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Existence and Multiplicity of Positive Solutions for Singular Semipositone p-Laplacian Equations

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Abstract. Positive solutions are obtained for the boundary value problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(t,u), \ t \in (0,1), p > 1 \\ u(0) = u(1) = 0. \end{cases}$$

Here $f(t,u) \geq -M$, (M is a positive constant) for $(t,u) \in [0,1] \times (0,\infty)$. We will show the existence of two positive solutions by using degree theory together with the upper-lower solution method.

1 Introduction

In this paper, we establish the existence of positive solutions for the p-Laplacian equation

(1.1) $$\begin{cases} -(\varphi_p(u'))' = \lambda f(t,u) & \text{for } t \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Here $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f\colon [0,1] \times (0,\infty) \to R$ is continuous and satisfies the following conditions.

(H1) There exists $M > 0$ such that

(1.2) $$f(t,y) \geq -M \quad \text{for } (t,y) \in [0,1] \times (0,\infty).$$

(H2)

(1.3) $$\limsup_{y\to\infty} \frac{\widetilde{f}(y)}{\varphi_p(y)} = \infty$$

where $\widetilde{f}(y) = \inf\{f(t,s) : (t,s) \in [0,1] \times [y,\infty)\}$ for $y > 0$.

(H3) $\exists a \in (0,\infty)$ such that

(1.4) $$f(t,y) \geq f(t,a) > 0 \quad \text{for } t \in [0,1],\ y \in (0,a].$$

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(H4) (i) $|f(t,y)| \le g(y) + h(y)$ on $[0,1] \times (0,\infty)$ with $g > 0$ continuous and nonincreasing on $(0,\infty)$, $h \ge 0$ continuous on $[0,\infty)$ and h/g nondecreasing on $(0,\infty)$;

(ii) for any $R > 0$, $1/g$ is differentiable on $(0,R]$ with $g' < 0$ a.e. on $(0,R]$, $\frac{|g'|^{1/p}}{g^{2/p}} \in L^1[0,R]$, and $\int_0^\infty \frac{|g'(t)|^{1/p}}{(g(t))^{2/p}}\,dt = \infty$;

and for any $D \ge 0$, there exists a sequence of numbers $\{M_n\}$ s.t. $\lim_{n\to\infty} M_n = \infty$ and

$$\lim_{n\to\infty} \frac{1}{\varphi_p^{-1}\left(1 + \frac{h(M_n)+D}{g(M_n)}\right)} \int_0^{M_n} \frac{dy}{\varphi_p^{-1}(g(y))} > \frac{p-1}{p} 2^{\frac{p}{1-p}}. \tag{1.5}$$

Remark 1.1 It is easy to see that if

$$\lim_{y\to 0^+} f(t,y) = +\infty \text{ uniformly on } [0,1], \tag{1.6}$$

then (H3) holds

Remark 1.2 Let $b = \min_{t\in[0,1]} f(t,a)/2$ (here a is as in (1.4)). Then

$$f(t,y) > b \text{ for } (t,y) \in [0,1] \times [0,a]. \tag{1.7}$$

Remark 1.3 From (1.3) and the definition of limit supremum, there exists a sequence $\{y_n\}$ with $0 < y_n < y_{n+1}$ for $n \in N$, $\lim_{n\to\infty} y_n = \infty$ and

$$\lim_{n\to\infty} \frac{\widetilde{f}(y_n)}{\varphi_p(y_n)} = \infty.$$

Now $\widetilde{f}(y_n) = \inf\{f(t,s) : (t,s) \in [0,1] \times [y_n,\infty)\}$, so we have

$$\widetilde{f}(y_1) \le \widetilde{f}(y_2) \le \cdots \le \widetilde{f}(y_n) \le \widetilde{f}(y_{n+1}) \le \cdots \tag{1.8}$$

and

$$\lim_{n\to\infty} \widetilde{f}(y_n) = \infty. \tag{1.9}$$

Now, since $f(t,y_n) \ge \widetilde{f}(y_n)$ for $t \in [0,1]$ and $n \in N$, we have

$$\lim_{n\to\infty} f(t,y_n) = \infty \text{ uniformly on } [0,1]. \tag{1.10}$$

Equations of the form (1.1) occur in the study of the p-Laplace equation, non-Newtonian fluid theory, and the turbulent flow of a gas in a porous medium [9]. Existence of positive solutions for problem (1.1) has been studied by many authors, usually under the condition

$$f(t,y) \ge 0 \text{ for } (t,y) \in [0,1] \times [0,\infty).$$

Recently, Anuradha et al. [3] studied the existence of positive solutions for second order boundary value problems with $p = 2$, if conditions (H1) and (H2) hold with $f\colon [0,1]\times[0,\infty)\to R$ continuous and $\lambda > 0$ small enough. Motivated by their work, we consider the p-Laplacian equation (1.1). We use degree theory to establish the existence of positive solutions, and we also discuss multiplicity.

For $p = 2$, problem (1.1) (with $f\colon [0,1]\times[0,\infty)\to R$ continuous) has a positive solution u if and only if $u+v := \overline{u}$ is a solution of

$$\begin{cases} -u'' = \lambda g(t, u-v), & t\in(0,1)\\ u(0) = u(1) = 0, \end{cases}$$

where v is a solution of the problem $-u'' = 1$, $u(0) = u(1) = 0$, and $g\colon [0,1]\times R\to R^+$ is defined by

$$g(x,y) = \begin{cases} f(x,y)+M & (x,y)\in[0,1]\times[0,\infty)\\ f(x,0)+M & (x,y)\in[0,1]\times(-\infty,0). \end{cases}$$

One can use a cone expansion/compression type theorem to establish an existence result when $p = 2$. However, no Green's function is available for general p. As a result, the method in [3] does not suit the p-Laplacian when $p\neq 2$.

Several results on the existence of positive solutions for the one dimensional p-Laplacian boundary value problems have been established in the literature (see [6, 7, 9, 10]. The key condition used is that the nonlinearity is nonnegative so the solution u is concave down; if the nonlinearity f is negative somewhere, then the solution u need no longer be concave down.

The main results of this paper are the following.

Theorem 1.4 *Assume* (H1), (H2), (H3), *and* (H4) *hold. Then the problem* (1.1) *has at least two positive solutions* $u_i\in C[0,1]\cap C^1(0,1)$ *with* $\varphi_p(u_i')\in C^1(0,1)$, $i = 1,2$ *for* $\lambda > 0$ *small enough.*

Theorem 1.5 *Assume* (H1), (H2), (H4), *and* (1.6) *hold. Then the problem* (1.1) *has at least two positive solution* $u_i\in C[0,1]\cap C^1(0,1)$ *with* $\varphi_p\left(u_i'\right)\in C^1(0,1)$, $i = 1,2$ *for* $\lambda > 0$ *small enough.*

Next we state three results from the literature [1, 2, 4] which we will use in Section 3. Consider the singular boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = q(t)f(t,u) \text{ for } t\in(0,1)\\ u(0) = u(1) = 0. \end{cases} \tag{1.11}$$

The singularity may occur at $u = 0$, $t = 0$, and $t = 1$, and the function f is allowed to change sign.

Lemma 1.6 ([2]) *Let* $n_0\in\{3,4,\dots\}$ *be fixed and suppose the following conditions are satisfied:*

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(i) *$f\colon [0,1]\times(0,\infty)\to R$ is continuous.*
(ii) *Let $n\in\{n_0,n_0+1,\dots\}$ and associated with each n we have a constant ρ_n such that $\{\rho_n\}$ is a nonincreasing sequence with $\lim_{n\to\infty}\rho_n = 0$ and such that for $0<t<1$ we have $q(t)f(t,\rho_n)\geq 0$*
(iii) *$q\in C(0,1)\cap L^1(0,1)$ with $q>0$ on $(0,1)$ and*

$$\int_0^{\frac12}\varphi_p^{-1}\Big(\int_s^{\frac12} q(r)\,dr\Big)\,ds+\int_{\frac12}^1\varphi_p^{-1}\Big(\int_{\frac12}^s q(r)\,dr\Big)\,ds<\infty.$$

(iv) *There exists a function $\alpha\in C[0,1]\cap C^1(0,1)$, $\varphi_p(\alpha')\in C^1(0,1)$, with $\alpha(0)=\alpha(1)=0$, $\alpha(t)>0$ on $(0,1)$ such that $q(t)f(t,\alpha(t))+(\varphi_p(\alpha'(t)))'\geq 0$ for $t\in(0,1)$.*
(v) *$|f(t,y)|\leq g(y)+h(y)$ on $[0,1]\times(0,\infty)$ with $g>0$ continuous and nonincreasing on $(0,\infty)$, $h\geq 0$ continuous on $[0,\infty)$, and h/g nondecreasing on $(0,\infty)$.*
(vi) *For any $R>0$, $1/g$ is differentiable on $(0,R]$ with $g'<0$ a.e. on $(0,R]$,*

$$\frac{|g'|^{1/p}}{g^{2/p}}\in L^1[0,R]\quad\text{and}\quad\int_0^\infty\frac{|g'(t)|^{1/p}}{(g(t))^{2/p}}\,dt=\infty.$$

(vii) *In addition assume there exists $M>\sup_{t\in[0,1]}\alpha(t)$ with*

$$\frac{1}{\varphi_p^{-1}\big(1+\frac{h(M)}{g(M)}\big)}\int_0^M\frac{dy}{\varphi_p^{-1}\left(g\left(y\right)\right)}>b_0.$$

holding. Here

$$b_0=\max\Big\{\int_0^{\frac12}\varphi_p^{-1}\Big(\int_s^{\frac12}q(r)\,dr\Big)\,ds,\int_{\frac12}^1\varphi_p^{-1}\Big(\int_{\frac12}^s q(r)\,dr\Big)\,ds\Big\}.$$

Then (1.11) has at least one solution $u\in C[0,1]\cap C^1(0,1)$ with $\varphi_p(u')\in C^1(0,1)$ with $u(t)\geq\alpha(t)$ for $t\in[0,1]$.

Lemma 1.7 ([4]) *Let C be a bounded closed set in a Banach space X and $K\colon[\alpha,\beta]\times C\to C$, $\alpha<\beta$, a compact mapping. Then the set*

$$S_{\alpha,\beta}=\{(s,x)\in[\alpha,\beta]\times C\mid K(s,x)=x\}$$

of "fixed points" of K contains a component $C_{\alpha,\beta}$ which connects $\{\alpha\}\times C$ to $\{\beta\}\times C$.

Remark 1.8 Let $S_{-1}=S_{\alpha,\beta}\cap(\{\alpha\}\times C)$ and $S_{+1}=S_{\alpha,\beta}\cap(\{\beta\}\times C)$. Suppose the set $S_{\alpha,\beta}$ contains a component $C_{\alpha,\beta}$ which connects $\{\alpha\}\times C$ to $\{\beta\}\times C$ and $\Phi\colon S_{\alpha,\beta}\to R$ is a continuous map with $\Phi(S_{-1})\leq 0$ and $\Phi(S_{+1})\geq 0$. Then $\Phi(s,x)=0$ has at least one solution in $S_{\alpha,\beta}$.

Lemma 1.9 ([1]) *$u\in\{y\in C[0,1] : y(t)\geq 0$ for $t\in[0,1]$ and y is concave on $[0,1]\}$. Then $u(t)\geq t(1-t)\|u\|_\infty$.*

In Section 2, we give some preliminary lemmas. In Section 3, we will prove Theorem 1.4 and Theorem 1.5. Recall $C[0,1]=C([0,1],(-\infty,\infty))$, with the norm $\|u\|_\infty=\sup_{t\in[0,1]}|u(t)|$.

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2 Preliminary Lemmas

Lemma 2.1 *There exist* $0 < \alpha \leq 1$ *and* $\beta \geq 1$ *such that* $\varphi_p^{-1}(x-y) \geq \alpha\varphi_p^{-1}(x) - \beta\varphi_p^{-1}(y)$ *for* $x \geq 0$ *and* $y \geq 0$, *where* $\varphi_p^{-1}(s) = |s|^{\frac{1}{p-1}} \operatorname{sign}(s)$ *is an inverse of* φ_p.

Proof Let $x \geq 0$, $y \geq 0$. If $y \leq \frac{x}{2}$, then

$$\begin{aligned}\varphi_p^{-1}(x-y) \geq & \varphi_p^{-1}\Big(\frac{x}{2}\Big) \\ &= \varphi_p^{-1}\Big(\frac{1}{2}\Big)\varphi_p^{-1}(x).\end{aligned}$$

If $y > \frac{x}{2}$, then

$$\begin{aligned}\varphi_p^{-1}(x-y) &> -\varphi_p^{-1}(y) + \varphi_p^{-1}(x) - \varphi_p^{-1}(2y) \\ &= \varphi_p^{-1}(x) - (1+\varphi_p^{-1}(2))\varphi_p^{-1}(y),\end{aligned}$$

since φ_p^{-1} is odd and increasing. Therefore $\varphi_p^{-1}(x-y) \geq \alpha\varphi_p^{-1}(x) - \beta\varphi_p^{-1}(y)$ for $x, y \geq 0$, where $\alpha := \varphi_p^{-1}\big(\frac{1}{2}\big)$, $\beta = 1 + \varphi_p^{-1}(2)$. ■

Lemma 2.2 *Let* $\lambda > 0$ *be fixed and let* u *be a solution of*

$$\begin{cases} -(\varphi_p(u'))' = \lambda\rho(t), \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

here $\rho(t)$ *is a continuous function with* $\rho(t) \geq -\overline{M}$ *for* $t \in [0,1]$ *(and* $\overline{M} > 0$ *is a constant). If* $\|u\|_\infty \geq \frac{\beta}{\alpha}\varphi_p^{-1}(\lambda\overline{M})$, *then*

$$u(t) \geq \big(\alpha\|u\|_\infty - \beta\varphi_p^{-1}(\lambda\overline{M})\big)\ \min\{t, 1-t\} \text{ for } t \in [0,1];$$

here α *and* β *are as in Lemma* 2.1.

Proof Let u be the solution of (2.1). Then

$$u(t) = \int_0^t \varphi_p^{-1}\Big(A + \lambda\int_s^1 \rho(\tau)\,d\tau\Big)\,ds$$

where

$$\int_0^1 \varphi_p^{-1}\Big(A + \lambda\int_s^1 \rho(\tau)\,d\tau\Big)\,ds = 0.$$

We know A exists and is unique, see [9].

Let $\|u\|_\infty = |u(t_0)|$ for some $t_0 \in (0,1)$. Then (note $u'(t_0) = 0$)

$$u(t) = \int_0^t \varphi_p^{-1}\Big(\lambda\int_s^{t_0} \overline{\rho}(\tau)\,d\tau - \lambda\overline{M}(t_0 - s)\Big)\,ds \text{ for } t \in (0, t_0]$$

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where $\overline{\rho}(\tau) = \rho(\tau) + \overline{M} \geq 0$. By Lemma 2.1, we get

$$u(t) \geq \alpha \int_0^t \varphi_p^{-1}\Big(\lambda \int_s^{t_0} \overline{\rho}(\tau)\, d\tau\Big)\, ds - \beta \int_0^t \varphi_p^{-1}(\lambda \overline{M}(t_0 - s))\, ds, \quad t \in (0, t_0].$$

Now

$$\int_0^t \varphi_p^{-1}(\lambda \overline{M}(t_0 - s))\, ds \leq \varphi_p^{-1}(\lambda \overline{M})t, \quad t \in (0, t_0],$$

so

$$\begin{aligned} u(t) &\geq \alpha \overline{u}(t) - \beta \varphi_p^{-1}(\lambda \overline{M})t \\ &\geq -\beta \varphi_p^{-1}(\lambda \overline{M})t \\ &\geq -\beta \varphi_p^{-1}(\lambda \overline{M}) \end{aligned} \tag{2.2}$$

for $t \in (0, t_0]$; here $\overline{u}(t) = \int_0^t \varphi_p^{-1}(\lambda \int_s^{t_0} \overline{\rho}(\tau)\, d\tau)\, ds$. If $\|u\|_\infty \geq \frac{\beta}{\alpha}\varphi_p^{-1}(\lambda \overline{M}) > \beta \varphi_p^{-1}(\lambda \overline{M})$, then $\|u\|_\infty = u(t_0) > 0$. Note $\overline{u}$ satisfies

$$\begin{cases} -(\varphi_p(\overline{u}'))' = \lambda \overline{\rho}(t), & t \in (0, t_0] \\ \overline{u}(0) = 0, \overline{u}(t_0) \geq \|u\|_\infty = u(t_0). \end{cases}$$

In fact, $\overline{u}(t) \geq u(t)$ for $t \in (0, t_0]$.

We next prove that $\overline{u}(t) \geq v(t)$ for $t \in [0, t_0]$ where v satisfies

$$\begin{cases} -(\varphi_p(v'))' = 0, & t \in (0, t_0] \\ v(0) = 0,\ v(t_0) = \|u\|_\infty. \end{cases}$$

Suppose it is not true. Then $\overline{u} - v$ has a negative absolute minimum at $\tau \in (0, t_0)$. Now since $\overline{u}(0) - v(0) = 0$ and $\overline{u}(t_0) - v(t_0) \geq 0$, there exists $\tau_0, \tau_1 \in [0, t_0]$ with $\tau \in (\tau_0, \tau_1)$ and $\overline{u}(\tau_0) - v(\tau_0) = \overline{u}(\tau_1) - v(\tau_1) = 0$ and $\overline{u}(t) - v(t) < 0$ for $t \in (\tau_0, \tau_1)$. Then

$$(\varphi_p(\overline{u}'))' - (\varphi_p(v'))' = -\lambda \overline{\rho}(t) \leq 0 \quad \text{for } t \in (\tau_0, \tau_1).$$

Let $w = \overline{u}(t) - v(t) < 0$ for $t \in (\tau_0, \tau_1)$. Then

$$\int_{\tau_0}^{\tau_1} \big((\varphi_p(\overline{u}'(t)))' - (\varphi_p(v'(t)))'\big)\, w(t)\, dt \geq 0.$$

On the other hand, using the inequality $(\varphi_p(b) - \varphi_p(a))(b - a) \geq 0$ for $a, b \in R$ and the fact that there exists $\tau^* \in (\tau_0, \tau_1)$ with $\overline{u}'(\tau^*) \neq v'(\tau^*)$ we have

$$\begin{aligned} &\int_{\tau_0}^{\tau_1} \big((\varphi_p(\overline{u}'(t)))' - (\varphi_p(v'(t)))'\big)\, w(t)\, dt \\ &\qquad = -\int_{\tau_0}^{\tau_1} (\varphi_p(\overline{u}'(t)) - \varphi_p(v'(t)))(\overline{u}' - v')\, dt \\ &\qquad < 0, \end{aligned}$$

a contradiction. Consequently, $\overline{u}(t) \geq v(t)$ for $t \in [0, t_0]$.

For $t \in (0, t_0)$, notice

$$v(t) = \frac{\|u\|_\infty}{t_0} t.$$

Since $\overline{u} \geq v$ for $t \in (0, t_0]$ and $\alpha > 0$, we have from (2.2) that

$$u(t) \geq \Big(\frac{\alpha \|u\|_\infty}{t_0} - \beta \varphi_p^{-1}(\lambda \overline{M})\Big) t, \quad t \in (0, t_0],$$

Similarly,

$$u(t) \geq \Big(\frac{\alpha \|u\|_\infty}{1 - t_0} - \beta \varphi_p^{-1}(\lambda \overline{M})\Big) (1 - t), \quad t \in [t_0, 1).$$

If $\|u\|_\infty \geq \frac{\beta}{\alpha} \varphi_p^{-1}(\lambda \overline{M})$, then

$$u(t) \geq (\alpha \|u\|_\infty - \beta \varphi_p^{-1}(\lambda \overline{M})) \min\{t, 1 - t\}, \quad t \in [0, 1].$$

This completes the proof of Lemma 2.2. ■

By condition (H3) we have

$$f(t, u) \geq f(t, a) \text{ for } (t, u) \in [0, 1] \times (0, a]. \tag{2.3}$$

Let us consider the problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda f^*(t, u), & t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.4}$$

where

$$f^*(t, y) = \begin{cases} f(t, y) & \text{if } t \in [0, 1], y \geq a, \\ f(t, a) & \text{if } t \in [0, 1], y < a. \end{cases}$$

By (2.3), we have

$$f(t, y) \geq f^*(t, y) \text{ for } (t, y) \in [0, 1] \times (0, \infty). \tag{2.5}$$

Let

$$\overline{f^*}(t, y) = f^*(t, y) + M \geq 0 \text{ for } \forall (t, y) \in [0, 1] \times (-\infty, \infty) \tag{2.6}$$

and

$$\widehat{f^*}(y) = \sup\{\overline{f^*}(t, x) : 0 \leq t \leq 1, x \leq y\} \text{ for } y > 0. \tag{2.7}$$

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Remark 2.3 From (1.10) and

$$\begin{aligned}\widehat{f^*}(y_n) &= \sup\{\overline{f^*}(t,x) : 0 \le t \le 1, x \le y_n\}\\ &\ge \overline{f^*}(t,y_n)(\to \infty \text{ as } n \to \infty, \text{ uniformly on } [0,1])\end{aligned}$$

(here $\{y_n\}(n \in N)$ is as in Remark 1.3), we have

$$\lim_{n\to\infty} \widehat{f^*}(y_n) = \infty. \tag{2.8}$$

Also, for all n large enough, we obtain

$$\begin{aligned}\frac{\widehat{f^*}(y_n)}{\varphi_p(y_n)} &\ge \frac{\overline{f^*}(t,y_n)}{\varphi_p(y_n)} \ge \frac{f^*(t,y_n)}{\varphi_p(y_n)}\\ &= \frac{f(t,y_n)}{\varphi_p(y_n)} \ge \frac{\widetilde{f}(y_n)}{\varphi_p(y_n)} \quad (\to \infty \text{ as } n \to \infty).\end{aligned}$$

Thus, we have

$$\limsup_{y\to\infty} \frac{\widehat{f^*}(y)}{\varphi(y)} = \infty. \tag{2.9}$$

For $u \in C[0,1]$, define

$$Tu(t) = \int_0^t \varphi_p^{-1}\Big(A + \int_s^1 \lambda f^*(\tau, u(\tau))\,d\tau\Big)\,ds$$

where

$$\int_0^1 \varphi_p^{-1}\Big(A + \int_s^1 \lambda f^*(\tau, u(\tau))\,d\tau\Big)\,ds = 0.$$

We know A exists and is unique for every $u \in C[0,1]$, and $u = Tu$ is a solution of

$$\begin{cases} -(\varphi_p(u'))' = \lambda f^*(t,u), & t \in (0,1)\\ u(0) = u(1) = 0.\end{cases}$$

We know [9] that $T\colon C[0,1] \to C[0,1]$ is continuous and completely continuous.

Lemma 2.4 *Let $\lambda > 0$ be fixed but sufficiently small. Then there exists $C_\lambda > a$ such that for any $0 \le \theta \le 1$ the problem*

$$u = \theta T u \tag{2.10}$$

has no solution satisfying $\|u\|_\infty = C_\lambda$.

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Proof Let u be a solution of (2.10). Then

$$u(t) = \theta \int_0^t \varphi_p^{-1}\Big(\int_s^{t_0} \lambda(\overline{f^*}(\tau, u(\tau)) - M) d\tau \Big)\, ds,$$

here $\overline{f^*}(t,u)$ is as in (2.6), M is as in (1.2) and $t_0 \in (0,1)$ is such that $\|u\|_\infty = |u(t_0)|$. Therefore,

$$\begin{aligned}
\|u\|_\infty &\le \int_0^{t_0} \varphi_p^{-1}\Big(\int_s^{t_0} \lambda \widehat{f^*}(\|u\|_\infty)\, d\tau \Big)\, ds \\
&\le \int_0^{t_0} \varphi_p^{-1}\Big(\int_0^{t_0} \lambda \widehat{f^*}(\|u\|_\infty)\, d\tau \Big)\, ds \\
&\le \varphi_p^{-1}(\lambda)\varphi_p^{-1}\big(\widehat{f^*}(\|u\|_\infty)\big)\, t_0^2 \\
&< \varphi_p^{-1}(\lambda)\varphi_p^{-1}\big(\widehat{f^*}(\|u\|_\infty)\big) \qquad \text{(because } 0 < t_0 < 1\text{);}
\end{aligned}$$

here $\widehat{f^*}(u)$ is as in (2.7). Thus

$$\frac{1}{\lambda} < \frac{\widehat{f^*}(\|u\|_\infty)}{\varphi_p(\|u\|_\infty)}. \tag{2.11}$$

From (2.8), there exists $k_0 > \max\{\frac{\beta}{\alpha}\varphi_p^{-1}(M), a\}$ with $\widehat{f^*}(k_0) > 0$ (here a is as in (1.4)). Let

$$0 < \Lambda_1 \le \min\Big\{ 1, \frac{\varphi_p(k_0)}{\widehat{f^*}(k_0)} \Big\} \tag{2.12}$$

be fixed. Suppose $0 < \lambda < \Lambda_1$. Then

$$\frac{1}{\lambda} > \frac{\widehat{f^*}(k_0)}{\varphi_p(k_0)}.$$

By (2.9), there exists $y^* > k_0$ such that $\frac{\widehat{f^*}(y^*)}{\varphi_p(y^*)} > \frac{1}{\lambda}$. On the other hand, $\frac{\widehat{f^*}(y)}{\varphi_p(y)}$ is continuous on $[k_0, y^*]$. Thus, there exists $C_\lambda \in (k_0, y^*)$ such that

$$\frac{1}{\lambda} = \frac{\widehat{f^*}(C_\lambda)}{\varphi_p(C_\lambda)}. \tag{2.13}$$

Hence by (2.11), $\|u\|_\infty \ne C_\lambda$. Thus for any $0 \le \theta \le 1$ we have that $u \ne \theta T u$ for u with $\|u\|_\infty = C_\lambda$. ■

Remark 2.5 In the proof of Lemma 2.4 it is enough to take $k_0 > 0$, and

$$0 < \Lambda_1 \le \frac{\varphi_p(k_0)}{\widehat{f^*}(k_0)}.$$

However in Lemma 2.8 we will need k_0, and Λ_1, chosen as in the proof of Lemma 2.4.

Lemma 2.6 *Assume $\lambda \in (0, \Lambda_1)$ be fixed. Consider the problem*

$$\begin{cases} -(\varphi_p(u'))' = \lambda(f^*(t,u) + h), \\ u(0) = u(1) = 0, \end{cases} \tag{2.14}$$

where $h > 2M$ (here M is as in (1.2)*) is a constant. Then there exists $h_0 > 2M$ such that the problem* (2.14) *(with h replaced by h_0) has no solution.*

Proof Let $h > 2M$ (here M is as in (1.2)). Then

$$\begin{aligned} f^*(t,y) + h &= f^*(t,y) + \frac{h}{2} + \frac{h}{2} \\ &> f^*(t,y) + M + \frac{h}{2} \\ &\geq \frac{h}{2} > 0 \qquad \text{(see (2.6))}, \end{aligned}$$

for all $(t,y) \in [0,1] \times (0,\infty)$. Suppose (2.14) has a solution u_h (associated to h) for all $h > 2M$. First, we prove that

$$\lim_{h\to\infty} \|u_h\|_\infty = \infty. \tag{2.15}$$

Fix $h > 2M$ and let $\|u_h\|_\infty = u_h(t_0) > 0$ for some $t_0 \in (0,1)$. Assume that $t_0 \geq \frac{1}{2}$. Then

$$\begin{aligned} \|u_h\|_\infty &= u_h(t_0) \\ &= \int_0^{t_0} \varphi_p^{-1}\Big(\lambda \int_s^{t_0} (f^*(\tau, u_h(\tau)) + h)\, d\tau\Big)\, ds \\ &\geq \int_0^{\frac14} \varphi_p^{-1}\Big(\lambda \int_s^{t_0} (f^*(\tau, u_h(\tau)) + h)\, d\tau\Big)\, ds \\ &\geq \int_0^{\frac14} \varphi_p^{-1}\Big(\lambda \int_s^{\frac12} (f^*(\tau, u_h(\tau)) + h)\, d\tau\Big)\, ds \\ &\geq \int_0^{\frac14} \varphi_p^{-1}\Big(\lambda \int_{\frac14}^{\frac12} \frac{h}{2}\, d\tau\Big)\, ds \\ &\geq \frac{1}{4}\varphi_p^{-1}(\lambda)\varphi_p^{-1}(h/8). \end{aligned}$$

Thus (2.15) holds. On the other hand, let

$$B = \frac{2\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M), \ \delta = \frac{\alpha}{8};$$

here α, β are as in Lemma 2.1 and Λ_1 is as in (2.12). By (2.15), there exist $H > 0$ such that for all $h > H$ we have

$$\|u_h\|_\infty \geq B.$$

Then

$$\begin{aligned}\|u_h\|_\infty &\geq \frac{2\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M)\\ &\geq \frac{\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M)\\ &\geq \frac{\beta}{\alpha}\varphi_p^{-1}(\lambda M).\end{aligned}$$

Thus, by Lemma 2.2, $u_h(t) > 0$ for $t \in (0,1)$. Also since $\alpha\|u_h\|_\infty \geq 2\beta\varphi_p^{-1}(\Lambda_1 M)$, we have

$$\frac{\alpha}{4}\|u_h\|_\infty - \frac{\beta}{4}\varphi_p^{-1}(\Lambda_1 M) \geq \frac{\alpha}{8}\|u_h\|_\infty.$$

Then for all $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, we have

$$\begin{aligned}u_h(t) &\geq (\alpha\|u_h\|_\infty - \beta\varphi_p^{-1}(\lambda M))\min\{t, 1-t\}\\ &\geq \frac{\alpha}{4}\|u_h\|_\infty - \frac{\beta}{4}\varphi_p^{-1}(\lambda M)\\ &\geq \frac{\alpha}{4}\|u_h\|_\infty - \frac{\beta}{4}\varphi_p^{-1}(\Lambda_1 M)\\ &\geq \frac{\alpha}{8}\|u_h\|_\infty = \delta\|u_h\|_\infty.\end{aligned}$$

Now for all $h > \max\{2M, H\}$ we have

$$\begin{aligned}\|u_h\|_\infty &= u_h(t_0)\\ &= \int_0^{t_0}\varphi_p^{-1}\Big(\lambda\int_s^{t_0}(f^*(\tau, u_h(\tau)) + h)\,d\tau)\,ds\\ &\geq \int_0^{\frac{1}{4}}\varphi_p^{-1}\Big(\lambda\int_s^{t_0}(f^*(\tau, u_h(\tau)) + h)\,d\tau\Big)\,ds\\ &\geq \int_0^{\frac{1}{4}}\varphi_p^{-1}\Big(\lambda\int_s^{\frac{1}{2}}(f^*(\tau, u_h(\tau)) + h)\,d\tau\Big)\,ds\\ &\geq \int_0^{\frac{1}{4}}\varphi_p^{-1}\Big(\lambda\int_{\frac{1}{4}}^{\frac{1}{2}}(f^*(\tau, u_h(\tau)) + h)\,d\tau\Big)\,ds\\ &\geq \frac{1}{4}\varphi_p^{-1}(\lambda)\varphi_p^{-1}(\widetilde{f^*}(\delta\|u_h\|_\infty)),\end{aligned}$$

where $\widetilde{f^*}(y) = \inf\{f^*(t,x) : (t,x) \in [0,1] \times [y,\infty)\}$ for $y > 0$. This yields

$$\frac{\widetilde{f^*}(\delta\|u_h\|_\infty)}{\varphi_p(\delta|u_h\|_\infty)} \le \frac{\varphi_p(4)}{\lambda\varphi_p(\delta)}. \tag{2.16}$$

We now prove that there exist $h_1 > \max\{2M, H\}$ with

$$\frac{\widetilde{f^*}(\delta\|u_{h_1}\|_\infty)}{\varphi_p(\delta\|u_{h_1}\|_\infty)} > \frac{\varphi_p(4)}{\lambda\varphi_p(\delta)}. \tag{2.17}$$

If this is true, we are finished. Let $h_* > \max\{2M, H, 2\}$ be fixed. By (1.3) and the definition of f^*, we have

$$\limsup_{y\to\infty} \frac{\widetilde{f^*}(y)}{\varphi_p(y)} = \infty.$$

Then there exists $C_* > \delta\|u_{h_*}\|_\infty$ with

$$\frac{\widetilde{f^*}(C_*)}{\varphi_p(C_*)} > \frac{\varphi_p(4)}{\lambda\varphi_p(\delta)}. \tag{2.18}$$

On the other hand, by (2.15), there exists $h^* > h_*$ such that $\delta\|u_{h^*}\|_\infty > C_*$.

We next prove that there exists $h_1 \in (h_*, h^*)$ so that the solution u_{h_1} of problem (2.14) (with h replaced by h_1) satisfies

$$C_* = \delta\|u_{h_1}\|_\infty.$$

By (1.5), there exist $M^* > \max\{\|u_{h_*}\|_\infty, \|u_{h^*}\|_\infty, a\}$ (here a is as in (1.4)) such that

$$\frac{1}{\varphi_p^{-1}\left(1 + \frac{\widehat{h}(M^*)+h^*}{g(M^*)}\right)} \int_0^{M^*} \frac{dy}{\varphi_p^{-1}(g(y))} > \frac{p-1}{p} 2^{\frac{p}{1-p}}; \tag{2.19}$$

here

$$\widehat{h}(u) = \begin{cases} h(u) & u \ge a, \\ h(a) & u \le a. \end{cases}$$

Let the function f^{**} be defined by

$$f^{**}(t,y) = \begin{cases} f(t,M^*) + r(M^* - y) & \text{for } y > M^* \text{ and } 0 \le t \le 1, \\ f(t,y) & \text{for } a \le y \le M^* \text{ and } 0 \le t \le 1, \\ f(t,a) & \text{for } y < a \text{ and } 0 \le t \le 1, \end{cases}$$

where $r\colon R \to [-1,1]$ is the radial retraction defined by

$$r(x) = \begin{cases} x & \text{for } |x| \le 1, \\ \dfrac{x}{|x|} & \text{for } |x| > 1. \end{cases}$$

For $u \in C[0,1]$ and $h \in [h_*, h^*]$, define

$$K(u,h)(t) = \int_0^t \varphi_p^{-1}\Big(A + \int_s^1 \lambda(f^{**}(\tau, u(\tau)) + h)\, d\tau\Big)\, ds \tag{2.20}$$

where

$$\int_0^1 \varphi_p^{-1}\Big(A + \int_s^1 \lambda(f^{**}(\tau, u(\tau)) + h)\, d\tau\Big)\, ds = 0.$$

We know A exists and is unique for every $u \in C[0,1]$, and notice $u = K(u,h)$ is a solution of

$$\begin{cases} -(\varphi_p(u'))' = \lambda(f^{**}(t,u) + h), & t \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{2.21}$$

We know [9] that $K\colon C[0,1] \times [h_*, h^*] \to C[0,1]$ is continuous and completely continuous.

Next we show any solution u, of the equation

$$u = K(u,h), \quad h \in [h_*, h^*] \text{ and } u \in C[0,1]$$

satisfies

$$\|u\|_\infty \le M^*. \tag{2.22}$$

Suppose it is false. Now since $u(0) = u(1) = 0$, there exist either (i): $t_1, t_2 \in (0,1)$ with $0 \le u(t) \le M^*$ for $t \in [0, t_2)$, $u(t_2) = M^*$ and $u(t) > M^*$ on (t_2, t_1) with $u'(t_1) = 0$ or (ii): $t_3, t_4 \in (0,1)$, $t_4 < t_3$ with $0 \le u(t) \le M^*$ for $t \in (t_3, 1]$, $u(t_3) = M^*$ and $u(t) > M$ on (t_4, t_3) with $u'(t_4) = 0$.

We can assume without loss of generality that either $t_1 \le 1/2$ or $t_4 \ge 1/2$. Suppose $t_1 \le 1/2$. Notice for $t \in (t_2, t_1)$ that we have

$$-(\varphi_p(u'))' = \lambda[f^{**}(t,u)+h] \le g(M^*)+h(M^*)+h^* = g(M^*)+\widehat{h}(M^*)+h^*. \tag{2.23}$$

Integrate (2.23) from t_2 to t_1 to obtain

$$\varphi_p(u'(t_2)) \le [g(M^*) + \widehat{h}(M^*) + h^*](t_1 - t_2),$$

and this together with the fact that $u(t_2) = M^*$ yields

$$\frac{\varphi_p(u'(t_2))}{g(M^*)} \le [1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}](t_1 - t_2). \tag{2.24}$$

Also for $t \in (0, t_2)$ we have

$$-(\varphi_p(u'(t)))' = \lambda[f^{**}(t, u(t)) + h] \le g(u(t)) + \widehat{h}(u(t)) + h^*.$$

and so

$$\frac{-(\varphi_p(u'(t)))'}{g(u(t))} = 1 + \frac{\widehat{h}(u(t)) + h^*}{g(u(t))} \le 1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}$$

for $t \in (0, t_2)$. Integrate from $t \in (0, t_2)$ to t_2 to obtain

$$\frac{-\varphi_p(u'(t_2))}{g(u(t_2))} + \frac{\varphi_p(u'(t))}{g(u(t))} + \int_t^{t_2} \Big[\frac{-g'(u(x))}{g^2(u(x))}\Big] |u'(x)|^p \, dx \le \Big[1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}\Big](t_2 - t),$$

and this together with (2.24) yields

$$\frac{\varphi_p(u'(t))}{g(u(t))} \le \Big[1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}\Big](t_1 - t) \text{ for } t \in (t, t_2).$$

Thus

$$\frac{u'(t)}{\varphi_p^{-1}(g(u(t)))} \le \varphi_p^{-1}\Big(1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}\Big)\varphi_p^{-1}(t_1 - t) \text{ for } t \in (t, t_2).$$

Integrate from 0 to t_2 to obtain

$$\begin{aligned}\int_0^{M^*} \frac{du}{\varphi_p^{-1}(g(u))} &\le \varphi_p^{-1}\Big(1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}\Big) \int_0^{\frac{1}{2}} \varphi_p^{-1}\Big(\frac{1}{2} - t\Big)\, dt \\ &\le \frac{p-1}{p} 2^{\frac{p}{1-p}} \varphi_p^{-1}\Big(1 + \frac{\widehat{h}(M^*) + h^*}{g(M^*)}\Big).\end{aligned}$$

This contradicts (2.19), so (2.22) holds (a similar argument yields a contradiction if $t_4 \ge \frac{1}{2}$). On the other hand, we can easily see that $f^{**}(t, u) + h > 0$, for $0 \le t \le 1$ and $u \in R$, since

$$\begin{aligned} f^{**}(t,u) + h &\ge f^{**}(t,u) + \frac{h_*}{2} + \frac{h_*}{2} \\ &\ge \min_{\substack{0 \le t \le 1 \\ a \le y \le M^*}} f(t, y) - 1 + M + \frac{h_*}{2} && (\text{since } h_* > 2M) \\ &\ge \frac{h_*}{2} - 1 && (\text{since } f(t,u) + M \ge 0) \\ &> 0 && (\text{since } h_* > 2) \end{aligned}$$

Thus $(\varphi_p(u'))' < 0$ for $t \in (0, 1)$, so $\varphi_p(u')$ is decreasing. As a result u' is decreasing, so u is concave on $[0, 1]$. Combining $u(0) = 0$, $u(1) = 0$ and Lemma 1.9, we see that $u(t) > 0$ for $t \in (0, 1)$. Thus we have

$$0 < u(t) < M^* + 1 \equiv M^{**} \text{ for } t \in (0, 1).$$

Let $C = \{x \in C[0,1] \mid \|x\|_\infty \le M^{**}\}$. By Lemma 1.7, the set

$$S_{h_*,h^*} = \{(s,x) \in [h_*,h^*] \times C \mid K(s,x) = x\}$$

contains a component C_{h_*,h^*} which connects $\{h_*\} \times C$ to $\{h^*\} \times C$ and $(h_*, u_{h_*}) \in S_{h_*,h^*}$, $(h^*, u_{h^*}) \in S_{h_*,h^*}$.

Define $\Phi\colon S_{h_*,h^*} \to R$ by

$$\Phi(u) = \|u\|_\infty - C^*/\delta;$$

here C^* is as in (2.18). Then Φ is a continuous map with $\Phi(S_{-1}) < 0$ and $\Phi(S_{+1}) > 0$ (see Remark 1.8 for definitions of S_{-1} and S_+). By Remark 1.8, there exist $h_1 \in (h_*, h^*)$ such that (2.21) (with h replaced by h_1) has a solution u_{h_1} satisfying

$$0 < u_{h_1}(t) < M^{**} \text{ for } t \in (0,1) \text{ and } \|u_{h_1}\|_\infty = C^*/\delta.$$

Thus, u_{h_1} is a solution of problem (2.14) (with h replaced by h_1) such that

$$C_* = \delta \|u_{h_1}\|_\infty.$$

As a result (2.17) is true. Thus there exists $h_0 > 2M$ such that the problem (2.14) has no solution. ■

Consider the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda[f^*(t,u) + \tau h_0], \ t \in (0,1) \\ u(0) = u(1) = 0; \end{cases} \tag{2.25}$$

here h_0 is as in Lemma 2.6. For $\forall \tau \in [0,1]$, define $S_\tau\colon C[0,1] \to C[0,1]$ by

$$(S_\tau u)(t) = \int_0^t \varphi_p^{-1}\Big(A + \lambda \int_s^1 [f^*(r,u(r)) + \tau h_0]\,dr\Big)\,ds \tag{2.26}$$

where

$$\int_0^1 \varphi_p^{-1}\Big(A + \lambda \int_s^1 [f^*(r,u(r)) + \tau h_0]\,dr\Big)\,ds = 0.$$

We know A exists and is unique for every $u \in C[0,1]$, and $u = S_\tau u$ is a solution of

$$\begin{cases} -(\varphi_p(u'))' = \lambda[f^*(t,u(t)) + \tau h_0] \quad t \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Also it is known [9] that $S_\tau\colon C[0,1] \to C[0,1]$ is continuous and completely continuous.

Lemma 2.7 *Let $0 < \lambda < \Lambda_1$ (here Λ_1 is as in (2.12)) be fixed, $0 \le \tau \le 1$ and h_0 be as in Lemma 2.6. Then the solutions of (2.25) are* a priori *bounded.*

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Proof Suppose the result of the lemma is false. Let

$$B = \frac{2\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M), \quad \delta = \frac{\alpha}{8};$$

here α, β are as in Lemma 2.1, Λ_1 is as in (2.12) and M is as in (1.2). Suppose u is a solution of (2.25) for some τ. Now either $\|u_\infty\| \geq B$ or $\|u\|_\infty < B$. Suppose

$$\|u\|_\infty \geq B.$$

Then

$$\begin{aligned}\|u\|_\infty &\geq \frac{2\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M)\\ &\geq \frac{\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M)\\ &\geq \frac{\beta}{\alpha}\varphi_p^{-1}(\lambda M).\end{aligned}$$

By Lemma 2.2 (see the proof of Lemma 2.6) we have

$$u(t) > 0 \text{ for } t \in (0,1) \quad \text{and} \quad u(t) \geq \delta\|u\|_\infty \text{ for } t \in \left[\frac{1}{4}, \frac{1}{2}\right].$$

Suppose $\|u\|_\infty = u(t_0)$ for some $t_0 \in (0,1)$ and $t_0 \geq \frac{1}{2}$. Using Lemma 2.1 (see the proof of Lemma 2.2) we get

$$\begin{aligned}\|u\|_\infty &= u(t_0)\\ &\geq \alpha \int_0^{t_0} \varphi_p^{-1}\Big(\lambda \int_s^{t_0} \overline{f^*}(t,u(t))\,dt\Big)\,ds - \beta\varphi_p^{-1}(\lambda M)\\ &\geq \alpha \int_0^{\frac{1}{4}} \varphi_p^{-1}\Big(\lambda \int_s^{t_0} \overline{f^*}(t,u(t))\,dt\Big)\,ds - \beta\varphi_p^{-1}(\lambda M)\\ &\geq \alpha \int_0^{\frac{1}{4}} \varphi_p^{-1}\Big(\lambda \int_{\frac{1}{4}}^{\frac{1}{2}} \overline{f^*}(t,u(t))\,dt\Big)\,ds - \beta\varphi_p^{-1}(\lambda M)\\ &\geq \Big(\frac{\alpha}{4\varphi_p^{-1}(4)}\varphi_p^{-1}\big(\widetilde{\overline{f^*}}(\delta\|u\|_\infty)\big) - \beta\varphi_p^{-1}(M)\Big)\varphi_p^{-1}(\lambda)\end{aligned}$$

and so

$$(2.27) \qquad \Big(\frac{\alpha}{4\varphi_p^{-1}(4)}\frac{\varphi_p^{-1}\big(\widetilde{\overline{f^*}}(\delta\|u\|_\infty)\big)}{\|u\|_\infty} - \frac{\beta\varphi_p^{-1}(M)}{\|u\|_\infty}\Big)\varphi_p^{-1}(\lambda) \leq 1 \text{ if } \|u\|_\infty \geq B;$$

here $\widetilde{\overline{f^*}}(y) = \inf\big\{\overline{f^*}(t,x) : (t,x) \in [0,1]\times[y,\infty)\big\}$ for $y > 0$.

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By (2.6) we have

$$\widetilde{\widetilde{f^*}}(y) = \widetilde{f^*}(y) + M.$$

From (1.3) and the definition of f^*, we have

$$\begin{aligned}
\limsup_{y\to\infty} \frac{\varphi_p^{-1}\big(\widetilde{\widetilde{f^*}}(y)\big)}{y} &= \limsup_{y\to\infty} \frac{\widetilde{f^*}(y)+M}{y}\\
&= \limsup_{y\to\infty} \frac{\widetilde{f^*}(y)}{y}\\
&= \infty.
\end{aligned}$$

Thus

$$\frac{\delta\alpha}{4\varphi_p^{-1}(4)} \limsup_{y\to\infty} \frac{\varphi_p^{-1}\big(\widetilde{\widetilde{f^*}}(\delta y)\big)}{\delta y} = \infty$$

where $\delta > 0$ is defined above. Let $\tau_0 \in [0, 1]$. If (2.25) has a solution $u_{\lambda\tau_0}$, then (2.27) holds if we assume $\|u_{\lambda\tau_0}\|_\infty \geq B$. The equality above implies that there exists $c_1 > \max\{B, \|u_{\lambda\tau_0}\|_\infty\}$ with

$$\Big(\frac{\alpha}{4\varphi_p^{-1}(4)} \frac{\varphi_p^{-1}\big(\widetilde{\widetilde{f^*}}(\delta c_1)\big)}{c_1} - \frac{\beta\varphi_p^{-1}(M)}{c_1}\Big)\varphi_p^{-1}(\lambda) > 1. \tag{2.28}$$

Now since we assume the result of the lemma is false, there exists $\tau_1 \in [0, 1]$ so that the solution $u_{\lambda\tau_1}$ (associated to λ, τ_1) of (2.25) satisfies

$$\|u_{\lambda\tau_1}\|_\infty > c_1 > \|u_{\lambda\tau_0}\|_\infty.$$

A similar argument as in Lemma 2.6 implies that there exist $\tau_2 \in (\tau_0, \tau_1)$ (if $\tau_1 > \tau_0$) or $\tau_2 \in (\tau_1, \tau_0)$ (if $\tau_1 < \tau_0$) so that the solution $u_{\lambda\tau_2}$ satisfies

$$\|u_{\lambda\tau_2}\|_\infty = c_1.$$

From (2.28) we have

$$\Big(\frac{\alpha}{4\varphi_p^{-1}(4)} \frac{\varphi_p^{-1}\big(\widetilde{\widetilde{f^*}}(\delta\|u_{\lambda\tau_2}\|_\infty)\big)}{\|u_{\lambda\tau_2}\|_\infty} - \frac{\beta\varphi_p^{-1}(M)}{\|u_{\lambda\tau_2}\|_\infty}\Big)\varphi_p^{-1}(\lambda) > 1. \tag{2.29}$$

Now (2.27) and (2.29) yield a contradiction. Hence the assertion of Lemma 2.7 follows. ■

Lemma 2.8 *Let $0 < \lambda < \Lambda_1$ (here Λ_1 is as in* (2.12)*) be fixed. Then problem* (2.4) *has at least one solution $u_* \in C[0, 1]$, and $\|u_*\|_\infty \geq C_\lambda > a$ (here C_λ is as in Lemma* 2.4*) with $u_*(t) > 0$ for $t \in (0, 1)$.*

Proof Let $0 < \lambda < \Lambda_1$ be fixed, and $0 \leq \theta \leq 1$. No solution of $(I - \theta T)u = 0$ lies on the boundary of $B(0, C_\lambda)$, by Lemma 2.4. Therefore

$$\deg(I - \theta T, B_{C_\lambda}, 0) = \text{constant}.$$

This gives

$$\begin{aligned}\deg(I - T, B_{C_\lambda}, 0) &= \deg(I - \theta T, B_{C_\lambda}, 0)\\ &= \deg(I, B_{C_\lambda}, 0)\\ &= 1.\end{aligned}$$

From Lemma 2.7, we can choose

$$R > C_\lambda \tag{2.30}$$

such that no solution of $S_\tau(u) = u$, $\tau \in [0, 1]$ lies on the boundary of B_R. Then

$$\deg(I - S_\tau, B_R, 0) = \text{constant}.$$

Thus by Lemma 2.6

$$\begin{aligned}\deg(I - T, B_R, 0) &= \deg(I - S_0, B_R, 0)\\ &= \deg(I - S_1, B_R, 0)\\ &= 0.\end{aligned}$$

Therefore

$$\deg(I - T, B_R \backslash B_{C_\lambda}, 0) = -1.$$

As a result there exist $u_* \in B_R \backslash B_{C_\lambda}$ such that

$$Tu_* = u_*.$$

That is,

$$\begin{cases}-(\varphi_p(u_*'))' = \lambda f^*(t, u_*), \ t \in (0, 1)\\ u_*(0) = u_*(1) = 0.\end{cases} \tag{2.31}$$

Clearly $\|u_*\|_\infty \geq C_\lambda$. From (2.12), we know that $k_0 > \frac{\beta}{\alpha}\varphi_p^{-1}(M) \geq \frac{\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M)$. Thus for all $\lambda \in (0, \Lambda_1)$, we have $\|u_*\|_\infty \geq C_\lambda > \frac{\beta}{\alpha}\varphi_p^{-1}(\Lambda_1 M) > \frac{\beta}{\alpha}\varphi_p^{-1}(\lambda M)$. Also by Lemma (2.2), for all $\lambda \in (0, \Lambda_1)$ we have

$$u_*(t) \geq (\alpha\|u_*\|_\infty - \beta\varphi_p^{-1}(\lambda M)) \min\{t, 1 - t\} \text{ for } t \in [0, 1];$$

here α and β are as in Lemma 2.1. In particular $u_*(t) > 0$ for $t \in (0, 1)$. ■

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3 Proof of Theorem 1.4

Let $\lambda \in (0, \Lambda_1)$ be fixed; here Λ_1 is as in (2.12). From (2.5) and (2.31) we have

$$0 = (\varphi_p(u_*'))' + \lambda f^*(t, u_*) \le (\varphi_p(u_*'))' + \lambda f(t, u_*) \text{ for } t \in (0,1);$$

here u_* is as in Lemma 2.8. Thus u_* is a lower solution of problem (1.1).

On the other hand, from (1.5), there exists $M > \sup_{t\in[0,1]} u_*(t)$ with

$$\frac{1}{\varphi_p^{-1}\left(1 + \frac{h(M)}{g(M)}\right)} \int_0^M \frac{dy}{\varphi_p^{-1}(g(y))} > \frac{p-1}{p} 2^{\frac{p}{1-p}}.$$

Let $\rho_n = \frac{a}{2^{n+1}}$, $(n \in N)$. From (1.4), we have $\{\rho_n\}$ is a nonincreasing sequence with $f(t, \rho_n) \ge f(t, a) > 0$, for $t \in [0, 1]$ (here a is as in (1.4)). Thus Lemma 1.6(ii) is true. Now Lemma 1.6 guarantees that (1.1) has a solution $u_1 \in C[0, 1] \cap C^1(0, 1)$ with $\varphi_p(u_1') \in C^1(0, 1)$ and $u_1(t) \ge u_*(t)$ for $t \in [0, 1]$. Also (from Lemma 2.8) $\|u_1\|_\infty \ge \|u_*\|_\infty \ge C_\lambda > a$ (here a is as in (1.4)). Next we prove problem (1.1) has another solution u_2 such that $0 < \|u_2\|_\infty \le a$. We consider the auxiliary equation

$$\begin{cases} -(\varphi_p(u'))' = \lambda g(t, u), & t \in (0, 1) \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where

$$g(t, y) = \begin{cases} f(t, y) & \text{for } (t, y) \in [0, 1] \times (0, a] \\ f(t, a) & \text{for } (t, y) \in [0, 1] \times [a, \infty). \end{cases} \tag{3.2}$$

Then $g(t, y) > b$ for $(t, y) \in [0, 1] \times (0, \infty)$, where b is given in (1.7).

Let $e_0 = \phi$, $e_n = [\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}]$, $n \ge 1$. Also we let

$$\theta_n(t) = \max\left\{\frac{1}{2^{n+1}}, \min\left\{t, 1 - \frac{1}{2^{n+1}}\right\}\right\}, \quad 0 \le t \le 1$$

and

$$f_n(t, y) = \max\{g(\theta_n(t), y), g(t, y)\},$$

Then $f_n \colon [0, 1] \times (0, \infty) \to (0, +\infty)$ is continuous.

Define

$$g_1(t, y) = f_1(t, y)$$
$$g_{n+1}(t, y) = \min\{g_n(t, y), f_{n+1}(t, y)\}.$$

Then $g_n \colon [0, 1] \times (0, \infty) \to (0, \infty)$ is continuous and

$$g(t, y) \le \cdots \le g_{n+1}(t, y) \le g_n(t, y) \le \cdots \le g_1(t, y)$$

for $(t, y) \in [0, 1] \times (0, \infty)$.

Let $\varepsilon_1 = \frac{a}{2}$, and $\varepsilon_n \downarrow 0$. Note that

$$g(t, y) > b, \quad (t, y) \in e_n \times (0, \varepsilon_n]. \tag{3.3}$$

Consider the problem

$$\begin{cases} l - (\varphi_p(u'))' = \lambda g_n(t, u), & t \in (0, 1) \\ u(0) = u(1) = \varepsilon_n. \end{cases} \tag{3.4$_n$}$$

Claim 3.1 *Let $c_n \in (0, \varepsilon_n]$ with $\alpha_n(t) = c_n, 0 \le t \le 1$. Then α_n is a lower solution of problem* $(3.4)_n$

Proof of Claim 3.1 We must show

$$g_n(t, c_n) \ge 0 \text{ for all } c_n \in (0, \varepsilon_n]. \tag{3.5}$$

We prove the validity of the above inequality for each $n \ge 1$, by induction. Let $c_1 \in (0, \varepsilon_1]$. Then (3.3) implies

$$\begin{aligned} g_1(t, c_1) &= f_1(t, c_1) \\ &= \max\{g(\theta_1(t), c_1), g(t, c_1)\} \\ &\ge g(\theta_1(t), c_1) \\ &\ge \min_{t \in e_1} g(t, c_1) \\ &> b > 0. \end{aligned}$$

Suppose that (3.5) holds for a given index $n \ge 1$. Let us check its validity for $n + 1$. If $c_{n+1} \in (0, \varepsilon_{n+1}] \subset (0, \varepsilon_n]$, then

$$\begin{aligned} g_{n+1}(t, c_{n+1}) &= \min\{g_n(t, c_{n+1}), f_{n+1}(t, c_{n+1})\} \\ &\ge \min\{0, \max\{g(\theta_{n+1}(t), c_{n+1}), g(t, c_{n+1})\}\} \\ &\ge \min\{0, b\} \\ &= 0. \end{aligned}$$

■

Claim 3.2 *If $z_n \in C^1[0, 1]$, $\varphi_p(z_n') \in C^1(0, 1)$ is a solution for problem $(3.4)_n$, then*

$$(\varphi_p(z_n'))' + \lambda g_{n+1}(t, z_n(t)) \le 0 \textit{ for } 0 < t < 1$$

(i.e., z_n is an upper solution of $(3.4)_n$).

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Proof of Claim 3.2

$$(\varphi_p(z_n'))' + \lambda g_{n+1}(t, z_n(t)) \le (\varphi_p(z_n'))' + \lambda g_n(t, z_n(t)) \\ = 0 \text{ for } 0 < t < 1. \qquad \blacksquare$$

Claim 3.3 *For all $n \ge 1$, $(3.4)_n$ has at least one solution $u_n \in C^1[0,1]$, $\varphi_p(u_n') \in C^1(0,1)$, with $\varepsilon_{n+1} \le y_{n+1}(t) \le y_n(t)$ for all $0 \le t \le 1$.*

Proof of Claim 3.3 Consider the problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda q(t), & t \in (0,1) \\ u(0) = u(1) = \varepsilon_1. \end{cases} \tag{3.6}$$

where

$$q(t) = \overline{q}(\theta_1(t)) + \overline{q}(t) \quad \text{and} \quad \overline{q}(t) = \max_{u \in [\frac{a}{2}, a]} f(t, y) \text{ for } t \in [0,1].$$

It is easy to check that (3.6) has a solution

$$z_0(t) = \begin{cases} \varepsilon_1 + \int_0^t \varphi_p^{-1}\big(\int_s^A \lambda q(r)\,dr\big)\,ds & 0 \le t \le A, \\ \varepsilon_1 + \int_t^1 \varphi_p^{-1}\big(\int_A^s \lambda q(r)\,dr\big)\,ds & A \le t \le 1, \end{cases}$$

where A satisfies

$$\int_0^A \varphi_p^{-1}\Big(\int_s^A q(r)\,dr\Big)\,ds = \int_A^1 \varphi_p^{-1}\Big(\int_A^s q(r)\,dr\Big)\,ds.$$

Let

$$\Lambda_2 = \varphi_p\Big(\frac{C^{-1}a}{2}\Big)$$

where a is as in (1.4) and

$$C = \max\Big\{\int_0^{\frac{1}{2}} \varphi_p^{-1}\Big(\int_s^{\frac{1}{2}} q(r)\,dr\Big)\,ds, \int_{\frac{1}{2}}^1 \varphi_p^{-1}\Big(\int_{\frac{1}{2}}^s q(r)\,dr\Big)\,ds\Big\}.$$

Let

$$\Lambda = \min\{\Lambda_1, \Lambda_2\}, \tag{3.7}$$

where Λ_1 is as in (2.12). Then for

$$\lambda \in (0, \Lambda] \tag{3.8}$$

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we have

$$\|z_0\|_0 = \varepsilon_1 + \varphi_p^{-1}(\lambda)\int_0^A \varphi_p^{-1}\Big(\int_s^A q(r)\,dr\Big)\,ds \tag{3.9}$$
$$= \varepsilon_1 + \varphi_p^{-1}(\lambda)\int_A^1 \varphi_p^{-1}\Big(\int_A^s q(r)\,dr\Big)\,ds$$
$$\le \frac{a}{2} + \varphi_p^{-1}(\lambda)C$$
$$\le \frac{a}{2} + \varphi_p^{-1}\Big(\varphi_p\Big(\frac{C^{-1}a}{2}\Big)\Big)C$$
$$\le a.$$

Moreover, $z_0 \in C^1[0,1]$ with $\varphi_p(z_0') \in C^1(0,1)$, and $z_0(t) \ge \varepsilon_1 = \frac{a}{2}$ for $0 \le t \le 1$. On the other hand,

$$(\varphi_p(z_0'))' + \lambda g_1(t,z_0) = -\lambda q(t) + \lambda g_1(t,z_0)$$
$$= -\lambda q(t) + \lambda \min\{f(\theta_1(t),z_0), f(t,z)\}$$
$$\le 0.$$

Thus, z_0 is an upper solution for problem $(3.4)_1$.

By Claim 3.1, $\alpha_n(t) = c_n \in (0,\varepsilon_n]$, $0 \le t \le 1$, is a lower solution of problem $(3.4)_n$ and

$$\varepsilon_1 \le z_0(t) \text{ for all } 0 \le t \le 1.$$

From [11, Lemma 4], we deduce that $(3.4)_1$ has at least one solution $z_1 \in C^1[0,1]$, such that $\varphi_p(z_1') \in C^1(0,1)$ and

$$\varepsilon_1 \le z_1(t) \le z_0(t) \text{ for all } 0 \le t \le 1.$$

Suppose now that $(3.4)_n$ has a solution $z_n \in C^1[0,1]$ such that $\varphi_p(z_n') \in C^1(0,1)$ and

$$\varepsilon_n \le z_n(t) \text{ for all } 0 \le t \le 1.$$

By Claim 3.2, $z_n(t)$ is an upper solution for problem $(3.4)_n$. Observe also that

$$\varepsilon_{n+1} \le \varepsilon_n \le z_n(t) \text{ for all } 0 \le t \le 1,$$

so [11, Lemma 4] guarantees that $(3.4)_n$ has at least one solution $z_{n+1} \in C^1[0,1]$, such that $\varphi_p(z_{n+1}') \in C^1(0,1)$ and $\varepsilon_{n+1} \le z_{n+1}(t) \le z_n(t)$ for all $0 \le t \le 1$. ■

Claim 3.4 *Suppose there exist $\nu^* \in C^1[0,1]$, $\nu^*(0) = \nu^*(1) = 0$, $\nu^*(t) > 0$, $0 < t < 1$ such that for all $h\colon (0,1)\times(0,\infty) \to (0,\infty)$ and $\bar z \in C^1[0,1]$, $\bar z(t) > 0$, $0 < t < 1$, $z(0) \ge 0$, $z(1) \ge 0$ the following conditions are satisfied:*

(i) $h(t,y) \ge g(t,y)$, $(t,y) \in (0,1)\times(0,\infty)$;

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(ii) $(\varphi_p(\overline{z}'(t)))' + \lambda h(t, \overline{z}(t)) = 0$, $0 < t < 1$.

Then $\overline{z}(t) \geq \nu^*(t)$, $0 \leq t \leq 1$.

Proof of Claim 3.4 Using [11, Lemma 2], we know there exists a function $\nu \in C^1[0,1]$, such that $\varphi_p(\nu') \in C^1(0,1)$ $M = \max_{0 \leq t \leq 1} |(\varphi_p(\nu'))'| > 0$, and $0 < \nu(t) < \varepsilon_n$ for all $t \in e_n \backslash e_{n-1}$, $n \geq 1$.

Let $m = \min\{1, (b/M)^{1/(p-1)}\}$. We prove

$$\overline{z}(t) - m\nu(t) \geq 0 \text{ for all } 0 \leq t \leq 1. \tag{3.10}$$

Suppose that there exists $t_0 \in (0,1)$ with

$$\min_{0 \leq t \leq 1} \{\overline{z}(t) - m\nu(t)\} = \overline{z}(t_0) - m\nu(t_0) < 0. \tag{3.11}$$

Note $\overline{z}'(t_0) - m\nu'(t_0) = 0$. Also there exists an $\varepsilon > 0$, with $\overline{z}'(t_\varepsilon) - m\nu'(t_\varepsilon) \geq 0$ for $t_\varepsilon \in (t_0, t_0 + \varepsilon)$. Since φ_p is an increasing function, we get

$$\begin{aligned}(\varphi_p(\overline{z}'(t)))'|_{t=t_0} &= \lim_{\varepsilon \to 0^+} \frac{\varphi_p(\overline{z}'(t_\varepsilon)) - \varphi_p(\overline{z}'(t_0))}{t_\varepsilon - t_0} \\ &\geq \lim_{\varepsilon \to 0^+} \frac{\varphi_p(m\nu'(t_\varepsilon)) - \varphi_p(m\nu'(t_0))}{t_\varepsilon - t_0} \\ &= (\varphi_p(m\nu'(t)))'|_{t=t_0}.\end{aligned}$$

Suppose $t_0 \in e_n \backslash e_{n-1}$. Then $0 < \nu(t_0) < \varepsilon_n$. By (3.11) we obtain $0 < \overline{z}(t_0) < m\nu(t_0) < \varepsilon_n$. Thus (3.3) with the above yields

$$\begin{aligned}b &< g(t_0, \overline{z}(t_0)) \leq h(t_0, \overline{z}(t_0)) \\ &= -(\varphi_p(\overline{z}'(t)))'|_{t=t_0} \leq -(\varphi_p(m\nu'(t)))'|_{t=t_0} \\ &\leq m^{p-1} |(\varphi_p(\nu'(t)))'|_{t=t_0}| \\ &\leq m^{p-1} M \\ &\leq b,\end{aligned}$$

a contradiction.

Let $\nu^*(t) \equiv m\nu(t)$.

By Claim 3.3, problem $(3.4)_n$ has at least one solution $u_n \in C^1[0,1]$, such that $\varphi_p(u_n') \in C^1(0,1)$, with

$$0 < \varepsilon_{n+1} \leq u_{n+1} \leq u_n \leq \cdots \leq u_1, \quad 0 \leq t \leq 1 \tag{3.12}$$

and

$$u_n(0) = u_n(1) = \varepsilon_n. \tag{3.13}$$

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By Claim 3.4, there exists $\nu^* \in C^1[0,1]$, $\nu^*(0) = \nu^*(1) = 0$, and $\nu^*(t) > 0$ for $0 < t < 1$ such that $u_n(t) \geq \nu^*(t)$, $0 \leq t \leq 1$, $n \geq 1$. Let

$$u(t) = \lim_{n\to\infty} u_n(t), \quad 0 < t < 1.$$

Now $u(t) \geq \nu^*(t)$ for $t \in (0,1)$. Also $u(0) = u(1)$ and $u(t) > 0$ for $t \in (0,1)$.

Now let $[c,d] \subset (0,1)$ be a compact interval. There is an index n^* such that $[c,d] \subset e_n$ for all $n > n^*$ and therefore, for these $n > n^*$,

$$(\varphi_p(u_n'(t))) + \lambda g(t, u_n(t)) = 0, \quad c \leq t \leq d. \tag{3.14}$$

On the other hand, $\nu^* \in C^1[0,1]$ and $\nu^*(t) > 0$ for all $0 < t < 1$. Let $r = \min_{c\leq t\leq d} \nu^*(t) > 0$. Moreover, by (3.2) there exist $q_r \in C[0,1]$ such that

$$g(t,y) \leq q_r(t), \quad (t,y) \in [0,1] \times [r, +\infty).$$

It is easy to see that there exists a continuous function $\widetilde{g}\colon [0,1] \times R \to R$ such that

$$|\widetilde{g}(t,y)| \leq q_r(t), \quad (t,y) \in (0,1) \times R,$$

and

$$\widetilde{g}(t,y) = g(t,y), \quad (t,y) \in (0,1) \times [r, +\infty).$$

It is clear that $u_n(t) \geq r$, $c \leq t \leq d$ for all $n \geq 1$. Moreover,

$$(\varphi_p(u_n'(t)))' + \lambda \widetilde{g}(t, u_n(t)) = 0, \quad c \leq t \leq d. \tag{3.15}$$

Now define $N_1\colon C^1[c,d] \to C^1[c,d]$ by

$$N_1(u(t)) = u(c) + \int_c^t \varphi_p^{-1}\Big(A_u + \int_s^d \lambda \widetilde{g}(\tau, u(\tau))\, d\tau\Big)\, ds,$$

where A_u is such that

$$\int_c^d \varphi_p^{-1}\Big(A_u + \int_s^d \lambda \widetilde{g}(\tau, u(\tau))\, d\tau\Big)\, ds = u(d) - u(c).$$

By (3.15), we have $N_1(u_n(t)) = u_n(t)$, $c \leq t \leq d$ for $n \geq n^*$.

Next, we notice for $n \geq n^*$ that

$$\max_{c\leq t\leq d} |u_n(t)| \leq \max_{c\leq t\leq d} |u_1(t)| < +\infty.$$

It is easy to see that there exists a subsequence S of $\{n_* + 1, n_* + 2, \dots\}$ with

$$\max_{c\leq t\leq d} |u_n(t) - u(t)| \to 0, \quad \text{and} \quad \max_{c\leq t\leq d} |u_n'(t) - u'(t)| \to 0 \text{ as } n \to \infty.$$

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Consequently, $\varphi_p(u') \in C^1(c,d)$, and

$$(\varphi_p(u'(t)))' + \lambda g(t,u(t)) = 0, \quad c \le t \le d.$$

Since $[c,d] \subset (0,1)$ is arbitrary, we find that

$$u \in C^1(0,1) \quad \text{and} \quad (\varphi_p(u'(t)))' + \lambda g(t,u(t)) = 0 \text{ for all } 0 < t < 1.$$

It remains to show the continuity of $u(t)$ at $t = 0$ and $t = 1$. This follows immediately from the fact that $u_n(t) \downarrow u(t)$ and $u_n(0) = u_n(1) = \varepsilon_n \downarrow 0$. Thus $u \in C[0,1]$.

On the other hand, (3.12) and (3.9) yield

$$0 < u(t) \le u_1(t) \le z_0(t) \le \|z_0\|_0 \le a \text{ for } t \in (0,1).$$

Then

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda f(t,u(t)) = 0 \text{ for all } 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases}$$

As a result $u(\cdot)$ is another solution of problem (1.1) with $0 < u(t) \le a$ on $[0,1]$. The proof of Theorem 1.4 is complete. ■

Proof of Theorem 1.5 By (1.6) there exist $a \in (0,\infty)$ such that

$$f(t,y) \ge f(t,a) \text{ for } (t,y) \in [0,1] \times [y,\infty).$$

Then the conditions of Theorem 1.4 are satisfied.

Example 1 Consider the problem

$$\begin{cases} -u'' = \lambda\left(\frac{1}{u} + q(u) - \mu^2\right) \text{ for all } 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{3.16}$$

where $\mu > 1$.

Define $\{x_n\}_{n=1}^{\infty}$ as $x_1 = 2$, $x_{2n} = x_{2n-1}^4$, $x_{2n+1} = x_{2n} + 1$, and

$$q(y) = \begin{cases} y^2 & \text{if } y \in [0,2], \\ x_{2n-1}^2 & \text{if } y \in [x_{2n-1}, x_{2n}], \\ \dfrac{x_{2n+1}^2 - \sqrt{x_{2n}}}{x_{2n+1} - x_{2n}}(y - x_{2n}) + \sqrt{x_{2n}} & \text{if } y \in [x_{2n}, x_{2n+1}]. \end{cases}$$

Then, (3.16) has two solutions $u_i \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(u_i') \in C^1(0,1)$ if $\lambda > 0$ is small enough.

To see this, we will apply Theorem 1.5 with

$$M = \mu^2, g(y) = \frac{1}{y} \quad \text{and} \quad h(y) = q(y) + \mu^2.$$

Notice

$$f(t,y) = \frac{1}{y} + q(y) - \mu^2 \geq -M \text{ for } (t,y) \in [0,1] \times (0,\infty).$$

Clearly (1.2) is satisfied. Now

$$\begin{aligned}\widetilde{f}_n(x_{2n+1}) &= \inf\{f(t,s) : (t,s) \in [0,1] \times [x_{2n+1},\infty)\}\\ &= x_{2n+1}^2 - \mu \text{ for } n \in \{2,3,\dots\}\end{aligned}$$

and

$$\lim_{n\to\infty} \frac{\widetilde{f}_n(x_{2n+1})}{x_{2n+1}} = \infty.$$

Then

$$\limsup_{y\to\infty} \frac{\widetilde{f}(y)}{y} = \infty.$$

On the other hand,

$$\lim_{y\to 0^+} f(t,y) = \infty \text{ uniformly on } [0,1].$$

Clearly (1.4), (H4)(i) and (ii) are satisfied. Let $D \geq 0$ be fixed. Let $M_n = x_{2n}$ for $n \in \{2,3,\dots\}$. Then $\lim_{n\to\infty} M_n = \infty$ and

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{1+\frac{h(M_n)+D}{g(M_n)}} \int_0^{M_n} \frac{dy}{g(y)} &= \lim_{n\to\infty} \frac{1}{1+M_n(h(M_n)+D)} \int_0^{M_n} y\,dy\\ &= \lim_{n\to\infty} \frac{x_{2n}^2}{2} \frac{1}{1+x_{2n}\left(\frac{x_{2n+1}^2-\sqrt{x_{2n}}}{x_{2n+1}-x_{2n}}(x_{2n}-x_{2n})+\sqrt{x_{2n}}+D\right)}\\ &= \lim_{n\to\infty} \frac{x_{2n}^2}{2(1+x_{2n}^{3/2}+Dx_{2n})}\\ &= \infty\\ &> \frac{1}{8} \quad \text{for } n \in \{2,3,\dots\}.\end{aligned}$$

The condition (1.5) is satisfied.

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