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THE LEBESGUE FUNCTION FOR HERMITE–FEJÉR INTERPOLATION ON THE EXTENDED CHEBYSHEV NODES

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Given $f \in C[-1,1]$ and n points (nodes) in $[-1,1]$, the Hermite–Fejér interpolation polynomial is the polynomial of minimum degree which agrees with f and has zero derivative at each of the nodes. In 1916, L. Fejér showed that if the nodes are chosen to be zeros of $T_n(x)$, the nth Chebyshev polynomial of the first kind, then the interpolation polynomials converge to f uniformly as $n \to \infty$. Later, D.L. Berman demonstrated the rather surprising result that this convergence property no longer holds true if the Chebyshev nodes are extended by the inclusion of the endpoints -1 and 1 in the interpolation process. The aim of this paper is to discuss the Lebesgue function and Lebesgue constant for Hermite–Fejér interpolation on the extended Chebyshev nodes. In particular, it is shown that the inclusion of the two endpoints causes the Lebesgue function to change markedly, from being identically equal to 1 for the Chebyshev nodes, to having the form $2n^2(1-x^2)\big(T_n(x)\big)^2 + O(1)$ for the extended Chebyshev nodes.

1. Introduction

Suppose f is a continuous real-valued function defined on the interval $[-1,1]$, and let

$$X = \{x_{k,n} : k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots\}$$

be an infinite triangular matrix such that, for all n,

$$1 \geqslant x_{1,n} > x_{2,n} > \ldots > x_{n,n} \geqslant -1.$$

The well-known Lagrange interpolation polynomial of f is the polynomial $L_n(X,f)(x) = L_n(X,f,x)$ of degree at most $n-1$ which satisfies

$$L_n(X,f,x_{k,n}) = f(x_{k,n}), \qquad 1 \leqslant k \leqslant n.$$

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It can be expressed as

$$L_n(X,f,x) = \sum_{k=1}^{n} f(x_{k,n})\ell_{k,n}(X,x), \tag{1}$$

where the fundamental polynomials $\ell_{k,n}(X,x)$ are the unique polynomials of degree at most $n-1$ which satisfy

$$\ell_{k,n}(X,x_{j,n}) = \delta_{k,j}, \qquad 1 \leqslant k,j \leqslant n.$$

(Here $\delta_{k,j}$ denotes the Kronecker delta.) Thus the $\ell_{k,n}$ are defined by

$$\ell_{k,n}(X,x) = \frac{\omega_n(X,x)}{(x-x_{k,n})\,\omega_n'(X,x_{k,n})}, \qquad 1 \leqslant k \leqslant n, \tag{2}$$

and

$$\omega_n(X,x) = \prod_{k=1}^{n}(x-x_{k,n}). \tag{3}$$

Consider the supremum norm

$$\|f\| = \max_{-1\leqslant x\leqslant 1}|f(x)|$$

on $C[-1,1]$. From (1) it follows that for given $x \in [-1,1]$,

$$\max_{\|f\|\leqslant 1}|L_n(X,f,x)| = \sum_{k=1}^{n}|\ell_{k,n}(X,x)|,$$

and so the quantity

$$\lambda_{0,n}(X,x) = \sum_{k=1}^{n}|\ell_{k,n}(X,x)|$$

is the norm of the linear functional

$$L_n(X,\,\cdot\,,x) : C[-1,1] \to \mathbf{R}$$

which is given by

$$L_n(X,\,\cdot\,,x)(f) = L_n(X,f,x).$$

The function $\lambda_{0,n}(X,x)$ is known as the Lebesgue function for Lagrange interpolation on X. Furthermore, the value

$$\Lambda_{0,n}(X) = \max_{-1\leqslant x\leqslant 1}\lambda_{0,n}(X,x),$$

which is termed the Lebesgue constant for Lagrange interpolation on X, is the norm of the linear operator

$$L_n(X,\,\cdot\,) : C[-1,1] \to C[-1,1]$$

which is defined by

$$L_n(X,\cdot)(f) = L_n(X,f).$$

For Lagrange interpolation it is known (see Rivlin [9, Section 1.3]) that there exists a positive constant c such that

$$\Lambda_{0,n}(X) > \frac{2}{\pi}\log n + c, \qquad n = 1,2,3,\ldots, \tag{4}$$

for any X. By the uniform boundedness theorem (see, for example, Powell [7, p. 203]), a consequence of (4) is the classic result, due to Faber [5], that for any matrix X there exists $f \in C[-1,1]$ so that $L_n(X,f)$ does not converge uniformly to f as $n \to \infty$. On the other hand, if T denotes the matrix of Chebyshev nodes

$$T = \left\{x_{k,n} = \cos\left(\frac{2k-1}{2n}\pi\right) : k = 1,2,\ldots,n;\ n = 1,2,3,\ldots\right\},$$

then

$$\Lambda_{0,n}(T) \leqslant \frac{2}{\pi}\log n + 1, \qquad n = 1,2,3,\ldots, \tag{5}$$

(see Rivlin [9, Theorem 1.2]). Furthermore, if the modulus of continuity $\omega(\delta;f)$ of f is defined by

$$\omega(\delta;f) = \max\left\{\left|f(s)-f(t)\right| : -1 \leqslant s,t \leqslant 1, |s-t| \leqslant \delta\right\},$$

it follows from (5) (see Rivlin [8, Section 4.1]) that if $f \in C[-1,1]$ satisfies the relatively weak additional condition $\omega(1/n;f)\log n \to 0$ as $n \to \infty$, the sequence of Lagrange interpolation polynomials $L_n(T,f)$ converges uniformly to f as $n \to \infty$. In view of these results, it can be seen that the Chebyshev nodes T are a good choice if uniform approximation by Lagrange interpolation polynomials is required. Consequently the Lebesgue function $\lambda_{0,n}(T,x)$ and Lebesgue constant $\Lambda_{0,n}(T)$ have been studied extensively — a comprehensive account of results appears in the survey paper by Brutman [4].

Given $f \in C[-1,1]$ and an interpolation matrix X, the Hermite–Fejér interpolation polynomial is the unique polynomial $H_n(X,f)(x) = H_n(X,f,x)$ of degree at most $2n-1$ which satisfies

$$\left\{\begin{array}{rcl} H_n(X,f,x_{k,n}) &=& f(x_{k,n}), \\ H_n'(X,f,x_{k,n}) &=& 0, \end{array}\right. \qquad 1 \leqslant k \leqslant n.$$

The Hermite–Fejér polynomial can be written as

$$H_n(X,f,x) = \sum_{k=1}^{n} f(x_{k,n})A_{k,n}(X,x), \tag{6}$$

where the fundamental polynomials $A_{k,n}(X,x)$ are the unique polynomials of degree at most $2n-1$ which satisfy

$$\left\{\begin{array}{rcl} A_{k,n}(X,x_{j,n}) &=& \delta_{k,j}, \\ A_{k,n}'(X,x_{j,n}) &=& 0, \end{array}\right. \qquad 1 \leqslant k,j \leqslant n. \tag{7}$$

It is readily verified that an explicit formula for $A_{k,n}(X,x)$ is

$$A_{k,n}(X,x) = \left(1 - \frac{\omega_n''(X,x_{k,n})}{\omega_n'(X,x_{k,n})}(x - x_{k,n})\right)\left(\ell_{k,n}(X,x)\right)^2, \tag{8}$$

where $\ell_{k,n}$ and ω_n are defined by (2) and (3).

The initial impetus for studying Hermite–Fejér interpolation came from the result of Fejér [**6**] in 1916, who showed that if $f \in C[-1,1]$, then

$$\lim_{n\to\infty} \left\| H_n(T,f) - f \right\| = 0.$$

Thus, for any $f \in C[-1,1]$, the Hermite–Fejér interpolation polynomials based on the Chebyshev nodes converge uniformly to f, a result which established the importance of the Hermite–Fejér method, and confirmed the utility of the Chebyshev node system, in polynomial interpolation.

The Lebesgue function for Hermite–Fejér interpolation on X is defined by

$$\lambda_{1,n}(X,x) = \sum_{k=1}^{n} \left| A_{k,n}(X,x) \right|, \tag{9}$$

while the Lebesgue constant is

$$\Lambda_{1,n}(X) = \max_{-1\leqslant x\leqslant 1} \lambda_{1,n}(X,x).$$

Now, as a result of uniqueness considerations, $H_n(X,1,x) \equiv 1$, and so (6) gives

$$\sum_{k=1}^{n} A_{k,n}(X,x) = 1.$$

Hence, for any X and x,

$$\lambda_{1,n}(X,x) = \sum_{k=1}^{n} \left| A_{k,n}(X,x) \right| \geqslant \sum_{k=1}^{n} A_{k,n}(X,x) = 1. \tag{10}$$

Note that by (7), equality holds in (10) at each interpolation node $x_{k,n}$.

By Fejér's result, the Lebesgue constants for Hermite–Fejér interpolation on the Chebyshev nodes T are uniformly bounded. In fact, much more can be stated, for it follows from (8) that

$$A_{k,n}(T,x) = \frac{1 - x x_{k,n}}{n^2}\left(\frac{T_n(x)}{x - x_{k,n}}\right)^2,$$

where $x_{k,n} = \cos(2k-1)\pi/(2n)$ and $T_n(x)$ denotes the nth Chebyshev polynomial

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leqslant x \leqslant 1$$

(see Rivlin [9, Section 1.4]). Thus the $A_{k,n}(T,x)$ are nonnegative on $[-1,1]$, and so by (10), $\lambda_{1,n}(T,x) \equiv 1$, and hence

$$\Lambda_{1,n}(T) = 1, \qquad n = 1,2,3,\ldots. \tag{11}$$

Thus the study of the Lebesgue function and constant for Hermite–Fejér interpolation on T is trivial.

Given that $\|H_n(T,f) - f\| \to 0$ as $n \to \infty$ for all $f \in C[-1,1]$, it was a rather unexpected development when, in a series of papers in the 1960s, Berman showed that even for very simple functions f, extending the Chebyshev nodes by including the endpoints -1 and 1 in the node system can result in an Hermite–Fejér interpolation process that has quite surprising divergence behaviour. To describe one of these results, let T_e denote the extended Chebyshev nodes

$$T_e = \{x_{k,n+2} : k = 0,1,\ldots,n+1;\ n = 1,2,3,\ldots\}, \tag{12}$$

where

$$\begin{cases} x_{0,n+2} = 1, \quad x_{n+1,n+2} = -1, \\ x_{k,n+2} = \cos\left(\dfrac{2k-1}{2n}\pi\right), \quad 1 \leqslant k \leqslant n. \end{cases} \tag{13}$$

In his paper [1, Theorem 2], Berman showed that if $f(x) = x^2$, the sequence $\{H_n(T_e,f,x)\}$ diverges as $n \to \infty$ for all $x \in (-1,1)$. (Note that this result cannot be extended to the entire interval $[-1,1]$ because ± 1 are interpolation nodes for all n.)

Subsequently, R. Bojanić [2] investigated this so-called 'Berman's phenomenon'. He showed that if $f \in C[-1,1]$, and if the left and right derivatives $f'_L(1)$ and $f'_R(-1)$ exist, then

$$\limsup_{n\to\infty} \left|H_n(T_e,f,x) - f(x)\right| = \frac{3}{4}(1-x^2)\left|(1+x)f'_L(1) - (1-x)f'_R(-1)\right|.$$

This result explains why $H_n(T_e,x^2,x)$ does not converge to x^2 for any $x \in (-1,1)$, and also shows that if $f'_L(1)$ and $f'_R(-1)$ exist, then a necessary and sufficient condition for the uniform convergence of $H_n(T_e,f)$ to f is that $f'_L(1) = f'_R(-1) = 0$. Further necessary and sufficient conditions for the uniform convergence of $H_n(T_e,f)$ to f were developed by Bojanić, Varma and Vértesi [3] in work that considered Hermite–Fejér interpolation on the extended Jacobi nodes, a more general setting than the extended Chebyshev nodes.

Our aim in this paper is to study the Lebesgue function $\lambda_{1,n+2}(T_e,x)$ and Lebesgue constant $\Lambda_{1,n+2}(T_e)$ for Hermite–Fejér interpolation on the extended Chebyshev nodes T_e. Figure 1 illustrates a typical Lebesgue function, with the oscillations in $\lambda_{1,n+2}(T_e,x)$ between each pair of nodes seemingly increasing in magnitude as x moves from the outside towards the centre of the interval $[-1,1]$. Note also that the graph of $\lambda_{1,n+2}(T_e,x)$ is qualitatively quite different at the ends of $[-1,1]$ than it is elsewhere on the interval.

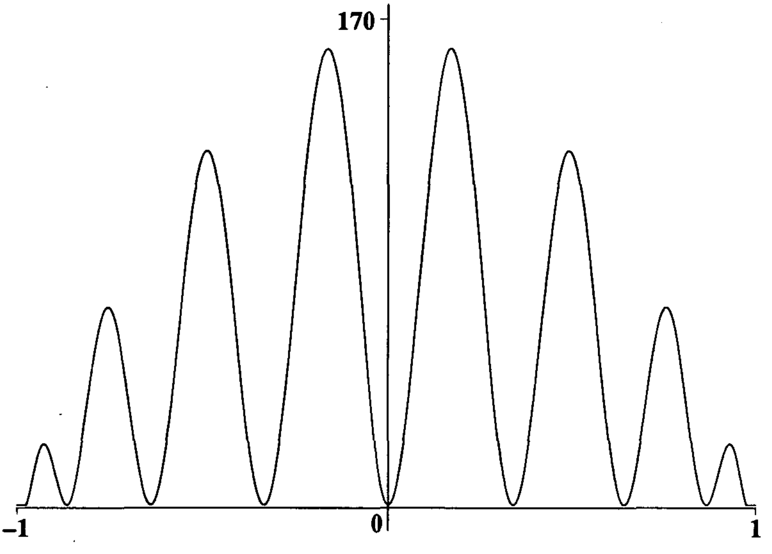

Figure 1: Graph of the Lebesgue function $\lambda_{1,11}(T_e, x)$

Our main results concerning the Lebesgue function are presented in the following two theorems, the proofs of which will be given in Section 2.

THEOREM 1. *For the extended Chebyshev nodes T_e defined by* (12) *and* (13), *there exists a positive constant C (independent of n and x) so that, for $-1 < x < 1$,*

$$(14)\qquad 2\big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big]+1-\frac{C\big(T_n(x)\big)^2}{n\big(1-|x|\big)^2} \leqslant \lambda_{1,n+2}(T_e,x) \leqslant 2\big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big]+1.$$

Observe that equality holds throughout (14) at the interpolation nodes $x = x_{k,n+2}$, $1 \leqslant k \leqslant n$.

The upper bound in (14) is, in fact, valid on $[-1,1]$. The next theorem provides a uniform lower bound.

THEOREM 2. *For $-1 \leqslant x \leqslant 1$,*

$$(15)\quad 2\big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big]-1 \leqslant \lambda_{1,n+2}(T_e,x) \leqslant 2\big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big]+1.$$

In Theorem 2, note that equality holds in the lower bound at the nodes $x = \pm 1$.

Our final theorem, which will be proved in Section 3, provides an estimate for the Lebesgue constant for Hermite–Fejér interpolation on the extended Chebyshev nodes. The proof of the theorem makes use of the observation from graphs such as Figure 1 that $\lambda_{1,n+2}(T_e,x)$ achieves its maximum value at or near the local maximum of $|T_n(x)|$ which is closest to 0 (which is $\sin \pi/(2n)$ if n is odd, and 0 if n is even).

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THEOREM 3. *The Lebesgue constant $\Lambda_{1,n+2}(T_e)$ satisfies*

$$\Lambda_{1,n+2}(T_e) = \begin{cases} 2n^2+3+O(1/n), & \text{if } n \text{ is even,} \\ 2n^2+3-\pi^2/2+O(1/n), & \text{if } n \text{ is odd.} \end{cases}$$

On comparing the results of Theorem 3 with (5) and (11), it is seen that the Lebesgue constants for Hermite–Fejér interpolation on the extended Chebyshev nodes are far larger than the corresponding Lebesgue constants for Lagrange and Hermite–Fejér interpolation on the (unextended) Chebyshev nodes. Such a result is not unexpected, given that the convergence behaviour of the Hermite–Fejér interpolation process on T_e is far worse than that of the Lagrange and Hermite–Fejér interpolation methods on T. It is also worth pointing out that Lebesgue constants much larger than those in Theorem 3 are known to occur for simple node systems. For instance, for Lagrange interpolation on equally-spaced nodes in $[-1,1]$, the Lebesgue constants grow exponentially with n (see Brutman's survey paper [**4**] for further details and references).

2. Proofs of Theorems 1 and 2

As shown by Bojanić [**2**], for the extended Chebyshev nodes T_e defined by (12) and (13), the fundamental polynomials are given by

$$(16)\qquad \begin{cases} A_{0,n+2}(T_e,x) &=& \big(1+(2n^2+1)(1-x)\big)\Big(\dfrac{1+x}{2}\,T_n(x)\Big)^2, \\ A_{n+1,n+2}(T_e,x) &=& \big(1+(2n^2+1)(1+x)\big)\Big(\dfrac{1-x}{2}\,T_n(x)\Big)^2, \\ A_{k,n+2}(T_e,x) &=& \dfrac{1+3xx_{k,n}-4x_{k,n}^2}{(1-x_{k,n}^2)^2}\Big(\dfrac{(1-x^2)T_n(x)}{n(x-x_{k,n})}\Big)^2, & 1\leqslant k\leqslant n. \end{cases}$$

For simplicity, we shall henceforth write $A_{k,n+2}(T_e,x)$ as $A_k(x)$ and $x_{k,n}$ as x_k. Let $x \in (-1,1)$. Note that $A_0(x)\geqslant 0$ and $A_{n+1}(x)\geqslant 0$. Also, for $1\leqslant k\leqslant n$, $1+3xx_{k,n}-4x_{k,n}^2 > 0$ if and only if $p(x)<x_k<q(x)$, where

$$(17)\qquad p(x)=\frac{3}{8}\Big(x-\sqrt{x^2+16/9}\Big),\qquad q(x)=\frac{3}{8}\Big(x+\sqrt{x^2+16/9}\Big).$$

Therefore, by (9),

$$(18)\qquad \lambda_{1,n+2}(T_e,x)=\big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big]+\sum_{k\in\mathcal{R}(x)}A_k(x)-\sum_{k\in\mathcal{S}(x)}A_k(x),$$

where

$$\begin{aligned} \mathcal{R}(x) &= \{k: 1\leqslant k\leqslant n,\ p(x)<x_k<q(x)\}, \\ \mathcal{S}(x) &= \{k: 1\leqslant k\leqslant n,\ x_k<p(x) \text{ or } x_k>q(x)\}. \end{aligned}$$

We shall employ the partial fraction representation

$$(19)\qquad \frac{4x_k^2-3xx_k-1}{(1-x_k^2)^2(x-x_k)^2}=\frac{1}{4(1-x)^2}\,\frac{1}{1-x_k}+\frac{3}{4(1-x)}\,\frac{1}{(1-x_k)^2}+\frac{1}{4(1+x)^2}\,\frac{1}{1+x_k} \\ +\frac{3}{4(1+x)}\,\frac{1}{(1+x_k)^2}-\frac{x}{(1-x^2)^2}\,\frac{1}{x-x_k}-\frac{1}{1-x^2}\,\frac{1}{(x-x_k)^2}.$$

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Applying this to $A_k(x)$ for $1 \leqslant k \leqslant n$ gives

$$\sum_{k\in\mathcal{R}(x)} A_k(x) - \sum_{k\in\mathcal{S}(x)} A_k(x) = \left(\frac{(1-x^2)T_n(x)}{n}\right)^2 [P(x) - Q(x)], \tag{20}$$

where

$$P(x) = \frac{1}{4(1-x)^2}\sum_{k=1}^{n}\frac{1}{1-x_k} + \frac{3}{4(1-x)}\sum_{k=1}^{n}\frac{1}{(1-x_k)^2} + \frac{1}{4(1+x)^2}\sum_{k=1}^{n}\frac{1}{1+x_k}$$
$$+ \frac{3}{4(1+x)}\sum_{k=1}^{n}\frac{1}{(1+x_k)^2} + \frac{x}{(1-x^2)^2}\sum_{k=1}^{n}\frac{1}{x-x_k} + \frac{1}{1-x^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}$$

and

$$Q(x) = \frac{1}{2(1-x)^2}\sum_{k\in\mathcal{R}(x)}\frac{1}{1-x_k} + \frac{3}{2(1-x)}\sum_{k\in\mathcal{R}(x)}\frac{1}{(1-x_k)^2} + \frac{1}{2(1+x)^2}\sum_{k\in\mathcal{R}(x)}\frac{1}{1+x_k}$$
$$+ \frac{3}{2(1+x)}\sum_{k\in\mathcal{R}(x)}\frac{1}{(1+x_k)^2} + \frac{2x}{(1-x^2)^2}\sum_{k\in\mathcal{S}(x)}\frac{1}{x-x_k} + \frac{2}{1-x^2}\sum_{k\in\mathcal{S}(x)}\frac{1}{(x-x_k)^2}.$$

On observing that

$$\frac{2x}{(1-x^2)^2}\sum_{k\in\mathcal{S}(x)}\frac{1}{x-x_k} + \frac{2}{1-x^2}\sum_{k\in\mathcal{S}(x)}\frac{1}{(x-x_k)^2} = \frac{2}{(1-x^2)^2}\sum_{k\in\mathcal{S}(x)}\frac{1-xx_k}{(x-x_k)^2},$$

we note for future reference that

$$Q(x) > 0, \qquad -1 < x < 1. \tag{21}$$

To simplify $P(x)$, we use some well-known properties of the Chebyshev polynomials (see, for example, Rivlin [**9**, Chapter 1]) and the symmetry of the interpolation nodes x_k about 0. Our starting point is the representation

$$\frac{T_n'(x)}{T_n(x)} = \sum_{k=1}^{n}\frac{1}{x-x_k}. \tag{22}$$

Therefore

$$\sum_{k=1}^{n}\frac{1}{1-x_k} = \sum_{k=1}^{n}\frac{1}{1+x_k} = \frac{T_n'(1)}{T_n(1)} = n^2. \tag{23}$$

Also, from (22),

$$\left(T_n(x)\right)^2\sum_{k=1}^{n}\frac{1}{(x-x_k)^2} = \left(T_n'(x)\right)^2 - T_n(x)T_n''(x), \tag{24}$$

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and so

$$\sum_{k=1}^{n} \frac{1}{(1-x_k)^2} = \sum_{k=1}^{n} \frac{1}{(1+x_k)^2} = \frac{2}{3}n^4 + \frac{1}{3}n^2. \tag{25}$$

Further, on employing $(1-x^2)T_n'' = xT_n' - n^2T_n$ and $(1-x^2)\big(T_n'(x)\big)^2 = n^2\Big[1-\big(T_n(x)\big)^2\Big]$, we obtain, from (24),

$$(1-x^2)\big(T_n(x)\big)^2 \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} = n^2 - xT_n(x)T_n'(x). \tag{26}$$

Therefore, by (22), (23), (25) and (26), it follows that

$$\Big(\frac{(1-x^2)T_n(x)}{n}\Big)^2 P(x) = \big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big] + 1,$$

and so, by (18) and (20),

$$\lambda_{1,n+2}(T_e, x) = 2\big(T_n(x)\big)^2\big[n^2(1-x^2)+1\big] + 1 - \Big(\frac{(1-x^2)T_n(x)}{n}\Big)^2 Q(x). \tag{27}$$

The upper bound for $\lambda_{1,n+2}(T_e, x)$ in Theorems 1 and 2 then follows from (21).

We now obtain the lower bound in Theorem 1. Because the interpolation nodes are symmetric about 0, the Lebesgue function $\lambda_{1,n+2}(T_e, x)$ is even, and hence there is no loss of generality in assuming $0 \leqslant x < 1$. Our aim is to obtain an upper bound on $[0, 1)$ for

$$\begin{aligned}(1-x^2)^2Q(x) = \sum_{k\in\mathcal{R}(x)} \Big(&\frac{(1+x)^2}{2}\frac{1}{1-x_k} + \frac{3(1-x^2)(1+x)}{2}\frac{1}{(1-x_k)^2} \\ &+ \frac{(1-x)^2}{2}\frac{1}{1+x_k} + \frac{3(1-x^2)(1-x)}{2}\frac{1}{(1+x_k)^2}\Big) + 2\sum_{k\in\mathcal{S}(x)} \frac{1-xx_k}{(x-x_k)^2}.\end{aligned}$$

Now, $p(x) \geqslant -1/2$ for $0 \leqslant x < 1$, so if $k \in \mathcal{R}(x)$, some simple estimations give

$$\begin{aligned}&\frac{(1+x)^2}{2}\frac{1}{1-x_k} + \frac{3(1-x^2)(1+x)}{2}\frac{1}{(1-x_k)^2} + \frac{(1-x)^2}{2}\frac{1}{1+x_k} \\ &\quad + \frac{3(1-x^2)(1-x)}{2}\frac{1}{(1+x_k)^2} \leqslant \frac{2}{1-q(x)} + \frac{3}{\big(1-q(x)\big)^2} + 7.\end{aligned} \tag{28}$$

Also, if $p(x)$ and $q(x)$ are given by (17), some elementary calculations show that

$$0 < q(x) - x \leqslant 1/2 \leqslant x - p(x), \qquad 0 \leqslant x < 1,$$

and so, if $k \in \mathcal{S}(x)$, then

$$\frac{2(1-xx_k)}{(x-x_k)^2} \leqslant \frac{4}{(q(x)-x)^2}. \tag{29}$$

On noting that $q(x) - x \leqslant 1 - q(x)$ for $0 \leqslant x < 1$, it follows from (28) and (29) that there exists an absolute constant K so that

$$(1-x^2)^2 Q(x) \leqslant \frac{Kn}{\left(q(x)-x\right)^2}.$$

Finally, from

$$q(x) - x = \frac{1}{8}\Big(\big(3\sqrt{x^2+16/9} - 3x - 2\big) + 2(1-x)\Big) > \frac{1-x}{4}, \qquad 0 \leqslant x < 1,$$

we see that there is an absolute constant C so that

$$(1-x^2)^2 Q(x) \leqslant \frac{Cn}{(1-x)^2}.$$

The lower bound in Theorem 1 then follows from (27).

It remains to obtain the lower bound in Theorem 2. From (9) and (16),

$$\lambda_{1,n+2}(T_e, x) \geqslant \left(T_n(x)\right)^2\left[n^2(1-x^2)+1\right] + \left(\frac{(1-x^2)T_n(x)}{n}\right)^2 \sum_{k=1}^{n} \frac{4x_k^2 - 3xx_k - 1}{(1-x_k^2)^2(x-x_k)^2}.$$

The lower bound in (15) then follows from the partial fraction representation (19), along with the summation formulas (22), (23), (25) and (26).

3. Proof of Theorem 3

Since $\left|T_n(x)\right| \leqslant 1$ on $[-1,1]$, it follows from Theorem 2 that $\lambda_{1,n+2}(T_e, x) \leqslant 2n^2 + 3$ for $-1 \leqslant x \leqslant 1$, and so

$$\Lambda_{1,n+2}(T_e) \leqslant 2n^2 + 3.$$

On the other hand, if n is even, then $\left|T_n(0)\right| = 1$, and so, by the lower bound in Theorem 1,

$$\Lambda_{1,n+2}(T_e) \geqslant \lambda_{1,n+2}(T_e, 0) \geqslant 2n^2 + 3 - \frac{C}{n},$$

which establishes Theorem 3 for even values of n.

Now suppose n is odd. By (15),

$$\lambda_{1,n+2}(T_e, x) \leqslant 2n^2(1-x^2)\left(T_n(x)\right)^2 + 3, \qquad -1 \leqslant x \leqslant 1.$$

Define $M_n > 0$ by

$$M_n^2 = \max_{-1 \leqslant x \leqslant 1} (1-x^2)\left(T_n(x)\right)^2,$$

so that

$$\Lambda_{1,n+2}(T_e) \leqslant 2n^2 M_n^2 + 3. \tag{30}$$

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From symmetry considerations, and because $1 - x^2$ is decreasing on $[0, 1]$, it is apparent that the maximum value of $(1 - x^2)\big(T_n(x)\big)^2$ on $[-1, 1]$ is achieved on the interval $\big[0, \sin\pi/(2n)\big]$. Then, by putting $x = \sin\theta$, we see that

$$M_n = \max_{0 \leqslant \theta \leqslant \pi/(2n)} \cos\theta \sin n\theta. \tag{31}$$

The maximum of $\cos\theta\sin n\theta$ in $0 \leqslant \theta \leqslant \pi/(2n)$ occurs at a point θ^* which is the solution of $n\cos n\theta\cos\theta - \sin n\theta \sin\theta = 0$, or, equivalently, $(n+1)\cos(n+1)\theta + (n-1)\cos(n-1)\theta = 0$. Hence $\cos(n+1)\theta^* < 0$, so

$$\frac{\pi}{2(n+1)} < \theta^* < \frac{\pi}{2n},$$

and thus

$$\theta^* = \frac{\pi}{2n} + O\Big(\frac{1}{n^2}\Big).$$

By (31) we conclude that

$$M_n \leqslant \cos\theta^* = 1 - \frac{\pi^2}{8n^2} + O\Big(\frac{1}{n^3}\Big),$$

and so, by (30),

$$\Lambda_{1,n+2}(T_e) \leqslant 2n^2 + 3 - \pi^2/2 + O(1/n).$$

On the other hand, if n is odd, then $\Big|T_n\big(\sin\pi/(2n)\big)\Big| = 1$, and so, by the lower bound in Theorem 1,

$$\Lambda_{1,n+2}(T_e) \geqslant \lambda_{1,n+2}\big(T_e, \sin\pi/(2n)\big) \geqslant 2n^2 + 3 - \pi^2/2 + O(1/n),$$

which completes the proof of Theorem 3. $\square$

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