HYPERPERIODS, ORBITAL STABILITY, AND SOLUTION OF THE PROBLEM OF KIRKWOOD GAPS

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I believe I have solved, at least in principle, the long-standing problem of the Kirkwood gaps, and have incidentally initiated a new approach to questions of orbital stability. I shall begin with the concept I call hyperperiod. A given periodic dynamical system S with period P may or may not have a latent long period - the hyperperiod P*. If P* exists, then any small displacement or variation, actual or virtual, once-for-all or recurrent, will induce a displacement y which will be periodic with period P* and will be of bounded amplitudes. We can then say that S is stable. If P* dose not exist, then y will eventually become indefinitely large - and we say that S is unstable.

Example 1. An idealised Sun-Jupiter-Halley system was idealised to be periodic with P = 154.2 yr. It was found (refs 1,3) that, for some initial configurations, P* exists and is about 600 yr; while for others, P* does not exist.

Example 2. An upright rod 13.3 cm long has its lower end moved up and down at 50 Hz (ref.1). Here, P = 0.02 sec. If the stroke exceeds 0.45 cm, then P* exists and is about 1.25 sec, and, given any small push, the rod will simply sway with that period (stable !). If the stroke is less than 0.45 cm, then P* does not exist and the rod falls over (unstable !).

The technique of finding P* is this: we first derive a Hill's equation for y (or a linear function thereof):

$$\frac{d^2y}{dt^2} + G(t)\, y = 0 \qquad (1)$$

where G(t) is a known period function of period P. The solution of (1) is of the form

$$y = e^{icx} \sum b_k e^{ikx}, \qquad (x = 2\pi t / P_0) \qquad (2)$$

where c is a latent frequency of the system and the b's integration constants. I shall call c the Hill exponent. The evaluation of c was first given in ref. 4; but see my remarks in ref. 3. If c is real, then it defines a hyperperiod:

R. L. Duncombe (ed.), Dynamics of the Solar System, 227–230.

$$P^* = P/c \tag{3}$$

If c is imaginary, then y will eventually be dominated by the term $b_0 \exp(icx)$, and P* does not exist.

Thus, if, for a given periodic dynamic system S, we can write down a Hill's equatinn for the displacement or variation, we can then define the stability of S according as P* exists or not, or as c is real or imaginary.

Now, each orbit in what I call the Schubart diagram (ref. 3) is a periodic dynamic system. Unlike the examples above, there are no further parameters to be specified so that any verdict we may return on the stability character of a given orbit will be an unconditional one. Of course, in order to reach a verdict, we must first write down a Hill's equation. Here, again, the case of resonant asteroids is different from the examples given above. In the examples, the derivation follows entirely the classical treatment; but if we try to do the same in the present case, we shall never succeed. This is because the classical treatment of the present case, namely, a conservative, periodic system of one degree of freedom will lead to the general assertion that all such systems are unstable in the sense of asymptotic stability and stable in the sense of orbital stability (for the various types of stability, see ref. 5), whereas if we could write down a Hill's equation, then the stability character of a given orbit will have to depend on the individual properties of the orbit. In other words, the classical treatment is incompatible with the possibility of writing down a Hill's equation. To do the latter we must at some stage modify the classical treatment.

The problem of the Kirkwood gaps reaches its most acute form when we compare the Hecuba and Hilda regions in the intermediate eccentricity range. Nature suggests that, in this range, the Hilda librators are orbitally stable and the Hecuba librators are orbitally unstable. The classical treatment with its sweeping statements will never be able to resolve the antimony, while the method of Hill's equation or hyperperiods offers a possibility of doing so.

Let us see what modifications are necessary. An orbit in the Schubart diagram is defined by the canonical equations

$$\frac{dx}{dt} = F_2, \quad \frac{dy}{dt} = -F_1 \tag{4}$$

where the numerical suffix (1 for x, 2 for y) denotes partial differention of the Hamiltonian F and setting the variations u and v equal to zero. (x and y are the canoncial variables here; do not confuse with their previous usage). Differentiating (4), we have

$$\frac{d^2x}{dt^2} = F_{21} F_2 - F_{22} F_1 \tag{5}$$

Now, apply to (5) the variations

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$$x \longrightarrow x + u, \quad y \longrightarrow y + v. \tag{6}$$

Then, after some reduction, we have

$$\frac{d^2u}{dt^2} = \frac{dF_{21}}{dt} u + \frac{dF_{22}}{dt} v + (F_{12}^2 - F_{11}F_{22}) u \tag{7}$$

The equation of v can be obtained by interchanging u and v, and the suffixes 1 and 2. Because of the term in v, (7) is not a Hill's equation. We can therefore get a Hill's equation by dropping that term. We get another Hill's equation by dropping also the preceding term. I opt to do the latter and obtain

$$\frac{d^2u}{dt^2} + (F_{11}F_{22} - F_{12}^2) u = 0 \tag{8}$$

The reason for my choice is that the form (8) can also be derived from the following scheme:

$$\frac{d^2u}{dt^2} = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{d}{dt}\left(x + u + h\frac{du}{dt}\right) - \frac{d}{dt}(x+u) \right\} \tag{9}$$

in the evaluation of which we always set

$$\frac{du}{dt} = F_{21} u + F_{22} v, \quad \frac{dv}{dt} = -F_{11} u - F_{12} v \tag{10}$$

which are the classical variation equations of (4). In this scheme,no use is made of equation (5) so that an "Enabling Principle" can be ennunciated as follows: "In order to be able to write down a Hill's equation, we must refer the variation to an unaccelerated frame in the phase space". The classical treatment, by contrast, refers the variation to the accelerated frame "following the natural motion".

The results of applying (8) to the orbits in the Schubart diagram are that the Hilda librators are, in general, stable, and the Hecuba librators are, in general, unstable. Details will be published elsewhere and the qualitative results have been given in ref. 3.

The "Enabling Principle" stated above offers the possibility of a new approach to a general theory of stability of dynamical systems, i.e., not only of systems of one degree of freedom.

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DISCUSSION

Scholl: Schubart's model which you used does not yield orbits in the Hecuba gap which drift out of the gap. According to Schubart's figures you use, that is not possible. Apparently, your stability investigation about Schubart's dynamical system, which has one degree of freedom, predicts the behavior of a system with two or more degrees of freedom. This procedure is, however, very doubtful.

Kiang: My faith and intuition is that the stability of resonant asteroids in real life can be studied using the simplest model. It is the same kind of faith that made Poincaré concentrate so much on periodic orbits.

Message: Would each of the speakers give their views as to the extent to which the simplifications, required to reduce the problem to one degree of freedom, conceal features of the actual motion over very long periods of time?

Kiang: I have to refer you to my paper in Nature, in which I actually used the "simplifications" to sharpen my qualitative results in the Hecuba case.

Schubart: I think that one can try to use the model that is simplified to one degree of freedom, with an addition that follows from the model without difficulty. After solving the problem of one degree of freedom for the arguments of slow variation, the short-period argument follows by means of a quadrature, and this gives the period of revolution of the longitude of perihelion, $\tilde{\omega}$. If a period of libration follows from the solution of the basic problem, the ratio of the two periods may be a characteristic for the stability of the orbit under consideration against perturbations that are not considered in the simplified model.

Schubart's comments: [For references compare paper 5.1] I am not in favor of the designation "Schubart diagrams" for the figures in a former paper (Schubart, 1964) mentioned by Dr. Kiang. Poincaré (1902) drew the first diagram of this type, so it is better to use "Poincaré diagrams" or "Izsak diagrams," since the late Dr. I. Izsak had asked me to plot the diagrams in just this way. The one-degree-of-freedom problem mentioned by Dr. Kiang (Schubart, 1964) is well defined. A numerical test about the time dependence of the relative distance of two points in phase space can give evidence about "hyperperiods." Some test about Dr. Kiang's theoretical results is necessary, according to my opinion.

I like the suggestion given by Dr. Kiang's recent treatment, to think again about the differences between the Hecuba and Hilda commensurabilities, using gravitational theory alone.

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