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Nonequilibrium quantum processes in the early universe

As stated in the Preface, we intend the chapters in the last part of the book to illustrate how quantum field theoretical methods can be applied to nonequilibrium statistical processes in several areas of current research, specifically, particle–nuclear processes (in RHIC and DCC), dynamics of cold atoms (BEC) in AMO physics and quantum processes in the early universe (cosmology) and in this endeavor also try to present an introduction to an important subject matter in that area. With this specified emphasis on the applications of techniques of NEqQFT, these accounts are more in the nature of a research topic exercise or extended example than a full review, in that the topics are selected because of the NEqQFT context, and the presentations are illustrations of the methodology. Thus we suggest the reader refer to review articles or monographs to get a more balanced and complete view on different physical approaches to the same subject matter.

In this chapter on cosmology, after a brief introduction to inflationary cosmology, highlighting the stochastic inflation model, we discuss how NEqQFT impacts on some central issues in cosmology. The methodology introduced in Chapters 4–6 covering particle creation mechanisms and the nPI CTP-CGEA/IF functional formalisms for NEq processes can be applied to solve a number of basic problems in cosmology.

Some specific processes have been discussed in earlier parts of this book. In Chapter 5, with the aid of the CGEA and the influence functional [Hu94b] we learned the relationship between the processes of dissipation, fluctuation, noise and decoherence. Then, in Chapter 9, we examined, starting from first principles, under what circumstances the fluctuations of a quantum field transmute into classical, stochastic fluctuations. We used a simple model to illustrate how decoherence comes about in a quantum phase transition. We then used a partitioned interacting scalar field theory in de Sitter spacetime to show how in the stochastic inflation paradigm the long-wavelength sector gets decohered and becomes classical under the influence of the short-wavelength sector acting as noise (more precisely, the rms value of the fluctuations can be treated as classical). Here we continue this investigation in early universe quantum processes, focusing on three major topics: the origin and nature of noise from quantum fields, structure formation from colored noises, and reheating from particle creation after inflation.

15.1 Quantum fluctuations and noise in inflationary cosmology

15.1.1 Inflationary cosmology

In modern cosmology, before the advent of the inflationary universe, the widely accepted model which explains very well the present day observed universe (according to the high-precision experiments of the 1990s and 2000s such as COBE and WMAP) has been the so-called standard model [Pee80] based on the Friedmann–Lemaitre–Robertson–Walker (FLRW) universe. Filled with a classical matter source with equation of state pressure $p = \gamma\rho$ matter density, its scale factor $a(t)$ undergoes a power-law expansion $a(t) = t^\alpha$ in cosmic time t. Thus for a spatially flat FLRW universe, in the matter-dominated era $\gamma = 0, \;\; \alpha = 2/3$ for a pressureless fluid; and in the radiation-dominated era, $\gamma = 1/3, \alpha = 1/2$ for a relativistic fluid.

Since the 1980s the inflationary cosmology has become a widely accepted paradigm to explain the observed large-scale flatness and homogeneity of the universe [LytRio99, Rio02]. Inflation also provides an efficient mechanism for the magnification of quantum fluctuations to cosmological scales, and the generation of small curvature perturbations which in principle can produce the observed cosmic microwave background (CMB) temperature anisotropies and provide the seeds for the formation of large-scale structures from galaxies to superclusters in today's universe.

Inflationary cosmology can be represented by the same FLRW spacetime, but instead filled with a constant energy density source which drives the universe (again assuming a spatially flat metric) into a phase of exponential expansion, $a(t) = a_0 \exp(Ht)$, where $H = \dot{a}/a$ is the Hubble expansion rate (a dot over a quantity stands for a derivative with respect to cosmic time t). This is the Einstein–de Sitter model obtained by de Sitter in 1917 from a solution of Einstein's equation with a cosmological term. When interpreted as classical matter this constant energy density source corresponds to matter with an unphysical equation of state $p = -\rho$ because it admits acausal propagation. What turned the de Sitter universe into a viable cosmological model was when Guth in 1981 proposed that this constant energy density in the potential energy is associated with the expectation value of a quantum field (the Higgs or the gauge field) which mediates some particle physics symmetry-breaking process in the early universe. Inflation was originally motivated by the removal of monopole overabundance in the GUT epoch, which it does, but turned out to be highly successful in addressing the flatness and horizon issues which are the more significant and immediate problems in cosmology.

The quantum scalar field Φ which drives inflation, known as the inflaton, evolves according to the equation

$$\ddot{\Phi} + 3H\dot{\Phi} + dV[\Phi]/d\Phi = 0, \tag{15.1}$$

where the potential $V[\Phi]$ can take on a variety of forms, such as the Φ^4 double well potential in Guth's original "old" inflation [Guth81, Sato81]; an almost-flat

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Coleman–Weinberg potential (of a massless field with only radiative correction) in the "new" inflation of Albrecht-Steinhardt and Linde [AlbSte82, Lin82]; a $m^2\Phi^2$ potential in Linde's chaotic inflation [Lin85]; an exponential form giving rise to power-law inflation [LucMat85] and many more later models suggested for specific purposes. The main idea is to get the universe into a vacuum energy dominated stage (the *entry problem*), to find ways (or rationale) to sustain the inflation for at least 68 e-folding time so as to produce sufficient entropy content of our present universe, and to get it out of this supercooled stage (the *exit problem*) by reheating it to the radiation-dominated FLRW universe described by the standard model.

Issues in the three stages pertaining to NEqQFT

Much work in the 1980s till now was devoted to the second issue, i.e. finding the right potential for inflation to serve specific purposes (see, e.g. [SteTur84]). Serious work on reheating started in the mid-1990s, but somewhat surprisingly, the very first issue, *the entry problem,* i.e. how did the universe get into a vacuum dominated phase, has not been taken up and pursued in earnest in the inflationary cosmology community, except for a brief period in the early 1990s [SalBon91, Hod90, MMOL91, KBHP91]. In principle one expects this issue can be resolved if we know what had happened in an earlier epoch. In this regard there were studies in quantum cosmology in the 1980s pertaining to this question. There were claims from both the no-boundary wavefunctions proposal of Hartle and Hawking [HarHaw83] and the "birth" by tunneling idea of Vilenkin [Vil83b] that these scenarios admit the de Sitter solution. This is an important issue of principle, related to what metastable states can exist in the pre-inflationary stage, what mechanisms can induce the universe to become vacuum dominated, and the probability it actually did. At the level of ideas there were criticisms of principle and of practice (e.g. [GiHaSt87, HawPag88, HolWal02, KoLiMu02]) and there were many plausibility arguments presented. More quantitative methods involve the derivation and solution of a Fokker–Planck equation for the distribution function constructed out of the universe's quantum state, from which one can examine the likelihood the universe could enter into a metastable state (the false vacuum) and stay there long enough to start inflation. See, e.g. [Sta82].

For the second issue, on *the dynamics of inflation*, in Guth's original model (old inflation) with a double well potential, the universe gets out of the vacuum dominated stage by tunneling. However, the underlying nucleation process happens infrequently and gives rise to a highly inhomogeneous universe. This can be improved upon by invoking a nearly flat potential as in new inflation, or by allowing the inflaton to slowly roll down the quadratic potential as in chaotic inflation. In all these cases, a slow-roll condition is desirable to sustain the inflationary expansion for a reasonable duration.

For the third issue, in conjunction with the so-called "graceful exit" problem, much detailed consideration has been devoted in the last 10 years to the *post-inflation reheating* processes. This epoch after the inflationary expansion contains several stages: preheating, reheating and thermalization. These processes are important because the temperature and entropy generated as the universe reheats after inflation are important parameters which enter into all ensuing cosmological processes.

What we want to point out is that in all three stages, the basic issues can be formulated in the language of NEqQFT, and be addressed with the techniques of NEqQFT we have constructed in earlier chapters. For example, on the "entry" or "get-started" issue, a more productive approach to the investigation on whether any metastable state exists could be by means of the Fokker–Planck equation for the distribution function (or a related master equation for the density matrix) of the universe. The second issue on the energetics of inflation depends strongly on the nature and dynamics of phase transition, whether it is first order via nucleation, as in old inflation or second order via spinodal decomposition as in new inflation. Vital issues in the quantum theory of structure formation, such as when the long-wavelength sector of the inflaton becomes classical, and what kind of noise the short-wavelength sector of quantum fluctuations engender, if any, are fundamentally NEqQFT problems. The third stage of reheating involves particle creation from the rapidly changing inflaton field as it descends a steep potential well, and is reasonably well treated by the CTP 2PI effective action, as we will illustrate in the last part of this chapter.

Stochastic inflation

To address these issues in some detail and to seek solutions, we now specialize and delve into one such theory of inflation known as stochastic inflation which was proposed by Starobinsky [Sta86] (see also earlier work by Vilenkin [Vil83a, Vil83b]) and developed by many [BarBub87, Rey87, PolSta96, GoLiMu87, NaNaSa88, NamSas89, Nam89, LiLiMe94, Hab90, StaYok94, Mat97a, Mat97b, WinVil00]. In this theory the inflation field is divided into two parts at every instant according to their physical wavelengths, i.e.

$$\Phi(x) = \Phi_<(x) + \Phi_>(x) \tag{15.2}$$

The first part $\Phi_<$ (the "system field") consists of field modes whose physical wavelengths associated with physical momenta $p \equiv k/a$ are longer than the de Sitter horizon size, i.e. $p < \sigma H$ where σ is a parameter smaller than unity defining the size of the coarse-graining domain and the shape of the window function. The second part $\Phi_>$ (viewed as the "environment field") consists of field modes whose physical wavelengths are shorter than the horizon size whereby $p > \sigma H$. Inflation continuously shifts additional modes of the environment field into the system, stretching their physical wavelengths beyond the de Sitter horizon size. Technically the system field can be obtained from the total field by introducing a dynamic cut-off in momentum space through a suitable time-dependent window function that filters out the modes whose frequencies are lower than the comoving horizon size.

Due to the exponentially rapid expansion of spacetime, fluctuations of the inflaton field $\Phi(x)$ on super-horizon scales effectively "freeze" in a few Hubble times H^{-1} after they leave the horizon. For this reason, it is often said that after suitable smoothing on the super-horizon scales, the averaged field containing the long-wavelength modes (the system field) $\Phi_<$ can be considered to be classical. The quantum field comprising of shorter wavelength modes (the environment field) can effectively be viewed as a classical noise ξ driving the system field via a Langevin equation of the form

$$\dot{\Phi}_< + \frac{1}{3H}\frac{dV[\Phi_<]}{d\Phi_<} = \xi(\mathbf{x}, t) \tag{15.3}$$

where $V(\Phi)$ is the inflaton potential, and the "noise field" $\xi(\mathbf{x}, t)$ is assumed for simplicity (but not required – this would be true for free fields anyway) to be a Gaussian random field characterized by its two-point function $\langle \xi(\mathbf{x}, t)\, \xi(\mathbf{x}', t') \rangle$. This noise correlator plays a key role in stochastic inflation.

To examine the form of the noise field, one can first examine a free scalar field in the de Sitter spacetime, wherein the scale factor (assuming a spatially flat FLRW universe) $a(t) \sim e^{Ht}$. (In reality the scalar field can only be approximately massless and the spacetime approximately de Sitter, because otherwise the universe will be forever inflating.) In Starobinsky's original derivation [Sta86] the noise correlator is given by

$$\langle \xi(\mathbf{x}, t)\, \xi(\mathbf{x}', t') \rangle = \left(\frac{H}{2\pi}\right)^2 \frac{\sin\theta}{\theta}\delta(t - t') \tag{15.4}$$

where $\theta \equiv r/R; r \equiv |\mathbf{x} - \mathbf{x}'|$, and R is the spatial averaging scale for the inflaton field:

$$R(t) \equiv [\sigma H a(t)]^{-1} \tag{15.5}$$

with $\sigma \ll 1$.

This equation is the basis for the investigation of structure formation. Two basic issues are: How does the long-wavelength sector become classical, and what is the underlying mechanism? What makes the short-wavelength sector behave like noise, and what kind of noise is it? As will be shown below, the characteristics of the noise field play a pivotal role in determining the spectral function of structures and the decoherence of the system field.

In this model, the partition of the system and environment modes is a crucial element which affects the outcome of structure formation, since the noise generated from it after being amplified in the inflationary dynamics is responsible for the structure of the late universe. Following Starobinsky's proposal [Sta86] many papers have been written using a Langevin equation with a white noise source, but the justification was not so clearly understood. A few authors (e.g. [HuPaZh93b, CalHu95, CalGon97, Mat97a, Mat97b]) took exception to this way of noise generation and suggested that, rather than using a window function for free fields which contains an arbitrary parameter, an interacting quantum field (which the inflaton is assumed to be) when partitioned into two sectors can naturally produce noise which in general is colored and multiplicative. Recently it was pointed out [WinVil00] that the white noise originating from a sharp momentum cut-off (or the window function being a step function in Fourier space) has some pathological behavior, whereas a smooth window function will necessarily lead to a colored noise.

As noticed by Winitzki and Vilenkin (WV) [WinVil00], equation (15.4) shows a surprisingly slow decay of correlations at large distances. For comparison, the two-point function of the time derivatives of the unsmoothed field $\langle\dot{\phi}(\mathbf{x},t)\dot{\phi}(\mathbf{x}',t')\rangle$ at large separations r behaves as $\propto r^{-4}$ (here the angular brackets denote vacuum expectation value rather than statistical average). One would not expect a smearing of the field operators $\phi\,(\mathbf{x},t)$ on scales R to have such an effect on correlations at distances $r \gg R$.

The analysis of WV shows that the origin of the unusual behavior of the correlator found by Starobinsky is the sharp momentum cut-off in his smoothing procedure. With a smooth cut-off, WV recover the r^{-4} behavior independently of the cut-off window function and find that the time dependence of the noise correlator at large times is generically $\propto \exp\,(-2Ht)$ instead of a sharp δ-function dependence of equation (15.4).

For the correct prediction of the density contrasts in a quantum theory of structure formation in the early universe it is necessary to give a proper treatment of quantum and classical fluctuations and a correct identification of the origin and nature of noise. We have discussed the issue of decoherence in stochastic inflation in Chapter 9. We will discuss the issue of noise and structures in two separate sections below.

15.1.2 Noise in stochastic inflation

Noise from partitioning and smoothing a free field

Consider a free massive (m) scalar field $\Phi(\mathbf{x},t)$ in a spatially-flat Robertson-Walker (RW) spacetime with metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2) \tag{15.6}$$

where $a(t)$ is the scale factor and η is the conformal time defined by $a(t)d\eta = dt$.

Expanding Φ in normal modes with the basis spatial wavefunctions $e^{i\mathbf{k}\cdot\mathbf{x}}/(2\pi)^{3/2}$ of the spatially flat RW spacetime,

$$\Phi\,(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[a_{\mathbf{k}}\phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.}\right] \tag{15.7}$$

the amplitude function $\phi_{\mathbf{k}}(t)$ of the $\mathbf{k}$ mode obeys the equation of motion

$$\ddot{\phi}_{\mathbf{k}}(t) + 3H\dot{\phi}_{\mathbf{k}}(t) + \omega_{\mathbf{k}}^2\phi_{\mathbf{k}}(t) = 0 \tag{15.8}$$

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where $\omega_{\mathbf{k}}^2(t) \equiv p^2 + m^2$, $\;p \equiv k/a$, $\;k \equiv |\mathbf{k}|$ and an overdot here denotes derivatives with respect to cosmic time t.

In the conformally related field, the normal mode amplitude $\chi_{\mathbf{k}}(\eta) = \phi_{\mathbf{k}} a(\eta)$ corresponding to $\phi_{\mathbf{k}}$ obeys the equation of motion

$$\chi_{\mathbf{k}}''(\eta) + \left(k^2 + m^2 a^2 - \frac{a''}{a}\right) \chi_{\mathbf{k}}(\eta) = 0 \tag{15.9}$$

where a prime denotes taking the derivative with respect to the conformal time $\partial_\eta = a\partial_t$.

For the de Sitter universe, in a spatially flat RW coordinate representation,

$$a(t) = e^{Ht} \tag{15.10}$$

the expansion rate (Hubble parameter) $H \equiv \dot{a}/a$ is a constant in time and inflation goes on forever. In conformal time (ranging from $-\infty$ to 0)

$$\eta = -\frac{1}{a(t)H} \tag{15.11}$$

the evolution equation for the amplitude function $\chi_{\mathbf{k}}(\eta)$ of the conformally related field becomes

$$\chi_{\mathbf{k}}''(\eta) + \left[k^2 - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\right] \chi_{\mathbf{k}}(\eta) = 0 \tag{15.12}$$

where the parameter ν is defined as

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \equiv \frac{3}{2} - \epsilon_m \tag{15.13}$$

The generic solution to this equation can be expressed in terms of Bessel functions of the first and second kind,

$$c_1 \sqrt{|\eta|} J_\nu(k|\eta|) + c_2 \sqrt{|\eta|} Y_\nu(k|\eta|) \tag{15.14}$$

Requiring each $\chi_{\mathbf{k}}$ to match the plane wave solution $e^{-ik\eta}/\sqrt{2k}$ for $k \gg aH$, when wavelengths are too short to feel any spacetime curvature effects, produces the standard Bunch–Davies solution

$$\chi_{\mathbf{k}}(\eta) = \frac{\sqrt{\pi}}{2} \sqrt{|\eta|} H_\nu^{(1)}(k|\eta|) \tag{15.15}$$

where

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \tag{15.16}$$

is the Hankel function of the first kind. The amplitude function of the $\mathbf{k}$th normal mode of the original scalar field ϕ is given by

$$\phi_{\mathbf{k}}(\eta) = \frac{\sqrt{\pi}}{2} H |\eta|^{3/2} H_\nu^{(1)}(k|\eta|) \tag{15.17}$$

which in the massless case ($\nu = \frac{3}{2}$) becomes

$$\phi_{\mathbf{k}}(\eta) = H \frac{k\eta - i}{\sqrt{2k^3}} e^{-ik\eta} \tag{15.18}$$

In an expanding universe each mode will successively leave the horizon when its physical wavelength $p^{-1} = a/k$ reaches H^{-1}. Thus for a de Sitter universe, at the horizon crossing, $|k\eta| = 1$.

Spatial averaging and noise

Field fluctuations on super-horizon scales behave effectively as classical fluctuation modes with random amplitudes. This is conventionally described by averaging the field Φ in space over super-horizon scales and treating the resulting field $\Phi_<$ as a classical stochastic field satisfying a Langevin equation with a noise source described by a Gaussian random field of the shorter wavelength modes, given by equation (15.3); see [Sta86, GonLin86, NaNaSa88, NamSas89, Nam89, Mij90, SalBon91].

The averaging of the field Φ is performed by means of a suitable window function $W_s(\mathbf{x}; R)$ with a characteristic smoothing scale R,

$$\bar{\Phi}(\mathbf{x}, t) \equiv \int d^3\mathbf{x}' \phi(\mathbf{x}', t) W_s(\mathbf{x} - \mathbf{x}'; R) \tag{15.19}$$

Here, the physical smoothing scale is taken to be σ^{-1} times larger than the horizon size, with $\sigma \ll 1$. The corresponding comoving scale is

$$R(t) = \frac{1}{\sigma H a(t)} \tag{15.20}$$

The volume-averaged field has a mode expansion

$$\bar{\Phi}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[w(kR) a_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.} \right] \tag{15.21}$$

where $w(kR)$ is a suitable Fourier transform of the window function W_s. Starobinsky used a sharp step-function cut-off in Fourier space:

$$w(kR) = \theta(1 - kR) \tag{15.22}$$

The volume-averaged inflaton field is treated as a classical field $\Phi_<$ satisfying the Langevin (15.3) under the potential $V(\Phi)$ and an effective "noise field" source $\xi(\mathbf{x}, t)$. In the original proposal the noise source $\xi(\mathbf{x}, t)$ was heuristically defined as a stochastic field that corresponds to the quantum operator of the free field derivative $\dot{\bar{\Phi}}$, in the sense that any average of ξ, such as the correlator $\langle \xi(\mathbf{x}, t) \xi(\mathbf{x}', t') \rangle$, is assumed to be the same as the corresponding quantum expectation values of $\dot{\bar{\Phi}}$ in the vacuum state (which for de Sitter spacetime is the standard Bunch–Davies vacuum). The effective noise field ξ defined in this way is a Gaussian random field with zero mean, so the correlator $\langle \xi(\mathbf{x}, t) \xi(\mathbf{x}', t') \rangle$ completely describes its properties.

We show below the calculation of the noise correlator $\langle \xi(\mathbf{x}, t) \xi(\mathbf{x}', t') \rangle$ from a computation of the corresponding expectation value of the quantum "noise operator" $\dot{\bar{\Phi}}$ following WV [WinVil00]. The noise correlator generally depends on the particular window function $W_s(\mathbf{x}; R)$ and on the parameter σ. These parameters can in principle be related to observational data such as from the WMAP via the standard theory of structure formation, a topic we will come to in a later section.

Correlator of noise

Here we derive the correlators of the effective noise field $\xi(\mathbf{x}, t)$ for an arbitrary smoothing window. In stochastic inflation the noise field $\xi(\mathbf{x}, t)$ is defined through the time derivative of the averaged field $\dot{\bar{\Phi}}$ in mode expansion

$$\dot{\bar{\Phi}}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[v_k(\eta)\, a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.}\right] \tag{15.23}$$

where

$$v_k(\eta) \equiv \frac{d}{dt}\left[w(kR)\, \phi_k(\eta)\right] = \left[-HkRw'(kR)\, \phi_k(\eta) + w(kR)\, \dot{\phi}_k(\eta)\right] \tag{15.24}$$

In the limit of $\sigma \ll 1$ we may disregard the second term in the square brackets. The noise correlator then becomes

$$\langle \xi(\mathbf{x}_1 \eta_1)\, \xi(\mathbf{x}_2, \eta_2) \rangle = \frac{H^4 \eta_1 \eta_2}{4\pi^2 r \sigma^2} \int_0^\infty dk\, \sin kr\, h(k) \tag{15.25}$$

where $h(k)$ is a dimensionless function of two variables η_1, η_2

$$h(k) \equiv (1 + iy_1)(1 - iy_2)\, e^{ik(\eta_2 - \eta_1)} w'\left(-\frac{y_1}{\sigma}\right) w'\left(-\frac{y_2}{\sigma}\right) \tag{15.26}$$

where $y_1 = k\eta_1$, $y_2 = k\eta_2$. The asymptotic form of equation (15.25) at large r is given by

$$\langle \xi(\mathbf{x}_1, \eta_1)\, \xi(\mathbf{x}_2, \eta_2) \rangle = -\frac{\left(H^2 \eta_1 \eta_2\right)^2}{2\pi^2 r^4 \sigma^4} \left|w''(0)\right|^2 + O\left(r^{-6}\right) \tag{15.27}$$

Now examine the unsmoothed correlator of quantum field derivatives given at arbitrary space and time points by (Appendix C of WV):

$$\begin{aligned}\left\langle \dot{\Phi}(\mathbf{x}_1, t_1)\, \dot{\Phi}(\mathbf{x}_2, t_2) \right\rangle &= \frac{1}{2\pi^2} \int_0^\infty \dot{\phi}_k(t)\, \dot{\phi}_k^*(0) \frac{\sin kr}{r} k dk \\ &= \frac{H^4}{2\pi^2} (\eta_1 \eta_2)^2 \frac{3(\eta_1 - \eta_2)^2 + r^2}{\left[(\eta_1 - \eta_2)^2 - r^2\right]^3} \end{aligned} \tag{15.28}$$

As expected, it diverges on the lightcone where r becomes $|\eta_1 - \eta_2|$. The asymptotic form of equation (15.28) at large distances r is

$$\left\langle \dot{\Phi}(\mathbf{x}_1, t_1)\, \dot{\Phi}(\mathbf{x}_2, t_2) \right\rangle = -\frac{H^4 (\eta_1 \eta_2)^2}{2\pi^2 r^4} + O\left(r^{-6}\right) \tag{15.29}$$

We see that the stochastic source correlator (15.27) is very similar to the quantum field correlator (15.29). Note also that the asymptotic (15.27) is essentially independent of the shape of the window function, since the value $|w''(0)|$ as indicated by equation (15.21) has the meaning of the window-averaged squared

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distance and must be of order 1 because the window profile $W(q)$ starts to decay at $q \sim 1$ by construction.

We can obtain a simpler expression for the correlator in the limit when the smoothing parameter σ is small while the product $\sigma H r$ remains finite. A rescaling $r \to \sigma H r \equiv \rho$ and the corresponding change of variable $k \equiv \sigma H \kappa$ simplify equation (15.25) because we can omit terms of order σ and smaller; in particular, the product of mode functions is simplified to

$$\phi_k^*(\eta_1)\,\phi_k(\eta_2) = \frac{1}{2H\kappa^3\sigma^3}\left(1 + O\left(\sigma^2\right)\right) \tag{15.30}$$

The leading term in the correlator, expressed through κ and ρ, becomes

$$\langle \xi(\mathbf{x}_1, \eta_1)\,\xi(\mathbf{x}_2, \eta_2)\rangle = \frac{H^6\eta_1\eta_2}{4\pi^2\rho}\int_0^\infty d\kappa\,\sin\kappa\rho\, w'(-H\eta_1\kappa)\, w'(-H\eta_2\kappa) + O\left(\sigma^2\right) \tag{15.31}$$

Therefore, in the limit of small σ but finite $\sigma H r$, the correlator as a function of the "effective distance" ρ and the time difference (expressed by η_2/η_1) becomes independent of σ.

The expression in equation (15.31) allows us to compute the correlator at all distances in the limit of small σ. Under this condition, for a Gaussian smoothing window, $w(p) = \exp\left(-p^2/2\right)$, we obtain

$$\begin{aligned}
&\langle \xi(\mathbf{x}_1, \eta_1)\,\xi(\mathbf{x}_2, \eta_2)\rangle \\
&= \frac{\left(H^4\eta_1\eta_2\right)^2}{4\pi^2\rho}\int_0^\infty \exp\left[-H^2\frac{\eta_1^2+\eta_2^2}{2}\kappa^2\right]\kappa^2\sin\kappa\rho\, d\kappa \\
&= \frac{\left(H^4\eta_1\eta_2\right)^2\mu^4}{4\pi^2\rho^4}\left[1 - \left(\frac{1}{\mu} - \mu\right) i\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{i\mu}{\sqrt{2}}\right)\exp\left(-\frac{\mu^2}{2}\right)\right]
\end{aligned} \tag{15.32}$$

where

$$\mu \equiv \frac{\rho}{\sqrt{H^2\left(\eta_1^2+\eta_2^2\right)}} \tag{15.33}$$

is a dimensionless quantity. (A plot of this function for $\eta_1 = \eta_2$ can be found in WV.) The leading term of the expression in brackets in equation (15.32) at large μ is $\left(-2\mu^{-4}\right)$, and since for the Gaussian window $w''(0) = -1$, we recover equation (15.27). The value of the correlator at the coincident points ($\rho = 0$) as a function of time separation is

$$\langle \xi(0, \eta_1)\,\xi(0, \eta_2)\rangle = \frac{H^4\left(\eta_1\eta_2\right)^2}{2\pi^2\left(\eta_1^2+\eta_2^2\right)^2} = \frac{H^4}{8\pi^2}\frac{1}{\cosh^2 H\Delta t} \tag{15.34}$$

We can also obtain the leading asymptotics of the unequal-time correlator at large time separations. Again start with equation (15.25) and assume that the time separation is much greater than the Hubble time, $\eta_2/\eta_1 \equiv a^{-1} \ll 1$. For simplicity we can choose the initial time such that $H\eta_1 = -1$. Using an expansion (see equation (A12) of WV) for $w\left(a^{-1}k\right)$ at small $a^{-1}\kappa$ (since the integration is

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effectively performed over a fixed finite range of k) we obtain

$$\langle \xi(\mathbf{x}_1,\eta_1)\,\xi(\mathbf{x}_2,\eta_2)\rangle = \frac{H^2 w''(0)}{4\pi^2\sigma^3 a^2 H r}\int_0^\infty dk\, k\sin kr\, w'\left(\frac{k}{\sigma H}\right) \times e^{ik/H}\left(1+i\frac{k}{H}\right) + O\left(a^{-4}\right) \tag{15.35}$$

The integral in equation (15.35) is time-independent. Therefore the correlator decays as $a^{-2} = \exp(-2Ht)$ with time separation at any fixed distance. This derivation shows how a regular window function produces a colored noise.

Colored noise from coarse graining an interacting field

In addition to using a smoothing window function as illustrated in the above section [WinVil00, Mat93, Rio02] one could make a frequency or wavelength partition, splitting the short- and the long-wavelength sectors. This has been treated in Chapter 5 for a scalar field in Minkowski spacetime and in Chapter 9 for a conformally-related theory in de Sitter spacetime. We now turn to the issue of structure formation from a colored noise.

15.2 Structure formation: Effect of colored noise

A standard mechanism for structure formation is the amplification of primordial density fluctuations by the evolutionary dynamics of spacetime [Sak66, LifKal63, Bar80, Muk05]. In the lowest order approximation the gravitational perturbations (scalar perturbations for matter density and tensor perturbations for gravitational waves) obey linear equations of motion. Their initial values and distributions are stipulated, generally assumed to be a white noise spectrum. In these theories, fashionable in the 1960s and 1970s, the primordial fluctuations are classical in nature. In the standard model of FLRW cosmology, the scale factor of the universe growing in a power law of cosmic time generates a density contrast which turns out to be too small to account for the galaxy masses. The observed nearly scale-invariant spectrum [Har70, Zel72] also does not find any easy explanation in this model [Pee80, ZelNov85].

Inflationary models explain structure formation from amplification of vacuum fluctuations of a scalar field Φ, the inflaton; see [GuthPi82, Sta82, MukChi82, Haw82, BaStTu83, Bra83, MuFeBr92, DeGuLa92, YiViMi91, YiVis92, YiVis93, YiVis93b, GlMaRa82, BoVeHo94, Bur95, Muk05]. Consider the "eternal inflation" stage where the universe has locally a de Sitter geometry, with a constant Hubble radius (de Sitter horizon) $l_h = H^{-1}$. (In reality H cannot strictly be a constant, for otherwise the universe cannot reheat to our present FLRW state.) The physical wavelength l of a mode of the inflaton field is $l = p^{-1} = a/k$, where k is the wavenumber of that mode. As the scale factor increases exponentially, the wavelengths of many modes can grow larger than the horizon size. After the end of the de Sitter phase, the universe begins to reheat, turning into a

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radiation-dominated Friedmann universe with power law expansion $a(t) \sim t^n$. In this phase, the Hubble radius grows much faster than the physical wavelength, and some inflaton modes will reenter the horizon. The fluctuations of these long-wavelength inflaton modes that went out of the de Sitter horizon and later came back into the FLRW horizon play an important role in determining the large-scale density fluctuations of the early universe, which in time seeded the galaxies.

The stochastic inflation paradigm, after a proper treatment of decoherence of the long wavelength modes[1] and a first-principles derivation of noise (arising from the short wavelength sector), could thus provide a sound rationale for the Langevin equation depicting the dynamics of the inflaton perturbations or the Fokker–Planck equation describing the evolution of their probability distributions.

A key issue in the solution of the Langevin or Fokker–Planck equation is the choice of the initial conditions for the perturbations. Many authors (see [SalBon91, Hod90, MMOL91, KBHP91]) agree that it should be consistent to assume the spatial homogeneity of our observable local patch of the universe, and therefore the vanishing of all fluctuations right before the moment it crosses the horizon size, about 60 e-folds before the end of inflation, since at that time only fluctuations on larger scales could have grown significantly. Therefore, all points inside the present Hubble radius (at that time contained in the same coarse-graining domain) must have the same local value of the scalar field, although this value can be different from the one assumed in other regions of the Universe.

Even if it is generally assumed that inflation started well before the last 60 e-folds, for the white-noise case the evolution of fluctuations is completely insensitive to what happened before that epoch and the constraint really becomes a new initial condition. In contrast, non-Markovian fluctuations generated by colored noises [HuPaZh93b] will retain some memory of the evolution before the constraint.

The linkage of colored noise-generated structure to observations in WMAP was suggested in [MaMuRi04] (MMR), where evidence was found for a blue tilt in the power-spectrum on the largest observable scales as a consequence of the non-Markovian dynamics near the constraint. This is due to the fact that the increased noise correlation time (with respect to the white-noise case) acts as a sort of "inertia" against the growth of the perturbations after the constraint, thereby resulting in a suppression of the power-spectrum on the scales that crossed the horizon in the ensuing few Hubble times. This is an interesting

[1] There are different views on how the long-wavelength modes got decohered, including the extreme one that no dynamical explanation needs to be provided. This so-called *decoherence without decoherence* theme first proposed by Polarsky and Starobinsky [KiPoSt98] is attractive more because of its expedience than truth value. The original form has been revised after meeting with criticisms. For a more careful recent study on this proposal, see, e.g. [CamPar05]. A different approach is suggested by Woodard [TsaWoo05, Woo05a, Woo06]. For a recent review, see [Win06].

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feature, since the CMB anisotropy measurements made by WMAP [Spe03] give some evidence for a suppression of the low multipoles, consistent with earlier analogous results found by COBE [Ben96], although the statistical significance of such a suppression is not large [TeCoHa03, OTZH04, Efs03, Efs04, BiGoBa04].

A related paper [LMMR04] (LMMR) points out the low multipoles suppression might also be a consequence of the colored noise. Compared to white noise, a smooth choice of the window function will in fact slightly suppress the contribution to the noise given by the field modes whose frequency is immediately higher than the cut-off scale $\sigma(aH)$ (while enhancing the lower frequencies). Right after the time τ_* at which the homogeneity constraint is set on the comoving patch of the universe, fluctuations with $k \lesssim \sigma a_* H_*$ will grow less than in the white-noise case before freezing out, and if σ is not too small this suppression can be effective also on observable scales. Even in the Markovian case, the noise correlation function in configuration space has a dependence of σ beyond the second order which could show up in the power spectrum.

Before getting into the details we want to add a qualifying remark on how this mechanism is placed in relation to other mechanisms so as to avoid a skewed perspective. The colored noise explanation of the suppression of lower multipoles (blue tilt) mode is only one amongst many proposed. As cautioned in the beginning, we select this topic mainly to illustrate some key ideas in NEqQFT, in this case, the effect of quantum noise on structure formation in stochastic inflation. Adopting this perspective we hope that even if at the end the actual physical scenario may not survive over other competing theories, the readers can learn the physics of NEq quantum fields through a detailed analysis of these sample problems.

15.2.1 Colored noise from smooth window functions

Partitioning and smoothing

As discussed earlier, if one uses the cosmological horizon as the partition scale, the environment field $\Phi_>$ consisting of the subhorizon (short-wavelength) modes can be sieved out by the use of a suitable time-dependent high-pass filter in Fourier space. This is achieved by means of a different window function $\tilde{W}_\sigma(y), \quad y \equiv k\eta$ such that $\tilde{W}_\sigma(y) = 0$ for $k|\eta| \ll \sigma$ and $\tilde{W}_\sigma(y) = 1$ for $k|\eta| \gg \sigma$. (Note that this window function used by LMMR is complementary to the one used by WV discussed in the last section, which is a low-pass filter.) The parameter σ defines the size of the coarse-graining domain and an "effective horizon" $\sigma(aH)$:

$$\Phi_> = \int d^3\mathbf{k} \frac{\tilde{W}_\sigma(k\eta)}{(2\pi)^{3/2}} \left[a_\mathbf{k} \phi_\mathbf{k}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.}\right] \tag{15.36}$$

In the stochastic inflation paradigm, the quantum fluctuations on subhorizon scales act as a classical noise source ξ with a given probability distribution $P[\xi]$ in a Langevin equation which drives the super-horizon modes. Our discussions

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in the previous section on the origin and nature of noise from quantum fluctuations and on the decoherence of the long-wavelength mode by this noise may serve as justification for such a proposal. Technically, the quantum problem of computing the expectation value of the coarse-grained field is thus reduced to the classical problem of evaluating the mean of the solution to the stochastic evolution equation averaged over all possible noise configurations.

Following such a prescription, we can split the scalar field $\Phi = \bar{\phi} + \varphi$ into its statistical mean value $\bar{\phi}$ whose normal mode amplitudes satisfy the classical equation of motion (15.8) and a fluctuation field $\varphi[\xi]$, with zero mean over the distribution $P[\xi]$.[2] The stochastic equation of motion for the super-horizon fluctuations was shown before to be

$$\ddot{\varphi}_{\mathbf{k}} + 3H\dot{\varphi}_{\mathbf{k}} - \left(\frac{k^2}{a^2} - m^2\right)\varphi_{\mathbf{k}} = \frac{\xi_{\mathbf{k}}}{a^3} \tag{15.37}$$

or, for the conformally related field normal mode amplitude $\chi_{\mathbf{k}} = a\phi_{\mathbf{k}}$ in conformal time η, in a similar decomposition $\chi_{\mathbf{k}} = \bar{\chi}_{\mathbf{k}} + \tilde{\chi}_{\mathbf{k}}$ the mean field satisfies (15.9) and the fluctuation field modes $\tilde{\chi}_{\mathbf{k}}$ obeys

$$\tilde{\chi}''_{\mathbf{k}} + \left(k^2 + m^2a^2 - \frac{a''}{a}\right)\tilde{\chi}_{\mathbf{k}} = \xi_{\mathbf{k}} \tag{15.38}$$

The noise ξ is a Gaussian random field, whose configurations are weighted by the functional probability distribution

$$P[\xi] = N\exp\left[-\frac{1}{2}\int d^4x d^4x' \xi(x)\mathcal{N}^{-1}(x,x')\xi(x')\right] \tag{15.39}$$

$$= N\exp\left[-\frac{1}{2}\int d\eta d\eta' d^3\mathbf{k} d^3\mathbf{k}' \xi_{\mathbf{k}}(\eta)\mathcal{N}^{-1}_{\mathbf{k},\mathbf{k}'}(\eta,\eta')\xi_{\mathbf{k}'}(\eta')\right] \tag{15.40}$$

where $\mathcal{N}^{-1}_{\mathbf{k},\mathbf{k}'}(\eta,\eta')$ is the functional inverse of

$$\mathcal{N}_{\mathbf{k},\mathbf{k}'}(\eta,\eta') = \delta(\mathbf{k}+\mathbf{k}')\frac{\mathrm{Re}[f(y)f^*(y')]}{2k^3} \tag{15.41}$$

and, with $y \equiv k\eta$,

$$f(y) = \sqrt{2k^3}(\tilde{W}''_\sigma \chi_{\mathbf{k}} + 2\tilde{W}'_\sigma \chi'_{\mathbf{k}}) \tag{15.42}$$

This probability distribution allows us to calculate the statistical mean value $\langle\ldots\rangle_\xi$ of any ξ-dependent quantity averaged over the noise field configurations, defined as

$$\langle\ldots\rangle_\xi = \int\mathcal{D}[\xi]\ldots P[\xi] \tag{15.43}$$

[2] This is true at linear order because nonlinear corrections will shift the mean value. Also, if Φ is the inflaton field then $\bar{\phi}$ should be the homogeneous background, and as such have no Fourier decomposition. All is well in the case of a test field with no metric fluctuations, which is what we will assume here.

Then, by definition the mean $\langle \xi(\eta) \rangle_\xi$ of the noise vanishes at all times, while the two-point correlation function is by definition

$$\langle \xi_{\mathbf{k}}(\eta) \xi_{\mathbf{k}'}(\eta') \rangle_\xi = \mathcal{N}_{\mathbf{k},\mathbf{k}'}(t,t') \tag{15.44}$$

This correlation function, the noise kernel, completely characterizes the statistical properties of the Gaussian noise field. In configuration space it reads

$$\langle \xi(x) \xi(x') \rangle_\xi = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{2k^3} \mathrm{Re}[f(y) f^*(y')] \tag{15.45}$$

As we saw before the statistical behavior of the noise depends critically on the shape of the filter. Choosing the special window function $\tilde{W}_\sigma(k\eta) = \theta(k|\eta| - \sigma)$ leads to the standard white-noise two-point correlation function. For $\mathbf{x} = \mathbf{x}'$ it reads

$$\langle \xi(x) \xi(x') \rangle_\xi = \frac{H^3}{4\pi^2} \big(1 + \mathcal{O}(\sigma^2)\big) \delta(t - t') \tag{15.46}$$

which is highly divergent for $t = t'$ and has a vanishing characteristic correlation time. In contrast a smooth window function yields a correlation function with no divergence and a finite correlation time, therefore producing a colored noise, e.g. with

$$\tilde{W}_\sigma(y) = 1 - e^{-\frac{y^2}{2\sigma^2}} \tag{15.47}$$

the two-point correlation function at $r = 0$ is given by

$$\langle \xi(t) \xi(t') \rangle_\xi = \frac{H^4}{8\pi^2} \frac{1}{\cosh^2(H(t - t'))} + \mathcal{O}(\sigma^2) \tag{15.48}$$

which behaves like $e^{-2H(t-t')}$ asymptotically. This asymptotic behavior is quite general for a wide class of smooth window functions [WinVil00].

Fluctuations and structures

The particular solution of the evolution equation (15.38) for the fluctuations $\tilde{\chi}_{\mathbf{k}}$ sourced by the noise field ξ can be expressed in terms of the general solutions $\chi_1 = \sqrt{k|\eta|} J_\nu(|y|)$ and $\chi_2 = \sqrt{k|\eta|} Y_\nu(|y|)$ of the homogeneous equation (15.12). This solution reads

$$\tilde{\chi}_{\mathbf{k}}[\xi](\eta) = \int_{\eta_i}^{\eta} d\eta'\, g(y, y')\, \xi_{\mathbf{k}}(y') \tag{15.49}$$

where

$$g(y, y') = \frac{\chi_1(y)\chi_2(y') - \chi_2(y)\chi_1(y')}{\chi_1'(y')\chi_2(y') - \chi_2'(y')\chi_1(y')} \tag{15.50}$$

and η_i is the beginning of inflation, at which we set the initial condition $\tilde{\chi}_{\mathbf{k}}(\eta_i) = 0$.

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Keeping this assumption, LMMR impose the constraint that at a much later time η_* (roughly about 60 e-folds before the end of inflation) there are no fluctuations in that part of the universe corresponding to the present observable sky. This is motivated by the fact that all the points we observe today with substantial homogeneity were included at η_* in the same coarse-grained domain.[3] Physically this amounts to the assumption that at η_* the comoving patch of the universe we observe today has complete homogeneity and all fluctuations on smaller scales were generated later by the stochastic source represented by the noise term.[4] There is no assumption made on the behavior of larger unobservable scales.

We are thus led to consider (for a given noise configuration) a different solution for the subsequent evolution of the fluctuations, obtained as in (15.49) by starting the integration at η_*, when a new (stochastic) initial condition holds. In turn, $\tilde{\chi}_{\mathbf{k}}[\xi](\eta_*)$ is determined again from (15.49) with the usual vanishing initial condition at η_i. However, as long as we are dealing with points inside the present observable universe, we can skip the stochastic initial conditions η_* since their inverse Fourier transform is assumed to vanish. Therefore, in configuration space the subsequent evolution of the fluctuations will only contain noise modes integrated after η_*. Thus, for relevant $\mathbf{x}$'s we may write

$$\varphi(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{a} \int_{\eta_*}^{\eta} dy'\, g(y,y')\, \xi_{\mathbf{k}}(\eta') \tag{15.51}$$

and

$$\varphi(\mathbf{x},\eta_*) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{a_*} \int_{\eta_i}^{\eta_*} d\eta'\, g(y_*,y)\, \xi_{\mathbf{k}}(y') \tag{15.52}$$

where the first equation is only valid for scales inside our observed patch of the universe.

As expected, since the fluctuation $\varphi_{\mathbf{k}}[\xi]$ is linear in ξ, at all times we have that

$$\langle \varphi[\xi](\eta) \rangle_\xi = 0 \tag{15.53}$$

while the two-point correlation function in $\mathbf{x}_1$ and $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{r}$ can be obtained by integrating the noise correlation function (15.41). LMMR find

$$C(\mathbf{r},\eta) \equiv \langle \varphi[\xi](\mathbf{x}_1,\eta)\varphi[\xi](\mathbf{x}_2,\eta) \rangle_\xi = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{|I_1(k)|^2}{2k^3} \tag{15.54}$$

[3] Introduced first by Salopek and Bond [SalBon91], such a constraint is necessary if one wants to use the variance of the single-point probability distribution (which has no spatial information encoded) to extract some information on the cosmic microwave background. Without this constraint, the variance will be much larger because fluctuations (specially at the beginning of inflation) add up very rapidly over time. However, this variance can now only be used to model the structure on ultralarge scales (of the order of the wavelength of the first modes crossing the Hubble radius).

[4] In principle, solving the Langevin equation with the full space dependence may not require the imposition of this constraint, because the correlation function is able to distinguish the scales. At any time, as a consequence of the smoothing, fluctuations on scales smaller than the filtering scale will not appear (as in white noise) or will appear only in a finite frequency range around this scale (as with colored noise). In either case imposition of the window function could effectively serve the function of the homogeneity constraint.

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where

$$I_1(k) = \frac{\sqrt{2k^3}}{a} \int_{\eta_*}^{\eta} d\eta' \, g(y, y')(\tilde{W}''_\sigma \chi_{\mathbf{k}} + 2\tilde{W}'_\sigma \chi'_{\mathbf{k}}) \tag{15.55}$$

In the same way one can calculate the correlation function evaluated at η_*, yielding

$$C_*(\mathbf{r}, \eta_*) \equiv \langle \varphi[\xi](\mathbf{x}_1, \eta_*) \varphi[\xi](\mathbf{x}_2, \eta_*) \rangle_\xi = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{|I_2(k)|^2}{2k^3} \tag{15.56}$$

where $I_2(k)$ has the same form as $I_1(k)$ but it refers to the time interval $[\eta_i, \eta_*]$.

One can also define the mixed correlation function of the scalar field perturbations evaluated at different times:

$$C_\times(\mathbf{r}, \eta, \eta_*) \equiv \langle \varphi[\xi](\mathbf{x}_1, \eta) \varphi[\xi](\mathbf{x}_2, \eta_*) \rangle_\xi = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\mathrm{Re}[I_1(k) I_2^*(k)]}{2k^3} \tag{15.57}$$

With these correlation functions $C(\mathbf{r}, \eta)$, $C_*(\mathbf{r}, \eta_*)$, $C_\times(\mathbf{r}, \eta, \eta_*)$ one can proceed to calculate the conditional correlation function of the scalar field perturbations – conditional (subscript c) here referring to the constraint defined in the set-up of the initial conditions described above. In the physically reasonable limit of $\eta_i \ll \eta_*$ LMMR [LMMR04] obtained

$$\langle \varphi(\mathbf{x}_1) \varphi(\mathbf{x}_2) \rangle_{\mathrm{c}} \simeq C(\mathbf{r}) \tag{15.58}$$

This yields the power spectrum $\mathcal{P}_{\delta\varphi}(k)$ of the fluctuations, defined by

$$\langle \varphi(\mathbf{x}_1) \varphi(\mathbf{x}_2) \rangle_{\mathrm{c}} = \frac{1}{4\pi} \int d^3\mathbf{k} \, e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\mathcal{P}_{\delta\varphi}(k)}{k^3} \tag{15.59}$$

as

$$\mathcal{P}_{\delta\varphi}(k) = \frac{1}{4\pi^2} |I_1(k)|^2 \tag{15.60}$$

In the small-σ limit the standard scale-invariant result $\mathcal{P}_{\delta\varphi}(k) = H^2/4\pi^2$ is recovered.

15.2.2 Curvature perturbations and blue tilt

So far this treatment has been under the test-field approximation, meaning that the background spacetime where the quantum field propagates is assumed to be fixed, i.e. the de Sitter universe. But in reality the quantum field contributes to the energy–momentum tensor which determines the evolution of the scale factor, via the slow-roll Friedmann equation $H^2 \simeq (8\pi G/3)V(\phi)$, and the field perturbations induce fluctuations in the metric. These metric perturbations need be considered alongside the scalar field perturbations φ. Let ψ be the curvature perturbation, which is gauge dependent. To avoid spurious coordinate effects it is preferable to use the gauge-invariant comoving curvature perturbation $\mathcal{R} = \psi + H(\varphi/\dot{\phi})$ which measures the intrinsic spatial curvature on hypersurfaces of constant time [Rio02] as the physical degrees of freedom.

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Defining in conformal time $v = a^2\phi'/a'$, the variable $u = -v\,\mathcal{R}$ satisfies the equation of motion

$$u'' - \nabla^2 u - \left(\frac{v''}{v}\right) u = 0 \tag{15.61}$$

Expanding the last term to first order in the slow-roll parameters $\epsilon_V \equiv -3\dot{H}/H^2$; $\eta_V \equiv V''/3H^2$ formed from the Hubble parameter and the inflaton potential, one finds

$$\frac{v''}{v} \simeq \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right) \tag{15.62}$$

where $\nu \simeq \frac{3}{2} + 3\epsilon_V - \eta_V$.

We see that in the slow-roll approximation the gauge-invariant normal modes $u_{\mathbf{k}}$ satisfy the same equation of motion (15.12), the only difference enters in the definition of the parameter ν labeling the solutions:

$$u''_{\mathbf{k}} + \left[k^2 - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\right] u_{\mathbf{k}} = \xi_{\mathbf{k}} \tag{15.63}$$

We can then apply to $\mathcal{R}$ the results derived for the power spectrum of the perturbations of a test scalar field, concluding that for the curvature perturbation we also have $\mathcal{P}_{\mathcal{R}}(k) \propto |I_1(k)|^2$.

In the limit $k|\eta| \ll \sigma \lesssim 1$ which is reasonably satisfied on cosmological scales, the power spectrum simplifies to

$$\mathcal{P}_{\mathcal{R}}(k) = A^2_{\mathcal{R}} \tilde{W}^2_\sigma(k|\eta_*|)(k|\eta|)^{2\eta_V - 6\epsilon_V} \tag{15.64}$$

Since $\tilde{W}^2_\sigma < 1$, this shows a blue tilt on large observable scales with $k \sim \sigma a_* H_*$, corresponding to physical lengths about σ^{-1} times greater than the present Hubble radius. In the limit $\sigma \ll k|\eta_*|$ (since $\tilde{W}_\sigma(k|\eta_*|) \simeq 1$) we recover the ordinary result

$$\mathcal{P}_{\mathcal{R}}(k) = A^2_{\mathcal{R}}(k|\eta|)^{2\eta_V - 6\epsilon_V} \tag{15.65}$$

This blue tilt stems from the fact that a smooth window function does not make a sharp separation in Fourier space but it gradually weighs the modes, allowing for a small low-frequency contribution to the short-wavelength part of the field (in terms of which the noise is defined) while depleting modes whose wavelength is immediately smaller than the cut-off scale. The colored noise originated from such a window function is thus able to generate fewer fluctuations than a white noise on scales slightly smaller than the comoving coarse-graining domain.

As a consequence, under the constraint that in our comoving patch of the universe the fluctuations can grow only after η_*, the scales that are leaving the horizon in the following few Hubble times receive fewer "random kicks" before freezing out than in the white-noise case. Therefore, the power spectrum is a function of k smoothly interpolating between the values 0 and 1 it assumes for small and large k, respectively.

This power spectrum can be used to calculate the CMB multipoles predicted by a specific choice of the window function W. Quite generally, we expect to find a suppression of the lowest multipole, which is sensitive to a modification of the power spectrum on this very large scale. However, in order to quantify this suppression one needs to choose the shape of the window function and the precise time η_* at which the constraint is set. As mentioned before the significance of the low multipole suppression varies depending on the choice of the constraint time. Detailed description can be found in LMMR, where our exposition here is adapted from.

15.2.3 Structures from coarse graining an interacting field

As we learned earlier colored noise can also be generated by coarse graining a sector of one partitioned interacting quantum field [CalHu95, CalGon97, Mat97a, Mat97b]. In Chapter 5 we derived the influence functional describing the effect of high-frequency modes on the low-frequency sector. The real part of the influence action contains divergent terms and should be renormalized. The imaginary part is finite and is associated with the decoherence process. From this one can derive the renormalized semiclassical Langevin equation governing the system field (the long-wavelength sector) driven by a noise originating from coarse graining the environment field (the short-wavelength sector). We can use this equation to understand the generation of classical inhomogeneities from quantum fluctuations, obtaining their power spectrum and be able to compare with observational data such as from WMAP.

In the ϕ^4 model used by Lombardo and Nacir [LomNac05] we presented in Chapter 9, there are two such sources ξ_2 and ξ_3, associated with the interaction terms $\phi^2_<\phi^2_>$ and $\phi^3_<\phi_>$ respectively. The full influence function is given in (9.112). Reading the noise kernels off that equation, we may now treat the generation of inhomogeneities with noise arising from one interacting quantum field.

We are interested in finding the power spectrum of perturbations to the inflaton field up to $\hbar$ and λ^2 order. To carry this out, we split the system field as $\phi_< = \phi_0(\eta) + \varphi_<$, where we identify $\phi_0(\eta)$ as a classical background field which satisfies the slow-roll conditions. The power spectrum of the field fluctuations $\varphi_<$ may be expressed as $P_\varphi(k) = 2\pi^2 k^{-3}\Delta^2_\varphi(k)$, with $\Delta^2_\varphi(k)$ defined by

$$\langle \varphi_<(\mathbf{x})\varphi_<(\mathbf{x}')\rangle = \int d^3\mathbf{k}\, \frac{\Delta^2_\varphi(k)}{4\pi k^3} \exp(-i\mathbf{k}\cdot\mathbf{r}) \tag{15.66}$$

where $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}'$.

Expanding the semiclassical Langevin equation up to linear order in the mode amplitude of interest $\varphi_<(\vec{k})$, we obtain

$$\phi_0''(\eta) + 2\tilde{H}\phi_0'(\eta) + 4\lambda a^2\phi_0^3(\eta) = 0 \tag{15.67}$$

$$\varphi_<''(\vec{k},\eta) + [k^2 + 12\lambda a^2\phi_0^2(\eta)]\varphi_<(\vec{k},\eta) + 2\tilde{H}\varphi_<'(\vec{k},\eta) = -\frac{\xi_2(\vec{k},\eta)}{a^2}\phi_0(\eta) \tag{15.68}$$

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where terms which do not contribute to the power spectrum up to order $\hbar$ have been discarded. The term with the ξ_3 noise source gives a zero contribution due to our approximations and the orthogonality of the Fourier modes. Note the presence of the ξ_2 noise source, which is instrumental to the decoherence process.

A general solution $\varphi_<$ to equation (15.68) is made up of two parts: a part $\varphi_<^q$ which is a solution to the homogeneous equation (i.e. without the source term on the right-hand side) and a particular solution $\varphi_<^\xi$ with vanishing initial conditions. Namely, $\varphi_<(\vec{k},\eta) = \varphi_<^\xi(\vec{k},\eta) + \varphi_<^q(\vec{k},\eta)$. The first part is made up of "intrinsic fluctuations" which coincide with the quantum fluctuations of the free field. The second part is sometimes called "induced fluctuations" referring to the influences from the environment. Under some reasonable approximations the result is analogous to that for the linear quantum Brownian motion (QBM) [Zha90, HuPaZh92, HuPaZh93a, PaHaZu93, Paz94, HalYu96, KUMS97]. Correspondingly the quantity $\Delta_\varphi^2(k)$ has two contributions:

$$\Delta_\varphi^2(k) = \Delta_{\varphi^q}^2(k) + \Delta_{\varphi^\xi}^2(k) \tag{15.69}$$

Because in equation (15.68) the dissipation kernel is assumed to be small, the first part follows an almost unitary evolution of the initial density matrix, yielding the usual result for the case of the free field: $\Delta_{\varphi^q}^2(k) = (H/2\pi)^2\,(1 + k^2\eta^2)$. The second part is due to the ξ_2 noise source and can be expressed as

$$\begin{aligned}\Delta_{\varphi^\xi}^2(k) = -\lambda^2 \frac{144}{\pi^2} k^3 \int_{\eta_i}^{\eta} d\eta_1 \int_{\eta_i}^{\eta} d\eta_2\; a^4(\eta_1) a^4(\eta_2) \\ \times \phi_0(\eta_1)\phi_0(\eta_2) h(k,\eta,\eta_1) h(k,\eta,\eta_2) \\ \times Re G_F^{\Lambda 2}(\eta_1,\eta_2,\vec{k})\end{aligned} \tag{15.70}$$

where

$$h(k,\eta,\eta') \equiv \frac{1}{a(\eta)a(\eta')}\left[\frac{\sin[k(\eta-\eta')]}{k}\left(1+\frac{1}{k^2\eta\eta'}\right) - \frac{\cos[k(\eta-\eta')]}{k^2\eta\eta'}(\eta-\eta')\right] \tag{15.71}$$

On the other hand, the usual contribution $\Delta_{\varphi^q}^2(k)$ is independent of k for a fixed value of $k\eta$, corresponding to a nearly scale-invariant spectrum, whereas $\Delta_{\varphi^\xi}^2(k)$ depends on k and Λ.

Thus, concerning the influence of the environment on the power spectrum for some modes in the system, the results of Lombardo and Nacir [LomNac05] indicate that the contribution to the spectrum from the unitary evolution of the Bunch–Davies initial condition dominates over the contribution from the system–environment interaction.

15.2.4 Structures from interaction with other fields

In this last subsection we turn to structure formation from colored noise generated from coarse graining some other quantum field(s) the inflaton interacts with, using the two-field model discussed in Chapter 5. We report on the findings

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of Wu *et al.* [WuNgLeeLeeCha06], who show that the inflaton fluctuations driven by the colored noise are strongly dependent on the onset of inflation and become scale-invariant asymptotically at small scales. These induced fluctuations would grow with time only in a certain intrinsic time-scale. For this proposal to work, one needs to assume that the gravitational perturbations associated with the passive (or induced) quantum field fluctuations can become larger than the active (or intrinsic) fluctuations. Some mechanism should be present to suppress the active fluctuations for this assumption to be valid. Only in the (hitherto not easily explicable) case when the induced fluctuations contribute a significant portion to the density perturbation would they cause a suppression of the density power spectrum on large scales which shows up as a depression of low-l multipoles in CMB. Of special interest to colored-noise induced structure formation is that the observed low CMB quadrupole may open a window on the physics of the first few e-foldings of inflation.

Consider an inflaton field Φ with potential $V(\Phi)$ coupled to a massive scalar quantum field Ψ described by the Lagrangian

$$\mathcal{L} = \frac{-1}{2} g^{\mu\nu} \partial_\mu \Phi \, \partial_\nu \Phi + \frac{-1}{2} g^{\mu\nu} \partial_\mu \Psi \, \partial_\nu \Psi - V(\Phi) - \left(\frac{m_\Psi^2}{2} \Psi^2 + \frac{g^2}{2} \Phi^2 \Psi^2 \right) \tag{15.72}$$

where $V(\Phi)$ is the inflaton potential that complies with the slow-roll conditions and g is a coupling constant between Φ and Ψ. Thus, we can approximate the spacetime during inflation by a de Sitter metric given by equation (15.6). We can rescale a so that at the initial time of the inflation era, $\eta_i = -1/H$. In the influence functional approach [HuPaZh93b, CalHu94, CalHu95, CalGon97, KUMS97, Lee04, LomNac05], the environmental field Ψ is traced out up to the one-loop level. Assuming also that the quantum field has gone through the quantum-to-classical transition, the Langevin equation for Φ is given by:

$$\begin{aligned} \Phi'' + 2aH\Phi' - \nabla^2\Phi + a^2 \left[dV(\Phi)/d\Phi + g^2 \langle \Psi^2 \rangle \Phi \right] - g^4 a^2 \Phi \\ \times \int d^4x' a^4(\eta') \theta(\eta - \eta') \, i \, G_-(x, x') \Phi^2(x') = \frac{\Phi}{a^2} \xi \end{aligned} \tag{15.73}$$

where the prime denotes differentiation with respect to η. As we will see later, the quantum fluctuations of Φ will contribute to the mass correction of Ψ at one loop. The dissipation term in this Langevin equation is actually divergent. Wu *et al.* removed the divergence by using the regularization method that sets the ultraviolet cut-off $\Lambda = He^{Ht}$. They found that this term only contributes a mass correction of about $10^{-2} g^4 \bar{\phi}_0^2$ to $m_{\varphi\mathrm{eff}}^2$ (defined after equation (15.76)) as well as a small friction term of order $10^{-2} g^4 \bar{\phi}_0^2 a\dot{\phi}/H$ to equation (15.73). As we have seen before, the environment field Ψ engenders dissipative dynamics in the inflaton field Φ via the kernel G_- and produces a multiplicative colored noise ξ with correlator

$$\langle \xi(x) \xi(x') \rangle = g^4 a^4(\eta) a^4(\eta') G_+(x, x') \tag{15.74}$$

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The kernels $G_\pm$ in equations (15.73) and (15.74) can be constructed from the Green's function of Ψ with respect to a particular choice of the initial vacuum state to be specified. They were derived in Chapter 5:

$$G_\pm(x,x') = \langle \Psi(x)\Psi(x')\rangle^2 \pm \langle \Psi(x')\Psi(x)\rangle^2 \tag{15.75}$$

To focus on noise-generated structure, in the solution of equation (15.73), one can first ignore the dissipative term.

Following the stochastic inflation paradigm, after sufficient decoherence, we can decompose $\Phi(\eta,\vec{x}) = \bar{\phi}(\eta) + \varphi(\eta,\mathbf{x})$ into a mean field $\bar{\phi}$ and a fluctuation field φ which obeys the linearized Langevin equation

$$\varphi'' + 2aH\varphi' - \nabla^2\varphi + a^2 m^2_{\varphi\mathrm{eff}}\varphi = \bar{\phi}\xi/a^2 \tag{15.76}$$

where the effective mass is defined as $m^2_{\varphi\mathrm{eff}} = d^2V(\bar{\phi})/d\varphi^2 + g^2\langle\Psi^2\rangle$ and the time evolution of $\bar{\phi}$ is governed by $V(\bar{\phi})$. The equation of motion for Ψ from which we construct its Green's function can be read off from its quadratic terms in the Lagrangian (15.72) as

$$\Psi'' + 2aH\Psi' - \nabla^2\Psi + a^2 m^2_{\Psi\mathrm{eff}}\Psi = 0 \tag{15.77}$$

where $m^2_{\Psi\mathrm{eff}} = m^2_\Psi + g^2(\bar{\phi}^2 + \langle\varphi_q^2\rangle)$. Here $\langle\varphi_q^2\rangle$ denotes the active or intrinsic quantum fluctuations with a scale-invariant power spectrum given by $\Delta_k^q = H^2/(4\pi^2)$. Let us decompose

$$\Upsilon(x) = \int \frac{d\mathbf{k}}{(2\pi)^{\frac{3}{2}}} Y_{\mathbf{k}}(\eta)\, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{15.78}$$

where $\Upsilon = \varphi, \xi$, and correspondingly $Y = \varphi_{\mathbf{k}}, \xi_{\mathbf{k}}$

$$\Psi(x) = \int \frac{d\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[b_{\mathbf{k}}\psi_k(\eta)\, e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right] \tag{15.79}$$

where $b^\dagger_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are creation and annihilation operators satisfying $[b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k}-\mathbf{k}')$.

The solution to equation (15.76) is obtained as

$$\varphi_{\vec{k}} = -i\int_{\eta_i}^{\eta} d\eta'\,\bar{\phi}(\eta')\xi_{\vec{k}}(\eta')\left[\varphi_k^{(1)}(\eta')\varphi_k^{(2)}(\eta) - \varphi_k^{(2)}(\eta')\varphi_k^{(1)}(\eta)\right] \tag{15.80}$$

where the homogeneous solutions $\varphi_k^{(1),(2)}$ are given by

$$\varphi_k^{(1),(2)} = \frac{1}{2a}(\pi|\eta|)^{\frac{1}{2}} H_\nu^{(1),(2)}(k\eta) \tag{15.81}$$

Here $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of the first and second kinds respectively and $\nu^2 = 9/4 - m^2_{\varphi\mathrm{eff}}/H^2$. In addition, we have from equation (15.77) that

$$\psi_k = \frac{1}{2a}(\pi|\eta|)^{\frac{1}{2}}\left[c_1 H_\mu^{(1)}(k\eta) + c_2 H_\mu^{(2)}(k\eta)\right] \tag{15.82}$$

where the constants c_1 and c_2 are subject to the normalization condition, $|c_2|^2 - |c_1|^2 = 1$, and $\mu^2 = 9/4 - m^2_{\Psi\mathrm{eff}}/H^2$.

Low ℓ WMAP modes and running spectral index

From this we can calculate the power spectrum of the perturbation $\delta\varphi$. To maintain the slow-roll condition: $m^2_{\phi\mathrm{eff}} = m^2_{\varphi\mathrm{eff}} \ll H^2$ (i.e. $\nu = 3/2$), we require that $g^2 < 1$ and $m^2_\Psi > H^2$. The latter condition limits the growth of $\langle\Psi^2\rangle$ during inflation to be less than about $10^{-2}H^2$ [BunDav78, VilFor82, EnNgOl88]. Under this condition, $\langle\varphi_q^2\rangle$ grows linearly as $H^3t/4\pi^2$ [BunDav78, VilFor82, EnNgOl88] and thus $\langle\varphi_q^2\rangle \simeq H^2$ after about 60 e-foldings (i.e. $Ht \simeq 60$). Therefore, as long as $g^2\bar{\phi}^2 \le 2H^2$, one can conveniently choose $m^2_{\Psi\mathrm{eff}} = 2H^2$ (i.e. $\mu = 1/2$) for which Ψ takes a very simple form. Also, it was shown that when $\mu = 1/2$ one can select the Bunch–Davies vacuum (i.e. $c_2 = 1$ and $c_1 = 0$) [EnNgOl88]. Hence, using equations (15.74) and (15.80), one obtains

$$\langle\varphi_{\vec{k}}(\eta)\varphi^*_{\vec{k}'}(\eta)\rangle = \frac{2\pi^2}{k^3}\Delta^\xi_k(\eta)\delta(\vec{k} - \vec{k}') \tag{15.83}$$

where the noise-driven power spectrum is given by

$$\begin{aligned}\Delta^\xi_k(\eta) = \frac{g^4y^2}{8\pi^4}\int_{y_i}^y dy_1\int_{y_i}^y dy_2\bar{\phi}(\eta_1)\bar{\phi}(\eta_2)\frac{\sin y_-}{y_1y_2y_-}\\ \times\left[\sin(2\Lambda y_-/k)/y_- - 1\right]F(y_1)F(y_2)\end{aligned} \tag{15.84}$$

where $y_- = y_2 - y_1$, $y = k\eta$, $y_i = k\eta_i = -k/H$, Λ is the momentum cut-off introduced in the evaluation of the ultraviolet divergent Green's function in equation (15.75), and

$$F(x) = \left(1 + \frac{1}{xy}\right)\sin(x-y) + \left(\frac{1}{x} - \frac{1}{y}\right)\cos(x-y) \tag{15.85}$$

Note that the term $\sin(2\Lambda y_-/k)/y_- \simeq \pi\delta(y_-)$ when $\Lambda \gg k$, so $\Delta^\xi_k(\eta)$ is insensitive to Λ. Both $\bar{\phi}(\eta_1)$ and $\bar{\phi}(\eta_2)$ in equation (15.84) can be approximated as a constant mean field $\bar{\phi}_0$, since we are concerned with large scales at which the rate of change of the mean field at horizon crossing, $d\bar{\phi}/d\ln k \simeq -\sqrt{-2\epsilon\dot{H}}M_{Pl}/H$, where $\epsilon \equiv -\dot{H}/H^2$ is the slow-roll parameter, is consistent with zero up to the scale near the first CMB Doppler peak in WMAP measurements [Spe03]. A plot by Wu *et al.* of $\Delta^\xi_k(\eta)$ at the horizon-crossing time (defined by $y = -2\pi$) versus k/H shows that the noise-driven fluctuations depend on the onset time of inflation and approach asymptotically to a scale-invariant power spectrum $\Delta^\xi_k \simeq 0.2g^4\bar{\phi}_0^2/(4\pi^2)$ at large k. Within the usual models of inflation, the possible interactions of the inflaton are too restricted for this effect to be observable; however, the fact that interactions do affect the spectrum of primordial fluctuations has some interest on its own.

On the other hand, if the effect of interactions is expected to be important, then a nonperturbative evaluation of the influence functional becomes necessary. We describe below a possible strategy [ZanCal07a].

15.2.5 Primordial spectrum from nonequilibrium renormalization group

The basic idea of RG for systems in equilibrium (where time does not enter in the description) is the coarse graining of the original system, i.e. the change in the resolution with which the system is observed [WilKog74]. Given a system with a range of scales which goes up to wavenumber Λ, if we are only interested in scales up to wavenumber $k < \Lambda$, we can separate the original system in two sectors: a lower wavenumber (soft) sector, with $k' < k$, the relevant system, and a higher wavenumber (hard) sector with $k < k' < \Lambda$, the environment. Once this division is done, the environment modes are eliminated from the description. In equilibrium, this is achieved by computing the coarse-grained "in-out" effective action for the lower sector, complemented with a rescaling of the fields and momenta that restores the cut-off and the coefficient of the q^2 term in the action to their initial values. The elimination of the modes between Λ and k proceeds by infinitesimal steps. In this way, the calculation involves only tree and one-loop diagrams, and the resulting equations form a set of differential equations for the parameters that define the effective action [WegHou73].

Essentially, the same scheme can be used for nonequilibrium systems. We want to compute true expectation values at given times, not transition amplitudes between "in" and "out" asymptotic states, far away in the future and in the past. We want to follow the real and causal evolution of expectation values, for which the usual "in-out" representation is not appropriate. A suitable description of nonequilibrium systems is given within the "closed time path" (CTP) formalism.

It is important to stress two basic differences between the nonequilibrium and equilibrium RG [Lit98]. The IF may be regarded as an action for a theory defined on a "closed time path" (CTP) composed of a first branch (going from the initial time $t = 0$ to a later time $t = T$ when the relevant observations will be performed – that is why we need the density matrix at T) and a second branch returning from T to 0. Thus each physical degree of freedom on the first branch acquires a twin on the second branch – we say the number of degrees of freedom is doubled. The IF is not just a combination of the usual actions for each branch, but also admits direct couplings across the branches. The damping constant κ and the noise constant ν are associated to two of those "mixed" terms. Therefore, the structure of the IF (from now on, CTP action, to emphasize this feature) is much more complex than the usual Euclidean or "in-out" action.

The second fundamental difference is the presence of the parameter T itself. In nonequilibrium evolution, it is important to specify the time-scale over which we shall observe the system. The CTP action contains this physical time-scale T. From the point of view of the RG, this adds one more dimensional parameter to the theory, much as an external field in the Ising model. Physically, because time integrations are restricted to the interval $[0, T]$, energy conservation does not hold at each vertex. This is of paramount importance regarding damping.

The RG for the CTP effective action (obtained by taking the limit $T \to \infty$) was studied by Dalvit and Mazzitelli [DalMaz96, Dal98]; see also [CaHuMa01] and [Pol06, ZanCal07a, ZanCal06b].

In formulating a nonequilibrium RG, we must deal with the fact that the CTP action may have an arbitrary functional dependence on the fields and be nonlocal both in time and space. In principle, one can define an exact RG transformation [DalMaz96], where all three functional dependencies are left open. However, the resulting formalism is too complex to be of practical use. Fortunately, the special properties of the application to thermalization allow for substantial simplifications.

The full RG equations for this theory is given in [ZanCal06b]. Here we shall only highlight those aspects of the calculation most relevant to the application to primordial fluctuation generation.

We shall work with the conformally scaled field $\chi = a\Phi$. For simplicity, we shall treat χ as a field on flat spacetime. This only induces an error of order 1 in the amplitude of the fluctuations at horizon exit.

Let us call $\chi^{1(2)}$ the field variable in the first (resp. second) branch of the CTP. To write down the CTP action, it is best to introduce average and difference variables

$$\begin{aligned} \chi_- &= \chi^1 - \chi^2 \\ \chi_+ &= \left(\chi^1 + \chi^2\right)/2 \end{aligned} \tag{15.86}$$

In terms of these variables, a generic CTP action may be written as

$$S_{\mathrm{CTP}} = S_0 + S_\lambda + S_{\mathrm{other}} \tag{15.87}$$

where S_0 is the CTP action functional for a free massless field theory, S_λ accounts for a $\lambda\chi^4$-type self-interaction and S_{other} includes all other possible terms. Momentum integrals are bounded by $k = \Lambda$. We shall assume that the initial condition for the RG flow is $S_{\mathrm{other}} = 0$ at the hard scale Λ, so that if it appears at soft scales, it is as a consequence of the RG running itself. Note that this is true, in particular, for the noise and dissipation terms.

To define the nonequilibrium RG we also need to specify the state of the field at the initial time $t = 0$. For simplicity, we shall assume this is the vacuum state for the free action S_0. Observe that this is a nonequilibrium state for the interacting theory.

The value λ_0 of the coupling constant λ at the hard scale Λ may be used as the small parameter in a perturbative expansion of the RG equation. To order λ_0^2, the RG equation for the quartic coupling decouples, and can be solved by itself. The result is that at soft scales k, λ is both scale and T dependent. There is no RG running if $T = 0$, while the usual textbook result is obtained as $T \to \infty$. For all values of T, λ is driven to zero as $k \to 0$ [ZanCal06b]. Thus it is consistent to assume that λ is uniformly small in the relevant scale range.

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In particular, in order to compute the RG equations to order λ^2, it will be enough to use in the Feynman graphs the zeroth order propagators, which are those of the massless free theory. The only exception is in computing the effective mass, but this calculation is decoupled from the noise and dissipation terms to order λ^2. Observe that it is at the same time a huge simplification and a strong limitation concerning the range of application of our results, as we expect substantial shifts in the propagators when T approaches the relaxation time of the theory.

Because of the nonzero initial value of λ, other couplings will appear as a result of the RG running. To order λ^2, it is enough to consider quadratic, quartic and six-point terms in the action. All these terms feed back into each other, so they must be taken self-consistently. To compute the amplitude of the fluctuations at horizon exit, however, it is enough to focus on the quadratic terms,

$$S_{\text{other}} \to S_2\left[\chi_-, \chi_+\right] = \int_0^T dt_1 \int_0^T dt_2 \int d^d\mathbf{k}\ \left[v_{21}(k;t_1,t_2)\,\chi_-(\mathbf{k},t_1)\,\chi_+(-\mathbf{k},t_2)\right. \\ \left. +\, i\, v_{22}(k;t_1,t_2)\,\chi_-(\mathbf{k},t_1)\,\chi_-(-\mathbf{k},t_2)\right] \tag{15.88}$$

In principle, the induced quadratic terms will be oscillatory functions of $\Lambda t_{1,2}$. However, by the time a mode reaches the horizon it becomes insensitive to high frequencies. To focus on the slow dynamics, we may project out the mass, dissipation and noise terms on which the oscillations are mounted.

To this end, we introduce two projectors. Given a function of two times $v(k;t_1,t_2)$, we define

$$\mathbf{P}v(k;t_1,t_2) = \mathcal{P}v(k)\ \delta(t_1-t_2) \tag{15.89}$$

and, if $v(k;t_1,t_2) = 0$ for $t_2 > t_1$,

$$\mathbf{Q}v(k;t_1,t_2) = \mathcal{Q}v(k)\ \left[2\left(\frac{\partial}{\partial t_2} + \delta(t_2) - \delta(0)\right)\delta(t_1-t_2)\right] \tag{15.90}$$

where

$$\mathcal{P}v(k) = \frac{1}{T}\int_0^T dt_1 \int_0^T dt_2\ v(k;t_1,t_2) \tag{15.91}$$

and

$$\mathcal{Q}v(k) = \frac{1}{T}\int_0^T dt_1 \int_0^T dt_2\ v(k;t_1,t_2)\ (t_2-t_1) \tag{15.92}$$

It is easy to verify that $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{Q}^2 = \mathbf{Q}$, and that $\mathbf{QP} = \mathbf{PQ} = 0$. This proves that the decomposition

$$v(k;t_1,t_2) = \mathbf{P}v(k;t_1,t_2) + \mathbf{Q}v(k;t_1,t_2) + \Delta v(k;t_1,t_2) \tag{15.93}$$

is unique. Defining

$$v_0 = \mathcal{P}v_{21} \tag{15.94}$$

and

$$v_1 = \mathcal{Q} v_{21} \tag{15.95}$$

we extract from $v_{21}(k; t_1, t_2)$ two quantities: $-v_0(k)$, which acts as a momentum-dependent mass squared term, and $-v_1(k)/2$, which is equivalent to a damping constant.

If we further expand in powers of wavenumber k

$$v_0(k) = v_0(0) + k\frac{\partial v_0(0)}{\partial k} + \frac{k^2}{2!}\frac{\partial^2 v_0(0)}{\partial k^2} + \dots \tag{15.96}$$

the linear term vanishes from symmetry, and the appearance of the quadratic term is prevented by performing a field rescaling as part of the RG transformation (thus the field acquires an anomalous dimension). The net effect is then to induce a mass term

$$m^2 = -v_0(0) \tag{15.97}$$

and a damping constant

$$\kappa = -v_1(0)/2 \tag{15.98}$$

The noise kernel is obtained in a similar way from the imaginary part of the CTP action, v_{22}.

After these considerations, the relevant CTP action for long-wavelength, slowly varying configurations reduces to

$$\begin{aligned} S_{\mathrm{CTP}}\left[\chi_-, \chi_+\right] = \int_0^T dt \int d^d\mathbf{k}\ \Big[&\dot{\chi}_-(\mathbf{k}, t)\ \dot{\chi}_+(-\mathbf{k}, t) - \chi_-(\mathbf{k}, t)\left(k^2 + m^2\right)\chi_+(-\mathbf{k}, t) \\ &- 2\kappa\ \chi_-(\mathbf{k}, t)\dot{\chi}_+(-\mathbf{k}, t) + \frac{i}{2}\nu\ \chi_-(\mathbf{k}, t)\chi_-(-\mathbf{k}, t)\Big] \end{aligned} \tag{15.99}$$

The flow of the RG drives the initial interacting theory towards the free theory (15.99), and allows us to find a relation between expectation values associated with each theory. The relation is

$$G\left(k, t, \mu(\Lambda, T)\right) = \left(\Lambda/k\right)^{\alpha(k,T)}\ G\left(\Lambda,\ \left(\Lambda/k\right)^{\beta(k,T)}\ t, \mu(k, T)\right) \tag{15.100}$$

On the left-hand side, G is the two-field expectation value computed for a mode k at time t, and $\mu(\Lambda, T)$ stands for the set of parameters which define the action at scale Λ. In our case the only parameter is the coupling constant λ. On the right-hand side, G is the expectation value of the theory defined by the set of parameters $\mu(k, T)$, reached after modes between k and Λ have been eliminated. The relevant parameters in $\mu(k, T)$ are $m^2(k, T)$, $\kappa(k, T)$, and $\nu(k, T)$. Finally, the exponents α and β depend on the trajectory followed by the action when it goes from scale Λ to k.

Now we connect to the original problem for the power spectrum of an interacting inflaton field. We must feed the RG group equations with an initial condition

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at scale Λ and then use the relation (15.100) to obtain the expectation value for the mode k as it exits the horizon. The initial condition, in terms of the conformal field, is given by the CTP action at scale Λ, where t has to be replaced by the conformal time η. The mode k exits the horizon when

$$\eta = -k^{-1} \tag{15.101}$$

If inflation starts at η^*, the time that the mode k spends inside the horizon is given by

$$\tau_k = -k^{-1} - \eta^* \tag{15.102}$$

For the physical field (subscript HE stands for horizon exit)

$$\langle \Phi(k,t)\Phi(k,t)\rangle_{\mathrm{HE}} = k^{-2}\, G\left(k, \tau_k, \lambda\right) \tag{15.103}$$

From equation (15.100), identifying t and T with τ_k, we get

$$\begin{aligned}\langle \Phi(k,t)\Phi(k,t)\rangle_{\mathrm{HE}} = {} & k^{-2}\left(\Lambda/k\right)^{\alpha(k,\tau_k)} \\ & G\left(\Lambda,\, \left(\Lambda/k\right)^{\beta(k,\tau_k)}\, \tau_k, m^2(k,\tau_k), \kappa(k,\tau_k), \nu(k,\tau_k)\right)\end{aligned} \tag{15.104}$$

Here, the relevant elements of $\mu(k, \tau_k)$ have been shown explicitly. The right-hand side of equation (15.104) can be calculated using the G corresponding to the action (15.99)

$$G\left(k, t, m^2, \kappa, \nu\right) = \left(\frac{2}{k} - \frac{\nu}{\kappa\omega_0^2}\right)\left[\frac{\omega_0^2}{\omega^2} - \frac{\kappa^2}{\omega^2}\cos(2\omega t) + \frac{\kappa}{\omega}\sin(2\omega t)\right] e^{-2\kappa t} + \frac{\nu}{\kappa\omega_0^2} \tag{15.105}$$

where $\omega_0^2 = m^2 + k^2$ and $\omega^2 = \omega_0^2 - \kappa^2$ [ZanCal06b].

The expressions for m^2, κ, and ν, and for the exponents α and β, as functions of k and τ_k, are given in [ZanCal06b]. The main effects are introduced by the mass term.

15.3 Reheating in the inflationary universe

We focus here on the so-called reheating regime when the universe began to warm up due to particle creation from excitations of the vacuum fluctuations of the inflaton field and other fields coupled to it. The back-reaction of created particles results in the decay of the inflaton mean field and the turnover of the universe from the inflationary state described by an approximate de Sitter solution to a radiation-dominated FLRW solution depicted in the so-called standard model.

As stated before, the inflationary scenarios can generically be divided into three eras: (1) entrance into a vacuum energy density dominated era, which can be a metastable state of the Higgs field in a GUT era, where the universe begins inflation; (2) a "slow roll" of the inflation field ϕ either from a relatively flat

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effective potential $V(\phi)$, or from a simple ϕ^2 potential, as in the new or chaotic inflationary cosmology; (3) exiting the inflationary era and entering into an era when the inflaton field undergoes rapid oscillations, where the vacuum energy density is transformed into radiation via particle creation and the universe begins to reheat to a radiation-dominated state.

One can also divide the reheating era roughly into two or even three stages, preheating, heating and thermalization. In the preheating stage the dominant effect is due to parametric particle creation [KoLiSt97]. Brandenberger, Traschen and Shtanov [ShTrBr95], Kofman, Linde and Starobinsky [KoLiSt94] and Boyanovsky *et al.* [BVHS96] first pointed out the importance of parametric resonance at work in this stage. We have explained this mechanism of a rather general nature in Chapter 4, e.g. the narrow and broad resonances. The thermalization process is a difficult and complex one. We discussed some aspects of it in Chapter 12, but the reader should consult representative papers (e.g. [BoVeSa05, Lin90, Muk05, BaTsWa06]) for a better understanding of the specific context of thermalization in post-inflationary reheating.

As explained earlier, since the purpose of these latter chapters is to illustrate how the methods in NEqQFT we have learned can be applied to treat relevant problems in different contexts, the discussions here on reheating are not meant to be of a review nature, where ideally all ideas and approaches ought to be represented. We refer the readers to reviews [BaTsWa06] for a more balanced overall perspective of the physical processes involved. For our more restricted aim here, we shall only describe two examples where we have some first-hand experience in which the full use of the methods of nonequilibrium quantum field theory plays an essential role. These examples concern the back-reaction of the created particles on the inflaton field during preheating, and the generation of primordial magnetic fields as a side-effect of reheating.

15.3.1 Case study I: Back-reaction of Fermi fields during preheating

The earliest analysis of the reheating stage assumed that the decay of the inflaton field could be described perturbatively, by computing the absorption parts of suitable Feynman graphs. That led to an apparent contradiction between the theories of reheating and structure formation, since the latter places very stringent limits on the possible couplings of the inflaton. Moreover, the generation of heavy particles was strongly suppressed, against the expectation that heavy bosons generated during reheating could play a role in baryogenesis.

This seeming difficulty was overcome when it was realized that the decay of the inflaton could proceed very efficiently through the parametric amplification of matter fields, which is an essentially nonperturbative process. As a matter of fact, in these new scenarios enough reheating is obtained even if the inflaton is not coupled to any other field at all, other than the gravitational field.

While the basic mechanism of parametric amplification during reheating are the broad and narrow resonances, they are also strongly affected by the expansion of the universe [RamHu97b]. As we have seen in Chapter 4, the evolution of the field under broad resonance may be described as a series of adiabatic evolutions punctuated by nonadiabatic transitions. The growth factor from one transition to the next depends on the accumulated phase of the field variable. The dynamic geometrical background induces changes in this phase, because the relevant parameters become time dependent, and thus affects the nature of resonance. The general case becomes a sequence of different resonance regimes due to the process of parametric resonance [KoLiSt97, GKLS97, ChNuMi05]. The nonlinearity of the inflaton oscillations also plays an important role. In the general case, the oscillating inflaton field will have a full frequency spectrum, not limited to a few narrow bands, and the amplification of the matter fields may be studied by the methods of particle production from a time-dependent background, which we have discussed in Chapters 4 and 8 [Bas98, ZMCB98, ZMCB99].

Eventually, all relevant matter field modes acquire high occupation numbers and a classical treatment becomes possible [CalGra02]. This observation has been key to progress in the analysis of the fully nonlinear regime, including inflaton fragmentation and the so-called turbulent reheating stage [KhlTka96, KoLiSt97, FelTka00, FeKoLi01, FelKof01, FGGKLT01, MicTka04, PFKP06]. We have discussed similar processes (albeit on nonexpanding spacetimes) in Chapter 12.

Nevertheless, the back-reaction of the created particles has a strong effect on the inflaton even before the classical approximation becomes reliable. The inflaton must be seen as an effectively open system – with all other matter fields providing an environment – and its dynamics is subject to dissipation and noise therefrom [Hu91, SinHu91, LomMaz96, DalMaz96, GreMul97].

It was earlier realized that a description of the inflaton dynamics based on the 1PI effective action (cf. Chapter 6) or similar constructs with a c-number inflaton field as the sole argument is not satisfactory. For one, inflaton fluctuations play a key role in the theory of structure formation and one should to follow their evolution through the reheating stage. Most importantly, the equations of motion as derived from the 1PI effective action are affected by secular terms and become unreliable after several inflaton oscillations. We have found a similar problem in the treatment of Bose–Einstein condensates in Chapter 13.

Although it is possible to extract useful information from these secular terms through dynamical renormalization group analysis, it is best to improve the model, by including the physical processes that cut off the growth of secular terms. The most efficient way of accomplishing this is by going over to a 2PI description (cf. Chapter 6), where the inflaton mean field and fluctuations are treated self-consistently. The 2PI effective action implements the resummation of secular terms, and also incorporates the basic processes that eventually could lead to thermalization, as we have discussed in Chapter 12.

As a concrete example of the application of 2PIEA techniques to the description of preheating, we shall analyze the evolution of the inflaton field coupled to N Fermi fields. Our treatment here follows [RaStHu98]. While Fermi fields are subject to Pauli blocking which, unlike the stimulated emission of Bose fields, opposes particle creation (cf. Chapter 4), the fact that most matter fields in the standard model are fermionic makes them a proper subject of study [KoLiSt97].

We consider a model of a scalar inflaton field Φ (with $\lambda\Phi^4$ self-coupling) interacting with a spinor field via Yukawa coupling. The system consists of the inflaton mean field and variance, and the environment consists of the spinor field(s) Ψ. We construct the CTP-2PI-CGEA, and derive from it the effective dynamical equations for the inflaton field, taking into account its effect on the environment, and back-reaction therefrom in a self-consistent manner.

This problem is a good example of how to apply many of the concepts and techniques presented in earlier chapers. The first step is to derive a set of coupled nonperturbative equations for the inflaton mean field and variance at two loops. Only beginning at two loops will both the inflaton mean field *and the inflaton variance* couple to the spinor degrees of freedom. They are damped by back-reaction from fermion particle production. (Calculations using the 1PI effective action will miss this important effect.) The equations of motion are real and causal, and the gap equation for the two-point function is dissipative due to fermion particle production.

As we emphasized in Chapter 9, there is a subtle yet important distinction between the system–environment division in nonequilibrium statistical mechanics and the system–bath division assumed in thermal field theory. In the latter, one assumes that the propagators for the bath degrees of freedom are *fixed,* finite-temperature equilibrium Green functions, whereas in the case of the CTP-CGEA, the environmental propagators are *slaved* (in the sense of [CalHu95a]) to the dynamics of the system degrees of freedom, and are not fixed a priori to be equilibrium Green functions for all time. This distinction is important for discussions of fermion particle production during reheating, because it is only when the inflaton mean field amplitude is small enough for the use of perturbation theory, that the system–bath split implicit in thermal field theory can be used. Otherwise, one must take into account the effect of the inflaton mean field on the bath (spinor) *propagators.*

As we saw in Chapter 6, the use of the closed time path (CTP) formalism allows formulation of the nonequilibrium dynamics of the inflaton from an appropriately defined initial quantum state. At the onset of the reheating period, the inflaton field's zero mode typically has a large expectation value, whereas all other fields coupled to the inflaton, as well as inflaton modes with momenta greater than the Hubble constant, are to a good approximation in a vacuum state [Bra85].

The model consists of a scalar field Φ (the inflaton field) which is Yukawa-coupled to a spinor field Ψ, in a curved, dynamical, classical background spacetime. The total action

$$S[\Phi, \bar{\Psi}, \Psi, g^{\mu\nu}] = S^{\mathrm{G}}[g^{\mu\nu}] + S^{\mathrm{F}}[\Phi, \bar{\Psi}, \Psi, g^{\mu\nu}] \tag{15.106}$$

consists of a part depicting classical gravity, $S^{\mathrm{G}}[g^{\mu\nu}]$, and a part for the matter fields,

$$S^{\mathrm{F}}[\Phi, \bar{\Psi}, \Psi, g^{\mu\nu}] = S^{\Phi}[\Phi, g^{\mu\nu}] + S^{\Psi}[\bar{\Psi}, \Psi, g^{\mu\nu}] + S^{\mathrm{Y}}[\Phi, \bar{\Psi}, \Psi, g^{\mu\nu}] \tag{15.107}$$

whose scalar (inflaton), spinor (fermion), and Yukawa interaction parts are given by

$$S^{\Phi}[\Phi, g^{\mu\nu}] = -\frac{1}{2}\int d^{\,4}x\sqrt{-g}\left[\Phi(\nabla^2 + m^2 + \xi R)\Phi + \frac{\lambda}{12}\Phi^4\right] \tag{15.108}$$

$$S^{\Psi}[\bar{\Psi}, \Psi, g^{\mu\nu}] = \int d^{\,4}x\sqrt{-g}\left[\frac{i}{2}\left(\bar{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - (\nabla_{\mu}\bar{\Psi})\gamma^{\mu}\Psi\right) - \mu\bar{\Psi}\Psi\right] \tag{15.109}$$

$$S^{\mathrm{Y}}[\Phi, \bar{\Psi}, \Psi, g^{\mu\nu}] = -f\int d^{\,4}x\sqrt{-g}\Phi\bar{\Psi}\Psi \tag{15.110}$$

For this theory to be renormalizable in semiclassical gravity, the bare gravity action $S^{\mathrm{G}}[g^{\mu\nu}]$ of equation (15.106) should have the form [DeW75, BirDav82]

$$S^{\mathrm{G}}[g^{\mu\nu}] = \frac{1}{16\pi G}\int d^{\,4}x\sqrt{-g}\left[R - 2\Lambda_{\mathrm{c}} + cR^2 + bR^{\alpha\beta}R_{\alpha\beta} + aR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\right] \tag{15.111}$$

In equations (15.108)–(15.110), m is the scalar field "mass" (with dimensions of inverse length); ξ is the dimensionless coupling to gravity; μ is the spinor field "mass," with dimensions of inverse length; ∇^2 is the Laplace–Beltrami operator in the curved background spacetime with metric tensor $g_{\mu\nu}$; ∇_{μ} is the covariant derivative compatible with the metric; $\sqrt{-g}$ is the square root of the absolute value of the determinant of the metric; λ is the self-coupling of the inflaton field, with dimensions of $1/\sqrt{\hbar}$; and f is the Yukawa coupling constant, which has dimensions of $1/\sqrt{\hbar}$. In equation (15.111), G is Newton's constant (with dimensions of length divided by mass); R is the scalar curvature; $R_{\mu\nu}$ is the Ricci tensor; $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor; a, b, and c are constants with dimensions of length squared; and Λ_{c} is the cosmological constant, which has dimensions of inverse length squared. The curved spacetime Dirac matrices γ^{μ} satisfy the anticommutation relation

$$\{\gamma^{\mu}, \gamma^{\nu}\}_{+} = 2g^{\mu\nu}1_{\mathrm{sp}}, \tag{15.112}$$

in terms of the contravariant metric tensor $g^{\mu\nu}$. The symbol 1_{sp} denotes the identity element in the Dirac algebra.

In four spacetime dimensions the terms with constants a, b, and c are related by a generalized Gauss–Bonnet theorem [Che62], so we have the freedom to

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choose $a = 0$. It is assumed that there is a definite separation of time-scales between the stage of "preheating" (see, e.g. [RamHu97b]), and fermionic particle production. In addition, the fermion field mass μ is assumed to be much lighter than the inflaton field mass m, i.e. the renormalized parameters m and μ satisfy $m \gg \mu$.

We denote the quantum Heisenberg field operators of the scalar field Φ and the spinor field Ψ by $\Phi_{\rm H}$ and $\Psi_{\rm H}$, respectively, and the quantum state by $|s\rangle$. For consistency with the truncation of the correlation hierarchy at second order, we assume $\Phi_{\rm H}$ to have a Gaussian moment expansion in the position basis [MazPaz89], in which case the relevant observables are the scalar mean field

$$\bar{\phi}(x) \equiv \langle s|\Phi_{\rm H}(x)|s\rangle \tag{15.113}$$

and the mean-squared fluctuations, or variance, of the scalar field

$$\langle s|\Phi^2_{\rm H}(x)|s\rangle - \langle s|\Phi_{\rm H}(x)|s\rangle^2 \equiv \langle s|{\varphi_{\rm H}}^2(x)|s\rangle \tag{15.114}$$

where the last equality follows from the definition of the scalar *fluctuation field*

$$\varphi_{\rm H}(x) \equiv \Phi_{\rm H}(x) - \bar{\phi}(x) \tag{15.115}$$

As discussed above, at the end of the preheating period, the inflaton variance can be as large as the square of the amplitude of mean-field oscillations. On the basis of our assumption of separation of time-scales and the conditions which prevail at the onset of reheating, the initial quantum state $|s\rangle$ is assumed to be an appropriately defined vacuum state for the *spinor* field.

The construction of the CTP-2PI-CGEA for the $\Phi\bar{\Psi}\Psi$ theory in a general, curved, background spacetime closely parallels the construction of the CTP-2PI effective action for the O(N) model discussed in Chapter 6 [LomMaz98]. Within the spacetime manifold (whose dynamics must be determined self-consistently through the semiclassical gravitational field equation), let M be defined as the past domain of dependence of a Cauchy hypersurface $\Sigma_\star$, where $\Sigma_\star$ has been chosen to be far to the future of any dynamics we wish to study. We now define a "CTP" manifold $\mathcal{M}$ as the union of the two copies of M corresponding to the $\{+,-\}$ time branches, with their last Cauchy hypersurfaces $\Sigma_\star$ identified. As in Chapter 6, we define an action functional on the closed time path manifold as the difference of the actions evaluated on each branch. For a function Φ on $\mathcal{M}$, the restrictions of Φ to the $+$ and $-$ time branches are subject to the boundary condition $(\Phi_+)_{|\Sigma_\star} = (\Phi_-)_{|\Sigma_\star}$ at the hypersurface $\Sigma_\star$.

Following the general procedure in Chapter 6, we obtain the CTP-2PI-CGEA

$$\begin{aligned}\Gamma[\bar{\phi}, G, g^{\mu\nu}] = \mathcal{S}^{\Phi}[\bar{\phi}] - \frac{i\hbar}{2}\ln \det G_{ab} - i\hbar \ln \det F_{ab} + \Gamma_2[\bar{\phi}, G] \\ + \frac{i\hbar}{2}\int_M d^{\,4}x\sqrt{-g}\int_M d^{\,4}x'\sqrt{-g'}\mathcal{A}^{ab}(x',x)G_{ab}(x,x')\end{aligned} \tag{15.116}$$

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where $\mathcal{A}^{ab}$ is the second functional derivative of the scalar part of the classical action S^{Φ}, evaluated at $\bar{\phi}$,

$$i\mathcal{A}^{ab}(x,x') = \frac{1}{\sqrt{-g}}\left(\frac{\delta^2 \mathcal{S}^{\Phi}}{\delta\Phi_a(x)\delta\Phi_b(x')}[\bar{\phi}]\right)\frac{1}{\sqrt{-g'}}$$
$$= -\left[c^{ab}(\nabla_x^2 + m^2 + \xi R(x)) + c^{abcd}\frac{\lambda}{2}\bar{\phi}_c(x)\bar{\phi}_d(x)\right]\frac{\delta(x-x')}{\sqrt{-g'}} \tag{15.117}$$

The symbol F_{ab} denotes the one-loop CTP spinor propagator, which is defined by

$$F_{ab}(x,x') \equiv \mathcal{B}^{-1}_{ab}(x,x') \tag{15.118}$$

where we are suppressing spinor indices, and the inverse spinor propagator $\mathcal{B}^{ab}$ is defined by

$$i\mathcal{B}^{ab}(x,x') = \frac{1}{\sqrt{-g}}\left[\frac{\delta^2(\mathcal{S}^{\Psi}[\bar{\Psi},\Psi] + \mathcal{S}^{\Upsilon}[\bar{\Psi},\Psi;\bar{\phi}])}{\delta\Psi_a(x)\delta\bar{\Psi}_b(x')}\right]\frac{1}{\sqrt{-g'}}$$
$$= \left(c^{ab}(i\gamma^{\mu}\nabla'_{\mu} - \mu) - c^{abc}f\bar{\phi}_c(x')\right)\frac{\delta(x'-x)}{\sqrt{-g}}1_{\text{sp}} \tag{15.119}$$

It is clear from equation (15.119) that the use of the one-loop spinor propagators in the construction of the CTP-2PI-CGEA represents a nonperturbative resummation in the Yukawa coupling constant, which (as discussed above) goes beyond the standard time-dependent perturbation theory. The boundary conditions which define the inverses of equations (15.117) and (15.119) are the boundary conditions at the initial data surface in the functional integral which in turn define the initial quantum state $|s\rangle$. The one-loop spinor propagators F_{ab}, introduced in Chapter 10, are related to the expectation values of the spinor Heisenberg field operators in the presence of the c-number background field $\bar{\phi}$.

Only diagrams which are two-particle irreducible with respect to cuts of *scalar* lines contribute to Γ_2. The distinction between the CTP-2PI, *coarse-grained* effective action which arises here, and the fully two-particle irreducible effective action (2PI with respect to scalar *and* spinor cuts) is due to the fact that we only Legendre-transformed sources coupled to Φ; i.e. the spinor field is treated as the environment.

We evaluate the functional $\Gamma_2[\bar{\phi}, G, g^{\mu\nu}]$ in a loop expansion, starting with the two-loop term, $\Gamma^{(2)}$. The $\lambda\Phi^4$ self-interaction leads to two terms in the two-loop part of the effective action. They are the "setting sun" diagram, which is $O(\lambda^2)$, and the "double bubble," which is $O(\lambda)$, respectively (cf. Chapter 6). The Yukawa interaction leads to only one diagram in $\Gamma^{(2)}$

$$\frac{if^2}{2}c^{aa'a''}c^{bb'b''}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}G_{ab}(x,x')\text{tr}_{\text{sp}}\left[F_{a'b'}(x,x')F_{b''a''}(x',x)\right] \tag{15.120}$$

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where the trace is understood to be over the spinor indices which are not shown, and the three-index symbol c^{abc} is defined as in Chapter 6.

We treat the λ self-interaction using the time-dependent Hartree–Fock approximation [CoJaTo74], which is equivalent to retaining the $O(\lambda)$ (double bubble) graph and dropping the $O(\lambda^2)$ (setting sun) graph. We assume for the present study that the coupling λ is sufficiently small that the $O(\lambda^2)$ diagram is unimportant on the time-scales of interest in the fermion production regime of the inflaton dynamics. The mean-field and gap equations including both the $O(\lambda)$ and the $O(\lambda^2)$ diagrams have been derived in a general curved spacetime in [RamHu97a].

The (bare) semiclassical field equations for the two-point function, mean field, and metric can be obtained from the CTP-2PI-CGEA by functional differentiation with respect to G_{ab}, $\bar{\phi}_a$, and $g^{\mu\nu}$, followed by identifications of $\bar{\phi}$ and $g^{\mu\nu}$ on the two time branches [RamHu97b]. The field equation of semiclassical gravity (with bare parameters) is

$$G_{\mu\nu} + \Lambda_c g_{\mu\nu} + c\,{}^{(1)}H_{\mu\nu} + b\,{}^{(2)}H_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle \tag{15.121}$$

where ${}^{(1,2)}H_{\mu\nu}$ are tensors constructed from the covariant derivatives of the metric and connection forms (e.g. defined in [BirDav82]). The (unrenormalized) quantum energy–momentum tensor is defined by

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\,\frac{\delta\Gamma[\bar{\phi}, G, g_{\mu\nu}]}{\delta g_{\mu\nu +}}\Bigg|_{\bar{\phi}_+=\bar{\phi}_-=\bar{\phi}\quad g^{\mu\nu}_+=g^{\mu\nu}_-=g^{\mu\nu}} \tag{15.122}$$

The energy–momentum tensor $\langle T_{\mu\nu}\rangle$ is divergent in four spacetime dimensions, and must be regularized via a covariant procedure [BirDav82, RamHu97b].

Making the two-loop approximation to the CTP-2PI-CGEA, where we take $\Gamma_2 \simeq \hbar^2\Gamma^{(2)}$, and dropping the $O(\lambda^2)$ diagram from Γ_2, the mean-field equation becomes

$$\begin{aligned}\left(\nabla^2 + m^2 + \xi R(x) + \frac{\lambda}{6}\bar{\phi}^2(x) + \frac{\lambda\hbar}{2}G(x,x)\right)\bar{\phi}&\\ + \hbar f \,\mathrm{Tr}_{\mathrm{sp}}[F_{ab}(x,x)] - \hbar^2 g^3\Sigma(x) = 0&\end{aligned} \tag{15.123}$$

where $G(x,x)$ is the coincidence limit of $G_{ab}(x,x')$, and $\Sigma(y)$ is a (self-energy) function defined by

$$\begin{aligned}\Sigma(y) = -2\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,\mathrm{Re}\,\mathrm{Tr}_{\mathrm{sp}}\Big[\Big(\theta(x,x')G_{(1)}(x',x)F^{(1)}(x,x')&\\ - G_{\mathrm{R}}(x,x')^{\dagger}F_{\mathrm{R}}(x,x')\Big)F_{\mathrm{R}}(y,x')^{\dagger}F_{\mathrm{R}}(y,x)\Big]&\end{aligned} \tag{15.124}$$

where an index 1 refers to a Hadamard propagator (cf. Chapters 6 and 10), and a subindex R to a retarded propagator. It is clear that the integrand vanishes whenever x or x' is to the future of y. The equation for G_{ab} is given by

$$\begin{aligned}(G^{-1})^{ba}(x,x') = \mathcal{A}^{ba}(x,x') + \frac{i\lambda\hbar}{4}c^{ba}G_1(x,x)\frac{\delta(x-x')}{\sqrt{-g'}}&\\ + \hbar f^2 c^{aa'a''}c^{bb'b''}\mathrm{Tr}_{\mathrm{sp}}\left[F_{a'b'}(x,x')F_{b''a''}(x',x)\right]&\end{aligned} \tag{15.125}$$

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Multiplying equation (15.125) through by G_{ab}, performing a spacetime integration, and taking the 11 component, we obtain

$$\left(\nabla^2 + m^2 + \xi R + \frac{\lambda}{2}\bar{\phi}^2 + \frac{\lambda\hbar}{4}G_1(x,x)\right)G_F(x,x') \\ +\hbar f^2 \int dx''\sqrt{-g''}\mathcal{K}(x,x'')G_F(x'',x') = -i\frac{\delta(x-x')}{\sqrt{-g'}} \tag{15.126}$$

in terms of a kernel $\mathcal{K}(x,x'')$ defined by

$$\mathcal{K}(x,x') = -i\mathrm{Tr}_{\mathrm{sp}}\left[F_F(x,x')^2 - F^+(x,x')^2\right] = \mathrm{Re}\,\mathrm{Tr}_{\mathrm{sp}}\left[F_R(x,x')F_1(x',x)\right] \tag{15.127}$$

which shows that equation (15.126) is manifestly real and causal. The kernel $\mathcal{K}(x,x')$ is dissipative, and it reflects the back-reaction from fermionic particle production induced by the time-dependence of the inflaton *variance*. Equation (15.126) is therefore damped for modes above threshold, and this damping is not accounted for in the 1PI treatments of inflaton dynamics (where only the inflaton mean field is dynamical). As stressed above, the dissipative dynamics of the inflaton two-point function can be important when the inflaton variance is on the order of the square of the inflaton mean-field amplitude; such conditions may exist at the end of preheating.

The set of evolution equations (15.123) for $\bar{\phi}$ and (15.126) for G is formally complete to two loops. Dissipation arises due to the coarse graining of the spinor degrees of freedom. These dynamical equations are valid in a general background spacetime and are useful for reheating studies and more general purposes.

15.3.2 Case study II: Primordial magnetic field generation

Given the difficulties in constructing a suitable model of the reheating stage, to further elucidate its physics it helps to investigate other physical processes coexisting with the reheating of the universe which could have produced an observable imprint either on the CMB or today's large-scale structures.

The two processes most studied are the generation of spin-two and spin-one fields. The former concerns the possible processing of primordial gravitational fluctuations on super-horizon scales, while the latter addresses the feasibility of generating primordial magnetic fields during reheating. Gravitational fluctuations ought to have influenced the spectrum and polarization of the CMB, while a primordial magnetic field could serve as a seed for the magnetic fields observed today in cosmological structures, and should also have affected the CMB [Dod03, Lon98].

Fields with a strength of about a millionth of the Earth's magnetic field are observed both in galaxies and clusters of galaxies. There are at least three good reasons to believe these fields have a cosmological origin. First, the fact that they extend over huge scales. Second, fields are also observed at high red-shift, when dynamo mechanisms have less time to operate. This strongly suggests the field was "already there," though at the time of writing it is unclear exactly how fast dynamo amplification can be [BraSub05]. Third, that in any case "local" mechanisms such as a "galactic dynamo" could amplify an existing seed field, but not create a field from nothing [GraRub01].

The same reasons of scale make it tempting to place the origin of the field in the inflationary era (for primordial but not inflationary mechanisms see [BoVeSi03b, BoVeSi03a, Vac01, VilLea82]). However a large enough magnetic field is not expected to be generated during inflation because of the conformal invariance of the Maxwell field.[5]

We give a semiquantitative discussion here (adopting natural units ($\hbar = c = k_B = 1$)). As in Chapter 7, the field is described by a vector potential A_μ; we rescale the field by the gauge coupling constant, so that the curved space free Lagrangian density reads

$$\mathcal{L} = \frac{-\sqrt{-g}}{4} F^{\mu\nu} F_{\mu\nu} \tag{15.128}$$

with the abelian field tensor equation (7.5) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. If we assume a conformally flat FLRW metric written in conformal coordinates $(\eta, \vec{x})$ (cf. Section 4.6.2), then in four spacetime dimensions the conformal factor drops out of the free action.

The inhomogeneous Maxwell equations for a field driven by a current

$$J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta A_\mu} \tag{15.129}$$

are given by

$$F^{\nu\mu}_{;\nu} = -J^\mu \tag{15.130}$$

During the radiation-dominated era, the current is induced by the Lorentz force acting on the charged plasma, so we have a constitutive relation

$$J^\mu = \sigma F^{\mu\nu} u_\nu \tag{15.131}$$

where

$$u_\mu = a\eta_{0\mu} \tag{15.132}$$

is the 4-velocity of the plasma in conformal coordinates. The conductivity is $\sigma \propto T$ (see below) [GioSha00], so the combination $\bar{\sigma} = a\sigma$ is independent of the scale factor. Since we already noted that the free action is independent of the

[5] For the sake of discussion, we gloss over the fact that properly speaking we should not be concerned with a Maxwell field, but rather with a spin-one field which becomes electromagnetic after electroweak symmetry breaking.

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scale factor in four spacetime dimensions, the result is that a drops from the Maxwell equations, which read

$$A_{i,i} = 0 \tag{15.133}$$

$A_0 = 0$ and

$$A_{j,00} + \bar{\sigma} A_{j,0} - A_{j,ii} = 0 \tag{15.134}$$

For each Fourier mode, the corresponding amplitude behaves as a damped harmonic oscillator. If the comoving wavenumber $k < \bar{\sigma}/2$, the mode is overdamped. There is a fast decaying component

$$f_{\text{fast}} = e^{-\bar{\sigma}\eta} \tag{15.135}$$

and a slow decaying component

$$f_{\text{slow}} = e^{-k^2\eta/\bar{\sigma}} \tag{15.136}$$

For long enough wavelengths we may approximate $f_{\text{slow}} = 1$. The boundary conditions are that at the beginning of the radiation-dominated era there are no fields, so $A_j(0) = 0$. From the constitutive relation we get

$$A_{j,0}(0) = -\frac{a^2(0)}{\bar{\sigma}} J_j(0) \tag{15.137}$$

and so once f_{fast} decays the field settles down to a time-independent value

$$A_j(\infty) = -\frac{J_j(0)}{\sigma^2(0)} \tag{15.138}$$

Associated with the free action there is an energy–momentum tensor (15.122)

$$T_{\mu\nu} = F_{\lambda\mu} F^{\lambda}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \tag{15.139}$$

and an energy density

$$\rho = T_{\mu\nu} u^{\mu} u^{\nu} \tag{15.140}$$

In the asymptotic regime, ρ scales as a^{-4}. Therefore the ratio r between the energy density of the coherent Maxwell field and the total energy density of radiation is constant, provided the cosmic expansion is adiabatic. Disregarding the entropy generated during particle annihilations, we may say r is constant up to our times. A value of $r = 10^{-8}$ is strong enough to originate the galactic fields without further dynamo amplification [TurWid88]. The lowest value of r that could seed the galactic field through dynamo amplification is hard to estimate, as it depends on the details both of the galaxy formation process and of the cosmological model (i.e. the amount of dark energy or the space curvature) [DaLiTo99]. A primordial field should also leave an imprint on the cosmic microwave radiation, but present data only provide upper bounds [YIKM06].

To put these numbers in perspective, we may ask which value of r could be expected for fields coherent over a physical scale L_{phys}, given thermal equilibrium conditions. Since in the Rayleigh–Jeans part of the spectrum we may assume equipartition, the energy density associated with modes $k < L_{\mathrm{phys}}^{-1}$ is $T_{\mathrm{today}} L_{\mathrm{phys}}^{-3}$, and so $r \approx \left(L_{\mathrm{phys}} T_{\mathrm{today}}\right)^{-3}$. Using $T_{\mathrm{today}} = 10^{-4}\mathrm{eV}$, we get $L_{\mathrm{phys}} T_{\mathrm{today}} \approx 10^{24} \left(L_{\mathrm{phys}}/1\,\mathtt{Mly}\right)$. For a galaxy cluster-size scale, r is way below the interesting range. In this section, we shall use a subindex "today" to indicate that a quantity is evaluated at the present time (we assume $a_{\mathrm{today}} = 1$). Similarly, "reh" will denote the end of reheating, and "equiv" the time of equivalence between matter and radiation.

More generally, r remains constant when both the coherent magnetic field and the thermal cosmic background evolve in a conformally invariant way. So to increase the value of r, we must break conformal invariance. In their seminal work on magnetic field generation [TurWid88], Turner and Widrow have considered a number of possible conformal symmetry-breaking mechanisms.

The hardest way to break the symmetry is to add to the action a direct coupling of the Maxwell field to curvature, such as, for example, $R^{\mu\nu} A_\mu A_\nu$. However, this term breaks gauge along with conformal symmetry, and it is hard to generate in a natural way. Gauge symmetric terms such as $f\,(R)\,F^{\mu\nu}F_{\mu\nu}$ are more appealing, partly because they arise naturally from radiative corrections in a curved spacetime. Nevertheless, Mazzitelli and Spedalieri [MazSpe95] have observed that, after proper resummation of the leading quantum corrections, the dependence on curvature is at most logarithmic, and so it is hard to achieve efficient magnetic field generation. A similar conclusion, in a wider set of problems, has been reached recently by Weinberg [Wei05a, Wei06].

Over and above the details of each mechanism, we must consider that the quantity r generated during inflation may well be diluted at reheating. During reheating the density of radiation increases by a factor of at least e^{4N}, where N is the number of e-foldings. Unless the coherent field is also amplified, r decreases by the same amount.

When we consider the generation of magnetic fields during reheating, a new possibility opens up. The abrupt changes in metric during this stage may result in abundant particle creation of charged species. This would generate stochastic currents (recall Chapter 8), which eventually decay onto the Maxwell field [CaKaMa98].

Before we evaluate whether such a mechanism is feasible, let us observe the following. Because the inflaton is a gauge singlet, we do not expect it will decay directly into charged species. Therefore, the model assumes these charged particles are created from the gravitational field, which in turn responds to the changes in the equation of state of the inflaton [PeeVil99].

Spin 1/2 particles such as electrons would be conformally invariant at the high energies prevalent during inflation, so they are not created in large numbers. We must seek a fundamental charged scalar field, of which there is none in the standard model. There are suitable candidates in supersymmetric extensions of the standard model, however [KCMW00].

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An alternative which appeals only to known and proven physics is to replace the charged field by the gravitational field itself. Inflation generates tensor gravitational fluctuations, and therefore an inflationary universe is not strictly speaking conformally flat. The evolution of these gravitational fluctuations may result in amplification of the Maxwell field [BaTsWa06, BPTV01, TsaKan05]. However, due to the weakness of the gravitational couplings, it is hard to achieve the desired efficiency.

In the following we shall give an estimate of the field strength to be expected from particle creation of a charged, minimally coupled scalar field ϕ by the end of the reheating period. We decompose the field into its real and imaginary parts $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$. The current is

$$J^\mu = J_1^\mu + J_2^\mu \tag{15.141}$$

where

$$J_{1\mu} = e\left(\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1\right) \tag{15.142}$$

$$J_{2\mu} = -e^2 A_\mu \left(\phi_1^2 + \phi_2^2\right) \tag{15.143}$$

In a linearized analysis we set $J_2 = 0$. Each field is decomposed into modes

$$\phi_i = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{kx}}\, \phi_{i\mathbf{k}} \tag{15.144}$$

where

$$\phi_{i\mathbf{k}} = \phi_{\mathbf{k}} a_{i\mathbf{k}} + \phi_{\mathbf{k}}^* a_{i-\mathbf{k}}^+ \tag{15.145}$$

leading to a mode decomposition of the current. The spatial components become

$$\mathbf{J}_1 = ie \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{kx}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[2\mathbf{q} - \mathbf{k}\right] \phi_{1\mathbf{k}-\mathbf{q}} \phi_{2\mathbf{q}} \tag{15.146}$$

while the charge density is

$$J_{10} = -ie \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{kx}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[\omega_{2\mathbf{q}} - \omega_{1\mathbf{k}-\mathbf{q}}\right] \phi_{1\mathbf{k}-\mathbf{q}} \phi_{2\mathbf{q}} \tag{15.147}$$

where

$$\omega_{i\mathbf{k}} = \frac{i}{\phi_{i\mathbf{k}}} \frac{d\phi_{i\mathbf{k}}}{dt} \tag{15.148}$$

We are interested in the current averaged over a comoving scale L

$$\mathbf{J}_{1L} = ie \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, W_L[k] \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[2\mathbf{q} - \mathbf{k}\right] \phi_{1\mathbf{k}-\mathbf{q}} \phi_{2\mathbf{q}} \tag{15.149}$$

where W_L is a window function. If the initial state of the field is the vacuum, it is clear that $\langle \mathbf{J}_{1L} \rangle = 0$, but

$$\left\langle \mathbf{J}_{1L}^2 \right\rangle = e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, W_L[k]^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[2\mathbf{q} - \mathbf{k}\right]^2 \left|\phi_{\mathbf{q}}\right|^2 \left|\phi_{\mathbf{k}-\mathbf{q}}\right|^2 \tag{15.150}$$

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To see the meaning of this equation, let us consider (and reject) the case of a conformally coupled field. For conformal coupling, we simply have (cf. Chapter 4)

$$|\phi_{\mathbf{q}}|^2 = \frac{1}{2a^2 q} \tag{15.151}$$

It would seem that $\left\langle \mathbf{J}_{1L}^2 \right\rangle$ is dominated by very short modes. However, since short modes are supposed to thermalize during reheating, they cannot possibly be described within a linear theory. There must be a comoving cut-off Λ which marks the limit of the linearized approximation. Assuming however $\Lambda \gg L^{-1}$, we see that the dominant contribution to $\left\langle \mathbf{J}_{1L}^2 \right\rangle$ comes from modes where $q \approx \Lambda \gg k \approx L^{-1}$. The integrals decouple, and we get

$$\left\langle \mathbf{J}_{1L}^2 \right\rangle \approx \frac{e^2 \Lambda^3}{a^4 L^3} \tag{15.152}$$

Under the same approximations, the mean square value of the charge density vanishes.

To transform this into an estimate for the Maxwell field, we need the value of σ at the end of reheating. The usual estimate for the conductivity is $\sigma \approx e^2 n\tau/m$, where n and m are the density and rest mass of the dominant charge carriers, and τ a typical mean free time. If the dominant carriers are just electrons and positrons, then prior to annihilation we have $n \approx T^3$. If we assume that reheating ends as soon as thermal equilibrium is reached, then at that time we may approximate τ by the effective age of the universe $\tau \equiv H^{-1} = m_P T_{\text{reh}}^{-2}$.

To conclude, we evaluate the asymptotic vector potential from (15.138) and the corresponding energy density from (15.140), where we use the result that a space derivative is $\partial \approx L^{-1}$

$$\rho \approx \frac{\left\langle \mathbf{J}_{1L}^2 \right\rangle_{\text{reh}}}{a^4 L^2 \sigma_{\text{reh}}^4} \approx \frac{1}{a^4} \left[\frac{m^4 \Lambda^3 H_{\text{reh}}^4}{a_{\text{reh}}^4 L^5 e^6 T_{\text{reh}}^{12}} \right] \tag{15.153}$$

Combining these estimates we get

$$r \approx \left(\frac{m}{m_P e^{3/2}} \right)^4 \left(\frac{T_{\text{reh}}}{m_P} \right)^3 \left(\frac{\Lambda}{a_{\text{reh}} H_{\text{reh}}} \right)^3 \left(\frac{1}{L T_{\text{today}}} \right)^5 \tag{15.154}$$

which is far worse than our previous estimate based on equilibrium conditions.

It is clear from this analysis that to obtain a larger value of r we must amplify the scalar field fluctuations far above the conformal value. During the radiation-dominated era the scalar curvature vanishes and any scalar field is conformally invariant as long as it is effectively massless. But during inflation the behavior is totally different, because while the conformal fields evolve as a^{-1} throughout, the minimal fields freeze upon horizon exit and remain constant until the scalar curvature is suppressed enough during reheating. To see this, let us return to the mode equation (15.8). We write the mode functions as

$$\phi_{\mathbf{k}} = \frac{f_k}{a^{3/2}} \tag{15.155}$$

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Using the Friedmann equation $H^2 = \rho/m_P^2$, the continuity equation $\dot{\rho} = -3H(\rho + p)$ and the equation of state $p = \gamma\rho$, we transform the mode equation into (4.22)

$$\frac{d^2 f_k}{dt^2} + \Omega_k^2(t) f_k(t) = 0 \tag{15.156}$$

where

$$\Omega_k^2(t) = \frac{k^2}{a^2} + m^2(t) + \frac{9}{4}H^2\gamma \tag{15.157}$$

and we allow for the possibility of a time-dependent mass, for example, due to thermal corrections (cf. Chapter 10). During inflation $\gamma = -1$. Ω_k^2 starts positive and becomes negative upon horizon exit. Outside of the horizon there is a growing mode which remains frozen because the growth of the WKB solution just matches the $a^{-3/2}$ suppression, and a decaying mode which soon becomes irrelevant. At some point during reheating γ becomes positive and Ω_k^2 changes sign again; we say the mode "thaws." Neglecting the decaying solution and assuming a long enough wavelength, we have immediately after thawing the q-number amplitudes

$$\phi_{i\mathbf{k}} = A_k F(t)\left[a_{i\mathbf{k}} + a_{i-\mathbf{k}}^{+}\right] \tag{15.158}$$

where $|A_k|$ is of the order of magnitude of the amplitude at horizon exit. Observe that the time-dependent part $F(t)$ is essentially mode-independent: the field is performing "Sakharov" oscillations [Sak66]. This implies the vanishing of the induced charge density.

As a first approximation, we may assume that all modes thaw at the same time at the end of reheating. As compared with the conformal case, the minimally coupled mode amplitudes are amplified by a factor $a_{\rm reh}/a_{\rm exit} = a_{\rm reh}H_{\rm reh}/k$, where we assume that reheating is fast enough that the Hubble rate remains approximately time-independent throughout.

Our estimate for the current at reheating now reads

$$\left\langle \mathbf{J}_{1L}^2 \right\rangle_{\rm reh} \approx \frac{e^2 H_{\rm reh}^4}{4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, W_L[k]^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3}\,[2\mathbf{q}-\mathbf{k}]^2\,\frac{1}{q^3}\,\frac{1}{(|\mathbf{k}-\mathbf{q}|)^3} \tag{15.159}$$

The q integral is dominated by peaks at $\mathbf{q} = 0$ and $\mathbf{q} = \mathbf{k}$. They both contribute the same, as they are transformed into each other by the change of variables $\mathbf{q} \to \mathbf{k} - \mathbf{q}$. So it is enough to evaluate the contribution from $q \ll k$

$$\left\langle \mathbf{J}_{1L}^2 \right\rangle_{\rm reh} \approx \frac{e^2 H_{\rm reh}^4}{4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{W_L[k]^2}{k}\int \frac{dq}{q} \tag{15.160}$$

We evaluate the logarithmic integral as $\ln[q_{\rm max}/q_{\rm min}]$, where the q's are the longest and shortest modes to leave the horizon during inflation. Therefore

$$\int \frac{dq}{q} \approx N \tag{15.161}$$

where N is the number of e-foldings. We obtain

$$\left\langle \mathbf{J}_{1L}^2 \right\rangle_{\rm reh} \approx \frac{e^2 N H_{\rm reh}^4}{4L^2} \tag{15.162}$$

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The improved estimate for r is

$$r \approx N \left(\frac{m}{m_P e^{3/2}} \right)^4 \left(\frac{T_{\text{reh}}}{m_P} \right)^4 \left(\frac{1}{L T_{\text{today}}} \right)^4 \tag{15.163}$$

which is still a very small number.

Although prospects are understandably bleak, our argument has a loophole [Fin00]. This is the neglect of the "London" current (15.143). Because of this term, the heavily amplified long-wavelength modes of the scalar field act as a Landau–Ginzburg order parameter in a superconductor [Tin96]. As in the Meissner effect, the photon acquires a (here time-dependent) mass. Kandus *et al.* have shown that an exponential growth of the Maxwell field during reheating as a consequence of parametric amplification is possible [CalKan02]. However, in this case the actual growth factor is sensitive to the details of the reheating scenario, and so it is not possible to obtain generally valid estimates such as the above.

At the end of this discussion, we reach a situation remarkably similar to our description of early thermalization in RHICs in Chapter 14. Both the generation of a primordial magnetic field during reheating and ultrafast equilibration after the collision are demonstrably beyond the possibilities of weakly interacting fields, but could be allowed because of exponential instabilities in strongly nonlinear scenarios. In either problem, we do not have answers yet, but it is clear that finding those answers will require the full application of the methods of nonequilibrium field theory, whose basic principles we have attempted to present in this book.