper:

- Knots are always considered up to isotopy.
- When considering a rational number $p/q \in \mathbb{Q}$, we will always assume this to be written with p and q coprime and $q \geq 1$.
- When performing Dehn surgery on a knot K, we use p/q to denote the slope $p\mu + q\lambda$, where μ is the meridian and λ is the null-homologous longitude.
- Given two oriented 3-manifolds Y and Y', we will use $Y \cong Y'$ to denote the existence of an orientation-preserving homeomorphism between them.
- For a knot K, we will denote its *Alexander polynomial* by $\Delta_K(t)$. We will always assume this is normalized so that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$.
- Given a knot K in S^3, we will use mK to denote its mirror.
- An L-space knot is one which admits positive L-space surgeries.

2 Preliminaries

In this section, we gather together all the auxiliary results required for the proof of Theorem 1.1.

2.1 Knots of small volume

First, we use the fact that Gabai, Haraway, Meyerhoff, Thurston, and Yarmola have classified the hyperbolic 3-manifolds of small volume [GHM21].

Theorem 2.1 *If K is a hyperbolic knot in S^3 with* $\mathrm{vol}(K) \leq 3.07$, *then*

$$K \in \{4_1, 5_2, 12n242, m5_2, m12n242\}.$$

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Proof Gabai, Haraway, Meyerhoff, Thurston, and Yarmola have shown that there are exactly 14 one-cusped orientable hyperbolic 3-manifolds with hyperbolic volume less than or equal to 3.07 and that these are `m003`, `m004`, `m006`, `m007`, `m009`, `m010`, `m011`, `m015`, `m016`, `m017`, `m019`, `m022`, `m023`, and `m026` [GHM21, Theorem 1.5]. Precisely, three of these arise as the complements of knots in S^3: `m004`, `m015`, and `m016` are (ignoring orientations) the complements of 4_1, 5_2, and $12n242$, respectively. ∎

We will informally refer to the five knots in Theorem 2.1 as the "low volume knots" and the remaining hyperbolic knots as the "large volume knots." For our purposes, it will be useful to note that the volume of 4_1 satisfies

$$\mathrm{vol}(4_1) \approx 2.0988 \leq 2.1 \tag{2.1}$$

and the volume of $12n242$ satisfies

$$2.82 \leq \mathrm{vol}(12n242) \approx 2.821 \leq 2.83. \tag{2.2}$$

2.2 Slope lengths

Let K be a hyperbolic knot in S^3, that is, $S^3 \smallsetminus K$ admits a complete finite-volume hyperbolic structure with one cusp. Given a slope σ on K and horoball neighborhood N of the cusp, we can assign a length to σ by considering the minimal length of a curve representing σ on ∂N (measured in the natural Euclidean metric on ∂N). Since $S^3 \smallsetminus K$ has a unique cusp, there is a unique maximal horoball neighborhood of this cusp. We will use $\ell_K(\sigma)$ to denote the length of σ with respect this maximal horoball neighborhood.

Lemma 2.2 *Let K and K' be hyperbolic knots in S^3 with $\mathrm{vol}(K') < \mathrm{vol}(K)$. If r and r' are slopes such that $S^3_r(K) \cong S^3_{r'}(K')$, then*

$$\ell_K(r) < \frac{2\pi}{\sqrt{1 - \left(\frac{\mathrm{vol}(K')}{\mathrm{vol}(K)}\right)^{\frac{2}{3}}}}.$$

Proof Since the bound on the right-hand side is always strictly greater then 2π, we can assume without loss of generality that $\ell = \ell_K(r) > 2\pi$. By Perelman's resolution of the geometrization conjecture [Per02, Per03a, Per03b] and the 2π-theorem [BH96], this implies that $S^3_r(K)$ is a hyperbolic manifold. Furthermore, Futer, Kalfagianni, and Purcell have shown that we have the following volume bound [FKP08, Theorem 1.1]:

$$\mathrm{vol}(K)\left(1 - \left(\frac{2\pi}{\ell}\right)^2\right)^{\frac{3}{2}} \leq \mathrm{vol}(S^3_r(K)).$$

Moreover, since Thurston showed that volume strictly decreases under hyperbolic Dehn filling [Thu80, Theorem 6.5.6], we have that $\mathrm{vol}(S^3_r(K)) = \mathrm{vol}(S^3_{r'}(K')) < \mathrm{vol}(K')$. Together these bounds give

$$\mathrm{vol}(K)\left(1 - \left(\frac{2\pi}{\ell}\right)^2\right)^{\frac{3}{2}} < \mathrm{vol}(K'),$$

which can be easily rearranged to give the desired bound on $\ell_K(p/q)$. ∎

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Next, we need a mechanism for converting bounds on $\ell_K(p/q)$ into bounds on p and q.

Lemma 2.3 *Let* $K \subseteq S^3$ *be a hyperbolic knot of genus* $g(K)$*. Then:*

(a) $|q| \leq 1.79\ell_K(p/q)$ *and*
(b) $|p| \leq 1.79\ell_K(p/q)(2g(K) - 1)$.

Proof Let N be a horocusp neighborhood in the complement $S^3 \setminus K$ of K. Let A be the area of ∂N (equipped with its Euclidean metric). A simple geometric argument (e.g., as used by Cooper and Lackenby [CL98, Lemma 2.1]) shows that for any two slopes of K, we have

$$\ell_K(\alpha)\ell_K(\beta) \geq A\Delta(\alpha, \beta),$$

where $\Delta(\alpha, \beta)$ denotes the distance between α and β (cf. [Ago00, Lemma 8.1]). Since Cao and Meyerhoff have shown there always exists a horocusp neighborhood N with $\text{Area}(\partial N) \geq 3.35$ [CM01], this establishes the bound

$$\ell_K(\alpha)\ell_K(\beta) \geq 3.35\Delta(\alpha, \beta),$$

for all slopes α and β. Since $\Delta(1/0, p/q) = |q|$ and $\ell_K(1/0) \leq 6$ by the 6-theorem of Agol and Lackenby [Ago00, Lac03], this gives the bound (a). Since $\Delta(0/1, p/q) = |p|$ and $\ell_K(0/1) \leq 6(2g - 1)$ by [Ago00, Theorem 5.1], this also gives the bound (b). ∎

2.3 Hyperbolic surgeries on satellite knots

We will use the following result to understand noncharacterizing slopes coming from satellite knots.

Lemma 2.4 *Let* K *be a satellite knot such that* $S^3_{p/q}(K)$ *is hyperbolic for some* $p/q \in \mathbb{Q}$*. Then there is a hyperbolic knot* J *with* $g(J) < g(K)$ *and an integer* $w > 1$ *such that* $S^3_{p/q}(K) \cong S^3_{p/(qw^2)}(J)$*. Moreover, if* $q \geq 2$*, then* K *is a cable of* J *with winding number* w*.*

Proof Let T be an incompressible torus in $S^3 \setminus K$. We can consider K as a knot in the solid torus V bounded by T. Thus, we can consider K as a satellite with companion given by the core J of V. By choosing T to be "innermost," we can ensure that $S^3 \setminus J$ contains no further incompressible tori. That is, we can assume that J is not a satellite knot. By Thurston's trichotomy for knots in S^3, this implies that J is a torus knot or a hyperbolic knot [Thu82]. Since $S^3_{p/q}(K)$ is hyperbolic, it is atoroidal and irreducible. Consequently, after surgery the solid torus V must become another solid torus. However, Gabai has classified knots in a solid torus with nontrivial solid torus surgeries, showing that K is either a torus knot or a one-bridge braid in V [Gab89]. Moreover, since solid torus fillings on one-bridge braids only occur for integer surgery slopes, K is a cable of J unless $q = 1$. In either event, we have that

$$S^3_{p/q}(K) \cong S^3_{p/q'}(J),$$

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where the slope p/q' is determined by the curve bounding a disk after surgering V. Using a homological argument, one can show that $q' = qw^2$, where $w > 1$ is the winding number of K in V [Gor83, Lemma 3.3]. Since $S^3_{p/q}(K)$ is a hyperbolic manifold, J cannot be a torus knot. It follows that J must be a hyperbolic knot. The only remaining statement is the inequality $g(J) < g(K)$. This follows from Schubert's formula for the genus of a satellite knot [Sch53], which asserts that for a knot $K = P(J)$ with pattern P of winding number $w \geq 0$, there is a constant $g(P) \geq 0$ such that

$$g(K) = g(P) + wg(J).$$

We obtain the necessary inequality since $w \geq 2$. ∎

2.4 The Casson–Walker invariant

It will also be convenient to use surgery obstructions derived from the Casson–Walker invariant [Wal92]. For any rational homology sphere Y, this is a rational-valued invariant $\lambda(Y) \in \mathbb{Q}$. Boyer and Lines showed that this satisfies the following surgery formula [BL90]:

$$\lambda(S^3_{p/q}(K')) = \lambda(S^3_{p/q}(U)) + \frac{q}{2p}\Delta''_{K'}(1),$$

where $\Delta''_K(1)$ denotes the second derivative of the Alexander polynomial $\Delta_K(t)$ evaluated at $t = 1$. This formula immediately yields the following observation.

Lemma 2.5 *Let K and K' be knots. If there is a nonzero $p/q \in \mathbb{Q}$ such that $S^3_{p/q}(K) \cong S^3_{p/q}(K')$, then $\Delta''_K(1) = \Delta''_{K'}(1)$.*

Lemma 2.5 can be used to obstruct noncharacterizing slopes coming from cables.

Lemma 2.6 *Let K and K' be knots. If there is K'' a nontrivial cable of K' and a nonzero slope $p/q \in \mathbb{Q}$ such that $S^3_{p/q}(K) \cong S^3_{p/q}(K'')$, then there are coprime integers r, s, with $s \geq 2$ such that*

$$\Delta''_K(1) = \frac{(r^2-1)(s^2-1)}{12} + s^2\Delta''_{K'}(1).$$

Proof Suppose that K'' is the (r,s)-cable of K', where $s \geq 2$ is the winding number. By the usual formula for the Alexander polynomial of a satellite knot [Lic97, Theorem 6.15], we have that

$$\Delta_{K''}(t) = \Delta_{K'}(t^s)\Delta_{T_{r,s}}(t),$$

where $T_{r,s}$ denotes the (r,s)-torus knot. Taking second derivatives, we obtain[1]

$$\Delta''_{K''}(1) = \Delta''_{T_{r,s}}(1) + s^2\Delta''_{K'}(1). \tag{2.3}$$

[1]The reader should note that since $\Delta_K(t) = \Delta_K(t^{-1})$, we have that $\Delta'_K(1) = 0$.

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Since the torus knot $T_{r,s}$ has symmetrized Alexander polynomial

$$\Delta_{T_{r,s}}(t) = t^{-\frac{(r-1)(s-1)}{2}} \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)},$$

one can calculate that[2]

$$\Delta''_{T_{r,s}}(1) = \frac{(r^2-1)(s^2-1)}{12}. \tag{2.4}$$

Combining (2.3) and (2.4) with Lemma 2.5 gives the desired statement. ∎

We will be applying these obstructions to the knots 5_2 and $12n242$. These have symmetrized Alexander polynomials:

$$\Delta_{5_2}(t) = 2t^{-1} - 3 + 2t,$$
$$\Delta_{12n242}(t) = t^{-5} - t^{-4} + t^{-2} - t^{-1} + 1 - t + t^2 - t^4 + t^5.$$

Hence, one finds that

$$\Delta''_{5_2}(1) = 4 \quad \text{and} \quad \Delta''_{12n242}(1) = 24. \tag{2.5}$$

2.5 An obstruction from ν^+

Here, we take some input from knot Floer homology. Recall that for a knot K in S^3, Ni and Wu derived a nonincreasing sequence of nonnegative integers $V_0(K), V_1(K), \dots$ from the knot Floer chain complex which can be used to calculate the d-invariants of surgeries on K. For $p/q > 0$ and an appropriate identification of $\mathrm{Spin}^c(S^3_{p/q}(K))$ and $\mathrm{Spin}^c(S^3_{p/q}(U))$ with $\{0,1,\dots,p-1\}$, we have [NW15, Proposition 1.6]

$$d(S^3_{p/q}(K), i) = d(S^3_{p/q}(U), i) - 2\max\left\{V_{\lfloor \frac{i}{q} \rfloor}(K), V_{\lceil \frac{p-i}{q} \rceil}(K)\right\}. \tag{2.6}$$

Hom and Wu defined the invariant $\nu^+(K)$ to be the smallest index i for which $V_i = 0$ [HW16]. In particular, we have $\nu^+(K) = 0$ if and only if $V_0 = 0$.

Lemma 2.7 *Let K be a knot such that $\nu^+(K) > 0$ and $\nu^+(mK) = 0$. Then there is no nonzero slope $p/q \in \mathbb{Q}$ such that $S^3_{p/q}(K) \cong S^3_{p/q}(mK)$.*

Proof Since $-S^3_{p/q}(K) \cong S^3_{-p/q}(mK)$, we can assume that $p/q > 0$. Summing the formula (2.6) over all spin^c-structures on $S^3_{p/q}(mK)$ and $S^3_{p/q}(K)$, we see that

$$\begin{aligned}\sum_{i=0}^{p-1} d(S^3_{p/q}(mK), i) - \sum_{i=0}^{p-1} d(S^3_{p/q}(K), i) &= 2\sum_{i=0}^{p-1} \max\left\{V_{\lfloor \frac{i}{q} \rfloor}(K), V_{\lceil \frac{p-i}{q} \rceil}(K)\right\} \\ &\geq 2V_0 > 0,\end{aligned}$$

which implies that $S^3_{p/q}(mK)$ and $S^3_{p/q}(mK)$ cannot be homeomorphic. ∎

[2] Since the direct calculation is somewhat involved, we include a derivation for completeness, but relegate it to the Appendix.

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Remark 2.8 We note that Lemma 2.7 applies to any nontrivial L-space knot (and in particular $12n242$). For a nontrivial L-space knot, one has $\nu^+(K) = g(K) > 0$ [HW16] and $\nu^+(mK) = 0$ [Gai17, Lemma 16].

3 Proof of Theorem 1.1

Throughout this section, we take $K = 12n242$. Suppose that $p/q \neq 0$ is a noncharacterizing slope for K satisfying

$$\ell_K(p/q) \geq 14.17 > \frac{2\pi}{\sqrt{1 - \left(\frac{\mathrm{vol}(4_1)}{\mathrm{vol}(12n242)}\right)^{\frac{2}{3}}}}. \tag{3.1}$$

Let $K' \neq K$ be a knot in S^3 such that $S^3_{p/q}(K) \cong S^3_{p/q}(K')$.

The length bound (3.1) implies that the manifold $S^3_{p/q}(K)$ is hyperbolic and, using Lemma 2.2, that $S^3_{p/q}(K)$ cannot be obtained by any surgery on the figure-eight knot 4_1. By Thurston's trichotomy for knots in S^3, the knot K' is either a torus knot, a hyperbolic knot or a satellite knot. Since torus knots never yield a hyperbolic manifold by surgery [Mos71], we may ignore the first possibility and restrict our attention to the latter two options.

Claim 1 If K' is a hyperbolic knot, then

$$q < 49 \quad \text{and} \quad |p| < 49(2g(K') - 1).$$

Proof Suppose that K' is a hyperbolic knot. Condition (3.1) eliminates the possibility that K' is 4_1. By consideration of the Casson–Walker invariant as in Lemma 2.5, we see that K' is not 5_2 or $m5_2$. Using the ν^+ invariant as in Lemma 2.7, we see that K' is not $m12n242$. Thus, having exhausted all the low volume knots in Theorem 2.1, we may conclude that $\mathrm{vol}(K') > 3.07$. Thus, by Lemma 2.2, we have the bound

$$\ell_{K'}(p/q) < \frac{2\pi}{\sqrt{1 - \left(\frac{\mathrm{vol}(12n242)}{3.07}\right)^{\frac{2}{3}}}} < 27.34.$$

Using Lemma 2.3, this yields the required bound. ∎

Claim 2 If K' is a satellite knot and $q \geq 2$, then

$$q < 49 \quad \text{and} \quad |p| < 49(2g(K') - 1).$$

Proof Suppose that K' is a satellite knot and that $q \geq 2$. By (3.1), the manifold $S^3_{p/q}(K)$ is hyperbolic and Lemma 2.4 applies to show that K' is a cable of a hyperbolic knot J such that $g(J) < g(K')$ and $S^3_{p/q'}(J) \cong S^3_{p/q}(K)$ for some $q' > q$. By the assumption (3.1), we see that J is not 4_1. Furthermore, applying the Casson–Walker invariant as in Lemma 2.6, we see that J cannot be 5_2, $m5_2$, $12n242$ or $m12n242$. This is because there are no nontrivial integer solutions with $s \geq 2$ to the equations:

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$$24 = \frac{(r^2-1)(s^2-1)}{12} + 4s^2$$

and

$$24 = \frac{(r^2-1)(s^2-1)}{12} + 24s^2.$$

Thus, having ruled out all the knots of small volume in Theorem 2.1, the only remaining possibility is that J must be a knot with $\mathrm{vol}(J) > 3.07$. Thus, by Lemma 2.2, we have the bound

$$\ell_J(p/q') < \frac{2\pi}{\sqrt{1-\left(\frac{\mathrm{vol}(12n242)}{3.07}\right)^{\frac{2}{3}}}} < 27.34.$$

Applying Lemma 2.3 along with the inequalities $q < q'$ and $g(J) < g(K')$ give the required bounds. ■

Claim 3 If K' is a satellite knot and $p/q \geq 9$, then

$$|p| < 49(2g(K')-1).$$

Proof Suppose that K' is a satellite knot and $p/q \geq 2g(K)-1 = 9$. Since K is an L-space knot, this implies that $S^3_{p/q}(K)$ is a hyperbolic L-space. By Lemma 2.4, there is a hyperbolic knot J such that $g(J) < g(K')$ and $S^3_{p/q'}(J) \cong S^3_{p/q}(K)$ for some $q' > q$. Since $\Delta''_K(1) \neq 0$, [BL90, Proposition 5.1] shows that J is not $12n242$. Furthermore, since $S^3_{p/q'}(J)$ is an L-space and none of $4_1, 5_2, m5_2$ or $m12n242$ are L-space knots, Theorem 2.1 allows us to conclude that $\mathrm{vol}(J) > 3.07$. Thus, as before, we arrive at the bounds

$$\ell_J(p/q') < \frac{2\pi}{\sqrt{1-\left(\frac{\mathrm{vol}(12n242)}{3.07}\right)^{\frac{2}{3}}}} < 27.34.$$

Applying Lemma 2.3(b) and $g(J) < g(K')$ gives the required bounds. ■

We now convert these statements into results on characterizing slopes. The bound $q \geq 49$ is straight forward.

Claim 4 The slope p/q is a characterizing slope for K whenever $q \geq 49$.

Proof Together Claims 1 and 2 show that p/q is a characterizing slope for K whenever $\ell_K(p/q) \geq 14.17$ and $q \geq 49$. However, Lemma 2.3(a) shows that $\ell_K(p/q) \geq 14.17$ is automatically satisfied whenever $q \geq 49$. ■

In order to obtain the other conditions on characterizing slopes, we need to invoke results linking the genera of K and K'.

Claim 5 The slope p/q is a characterizing slope for K whenever $p \geq \max\{24q, 441\}$.

Proof Since $S^3_{18}(K)$ is a lens space, it bounds a sharp 4-manifold [OS03].[3] Thus, [McC21, Theorem 1.2] applies to show that $S^3_{p/q}(K)$ bounds a sharp 4-manifold for all $p/q \geq 18$. In particular, we may apply [McC21, Theorem 1.1] to show that if $p/q \geq 4g(K) + 4 = 24$, then $g(K') = g(K) = 5$. Thus, Claims 1 and 3 imply that p/q is a characterizing slope for K whenever the conditions $p \geq 24q$, $p \geq 49(2g(K) - 1) = 441$ and $\ell_K(p/q) \geq 14.17$ are all satisfied. Lemma 2.3(b) shows that the bound $\ell_K(p/q) \geq 14.17$ is redundant, being implied by $p \geq 441$. Thus, we have a characterizing slope for K if $p \geq 24q$ and $p \geq 441$. ∎

Claim 6 The slope p/q is a characterizing slope for K whenever

$$q \geq 2 \quad \text{and} \quad p \leq -\max\{12 + 4q^2 - 2q, 441\}.$$

Proof By [McC20, Theorem 1.8(ii)], we see that if $q \geq 2$ and $p \leq \min\{2q - 12 - 4q^2, -10q\}$, then $g(K') = g(K) = 5$. Thus, Claims 1 and 2 imply that p/q is a characterizing slope for K if the conditions $q \geq 2$, $p \leq -\max\{12 + 4q^2 - 2q, 10q\}$, $p \leq -441$, and $\ell_K(p/q) \geq 14.17$ are all satisfied. Since $12 + 4q^2 - 2q > 10q$ for all q and the condition $p \leq -441$ implies $\ell_K(p/q) \geq 14.17$, we see that the conditions $p \leq -12 + 4q^2 - 2q$, $q \geq 2$, and $p \leq -441$ are sufficient to imply that p/q is a characterizing slope for K. ∎

This completes the proof of all bounds in Theorem 1.1.

4 Constructing some noncharacterizing slopes

In this section, we construct some examples of hyperbolic knots with noncharacterizing slopes with arbitrarily large denominator. Brakes used an essentially identical construction to exhibit examples noncharacterizing slopes on satellite knots [Bra80]. Let $L = C' \cup K'$ be a link with two unknotted components and linking number $\mathrm{link}(C', K') = \omega$. Let Y be the manifold obtained by performing $1/n$-surgery on both components on L for some nonzero integer $n \in \mathbb{Z}$. Since C' and K' are both unknotted, performing $1/n$ surgery on one or other of them individually again results in S^3. Performing such a surgery shows that Y arises by $(n\omega^2 + \frac{1}{n})$-surgery on the knots K and C, where K is the image of K' in the copy of S^3 obtained by surgering C' and C is the image of C' after surgering K'. If one chose L wisely, then the knots K and C will be distinct and thus the slope $n\omega^2 + \frac{1}{n}$ will be noncharacterizing for K and C.

Using this idea, we can prove the following.

Proposition 4.1 *Let K be a knot with $g(K) \geq 2$ which can be unknotted by adding q positive full twists along two oppositely oriented strands. Then $\frac{1}{q}$ is a noncharacterizing slope for K.*

[3] Although the precise definition of sharpness plays no role in this article, we record it here for context. Intuitively a 4-manifold is sharp if it determines the Heegaard Floer homology d-invariants of its boundary. More precisely, a compact, smooth, oriented 4-manifold X with boundary Y is *sharp*, if its intersection is negative definite and for all spin^c-structures $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, there exists $\mathfrak{t} \in \mathrm{Spin}^c(X)$ such that $d(Y, \mathfrak{t}) = \frac{1}{4}(c_1(\mathfrak{s})^2 + b_2(X))$.

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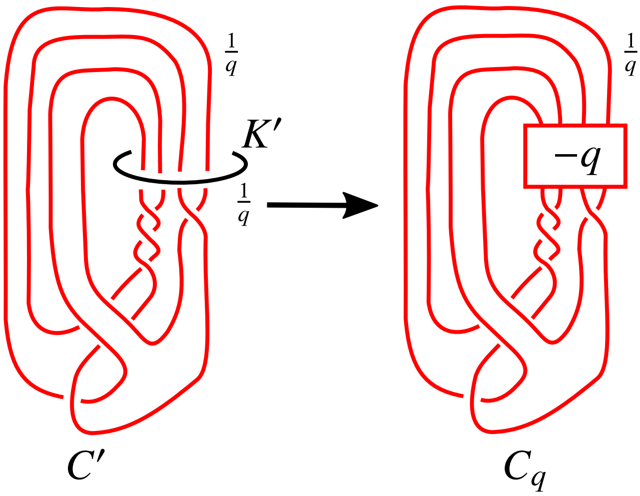

Figure 3: The link $K' \cup C'$ isotoped so that K' appears as a round unknot. A knot C_q such that $S^3_{\frac{1}{q}}(C_q) \cong S^3_{\frac{1}{q}}(K_q)$ is, thus, obtained by adding q negative full twists along C'.

Proof The hypothesis on unknotting implies that we can take a link $L = C' \cup K'$ with unknotted components such that (a) K can be obtained from K' by performing $1/q$-surgery on C' and (b) C' bounds a disk D that intersects K' in two oppositely oriented points. If we take the disk D and add a tube that follows an arc of K', we obtain an embedded genus one surface Σ with boundary C' which is disjoint from K'. Since Σ is disjoint from K', it is preserved under surgery on K' and hence shows that the knot C obtained by performing $1/q$ surgery on K' has genus at most one. Since K is assumed to have genus at least two, this implies that C is not isotopic to K and hence that $1/q$ is a noncharacterizing slope for K. ∎

Example 4.2 Using the preceding proposition, we can show that for every $q \geq 1$, there is a hyperbolic two-bridge knot K_q for which $\frac{1}{q}$ is a noncharacterizing slope. Figure 2 depicts a two-bridge knot K_q of genus two that can be unknotted by adding q positive full twists along two oppositely oriented strands. The genus of K_q can be easily verified, since Seifert's algorithm always yields a minimal genus Seifert surface when applied to an alternating diagram [Cro59, Mur58]. Thus, Proposition 4.1 applies to K_q. Figure 3 shows how one can obtain a knot C_q such that $S^3_{\frac{1}{q}}(C_q) \cong S^3_{\frac{1}{q}}(K_q)$.

We also note that sufficiently complicated knots with unknotting number one must always have an noncharacterizing slope. Since every slope is characterizing for the trefoil and the figure-eight knot [OS19], we see that the condition on the genus cannot be relaxed.

Corollary 4.3 *Let K be a knot with $g(K) \geq 2$ and $u(K) = 1$.*

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- *If K can be unknotted by changing a positive crossing, then $+1$ is noncharacterizing for K.*
- *If K can be unknotted by changing a negative crossing, then -1 is noncharacterizing for K.*

A Calculating $\Delta''_{T_{r,s}}(1)$

We conclude with a derivation of (2.4). It will be convenient to define, for any positive integer k, the function

$$Q_k(t) = t^{\frac{1-k}{2}}\left(\frac{t^k-1}{t-1}\right) = t^{\frac{1-k}{2}}\left(\sum_{i=0}^{k-1} t^i\right).$$

Using these, we can write the Alexander polynomial of a torus knot in the form:

$$\Delta_{T_{r,s}}(t) = \frac{Q_{rs}(t)}{Q_r(t)Q_s(t)}.$$

Since $Q_k(t) = Q_k(t^{-1})$, we have that

$$Q'_k(1) = 0.$$

Furthermore, we calculate that

$$Q_k(1) = k$$

and

$$\begin{aligned} Q''_k(1) &= \sum_{i=0}^{k-1}\left(i - \frac{k-1}{2}\right)\left(i - \frac{k+1}{2}\right) = \sum_{i=0}^{k-1}\left(i^2 - ki + \frac{(k-1)(k+1)}{4}\right) \\ &= \frac{k(k-1)(2k-1)}{6} - \frac{k^2(k-1)}{2} + \frac{k(k-1)(k+1)}{4} \\ &= \frac{k(k^2-1)}{12}. \end{aligned}$$

These identities allow us to calculate $\Delta''_{T_{r,s}}(1)$ implicitly. Differentiating the identity

$$Q_r(t)Q_s(t)\Delta_{T_{r,s}}(t) = Q_{rs}(t)$$

twice and evaluating at $t = 1$, we obtain

$$\begin{aligned} Q''_{rs}(1) &= \frac{rs(r^2s^2-1)}{12} \\ &= (Q_r(1)Q_s(1))''\,\Delta_{T_{r,s}}(1) + 2\,(Q_r(1)Q_s(1))'\,\Delta'_{T_{r,s}}(1) + Q_r(1)Q_s(1)\Delta''_{T_{r,s}}(1) \\ &= Q''_r(1)Q_s(1) + 2Q'_r(1)Q'_s(1) + Q_r(1)Q''_s(1) + rs\Delta''_{T_{r,s}}(1) \\ &= \frac{rs(r^2-1)}{12} + \frac{rs(s^2-1)}{12} + rs\Delta''_{T_{r,s}}(1). \end{aligned}$$

From this, one rearranges to obtain the desired formula:

$$\Delta''_{T_{r,s}}(1) = \frac{(r^2 - 1)(s^2 - 1)}{12}.$$

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References

[Ago00] I. Agol, *Bounds on exceptional Dehn filling*. Geom. Topol. **4** (2000), 431–449.
[BM18] K. Baker and K. Motegi, *Noncharacterizing slopes for hyperbolic knots*. Algebr. Geom. Topol. **18** (2018), no. 3, 1461–1480.
[BS22] J . Baldwin and S . Sivek, *Characterizing slopes for* 5_2. Preprint, 2022. arXiv:2209.09805
[BH96] S. A. Bleiler and C. D. Hodgson, *Spherical space forms and Dehn filling*. Topology. **35** (1996), no. 3, 809–833.
[BL90] S. Boyer and D. Lines, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*. J. Reine Angew. Math. **405** (1990), 181–220.
[Bra80] W. R. Brakes, *Manifolds with multiple knot-surgery descriptions*. Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 443–448.
[CM01] C . Cao and G. Robert Meyerhoff, *The orientable cusped hyperbolic 3-manifolds of minimum volume*. Invent. Math. **146** (2001), no. (3), 451–478.
[CL98] D . Cooper and M . Lackenby, *Dehn surgery and negatively curved 3-manifolds*. J. Differential Geom. **50** (1998), no. 3, 591–624.
[Cro59] R. Crowell, *Genus of alternating link types*. Ann. of Math. **2** (1959), no. 69, 258–275.
[FS80] R. Fintushel and R. J. Stern, *Constructing lens spaces by surgery on knots*. Math. Z. **175** (1980), no. 1, 33–51.
[FKP08] D. Futer, E. Kalfagianni, and J. S. Purcell, *Dehn filling, volume, and the Jones polynomial*. J. Differential Geom. **78** (2008), no. 3, 429–464.
[Gab89] D. Gabai, *Surgery on knots in solid tori*. Topology. **28** (1989), no. 1, 1–6.
[GHM21] D . Gabai, R . Haraway, R . Meyerhoff, N . Thurston, and A . Yarmola, *Hyperbolic 3-manifolds of low cusp volume*. Preprint, 2021. arXiv:2109.14570
[Gai17] F. Gainullin, *The mapping cone formula in Heegaard Floer homology and Dehn surgery on knots in* S^3. Algebr. Geom. Topol. **17** (2017), no. 4, 1917–1951.
[HW16] J. Hom and W. Zhongtao, *Four-ball genus bounds and a refinement of the Ozváth–Szabó tau invariant*. J. Symplectic Geom. **14** (2016), no. 1, 305–323.
[Lac03] M. Lackenby, *Exceptional surgery curves in triangulated 3-manifolds*. Pacific J. Math. **210** (2003), no. 1, 101–163.
[Lac19] M. Lackenby, *Every knot has characterising slopes*. Math. Ann. **374** (2019), nos. 1–2, 429–446.
[Gor83] C. McA. Gordon, *Dehn surgery and satellite knots*. Trans. Amer. Math. Soc. **275** (1983), no. 2, 687–708.
[McC19] D. McCoy, *On the characterising slopes of hyperbolic knots*. Math. Res. Let. **26** (2019), no. 5, 1517–1526.
[McC20] D. McCoy, *Non-integer characterizing slopes for torus knots*. Comm. Anal. Geom. **28** (2020), no. 7, 1647–1682.
[McC21] D. McCoy, *Surgeries, sharp 4-manifolds and the Alexander polynomial*. Algebr. Geom. Topol. **21** (2021), no. 5, 2649–2676.
[Mos71] L. Moser, *Elementary surgery along a torus knot*. Pacific J. Math. **38** (1971), 737–745.
[Mur58] K. Murasugi, *On the genus of the alternating knot. I, II*. J. Math. Soc. Japan **10** (1958), no. 3, 235–248.
[NW15] Y. Ni and Z. Wu, *Cosmetic surgeries on knots in* S^3. J. Reine Angew. Math. **706** (2015), 1–17.
[OS03] P . Ozsváth and Z . Szabó, *On the Floer homology of plumbed three-manifolds*. Geom. Topol. **7** (2003), 185–224 (electronic).

[OS19] P. Ozsváth and Z. Szabó, *The Dehn surgery characterization of the trefoil and the figure eight knot*. J. Symplectic Geom. **17** (2019), no. 1, 251–265.

[Per02] G . Perelman, *The entropy formula for the Ricci flow and its geometric applications*. Preprint, 2002. arXiv:0211159

[Per03a] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*. Preprint, 2003a. arXiv:0307245

[Per03b] G . Perelman, *Ricci flow with surgery on three-manifolds*. Preprint, 2003b. arXiv:0303109

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Springer, New York, NY, 1997.

[Sch53] H. Schubert, *Knoten und Vollringe*. Acta Math. **90** (1953), 131–286.

[Thu80] W. Thurston, *The geometry and topology of three-manifolds.* 1980. Notes available at http://library.msri.org/books/gt3m/

[Thu82] W. P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*. Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381.

[Wal92] K . Walker, *An extension of Casson's invariant*. Annals of Mathematics Studies, 126, Princeton University Press, Princeton, NJ, 1992.

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