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A NOTE ON MATRIX APPROXIMATION IN THE THEORY OF MULTIPLICATIVE DIOPHANTINE APPROXIMATION

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Abstract

We prove the Hausdorff measure version of the matrix form of Gallagher's theorem in the inhomogeneous setting, thereby proving a conjecture posed by Hussain and Simmons ['The Hausdorff measure version of Gallagher's theorem—closing the gap and beyond', *J. Number Theory* **186** (2018), 211–225].

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1. Introduction

Throughout, let $m \geq 1$ be an integer, $\mathbb{I}^m$ the unit cube $[0,1]^m$ and $\|\cdot\|$ the distance to the nearest integer in $\mathbb{Z}$. Let $\psi : \mathbb{N} \to [0,\infty)$ be a monotonically decreasing function, which we call an approximating function, and let $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ be a given point in $\mathbb{R}^m$. Denote by $\mathcal{M}_m^{\mathbf{y}}(\psi)$ the set of $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ for which

$$\|qx_1 - y_1\| \cdot \|qx_2 - y_2\| \cdots \|qx_m - y_m\| < \psi(q)$$

holds for infinitely many $q \in \mathbb{N}$, that is, the set of multiplicatively ψ-approximable points.

Multiplicative Diophantine approximation deals with the properties of the sets $\mathcal{M}_m^{\mathbf{y}}(\psi)$ and is an active area of research. In particular, the long-standing conjecture of Littlewood that $\mathcal{M}_2^{0}(q \mapsto \varepsilon \cdot q^{-1}) = \mathbb{R}^2$ for any $\varepsilon > 0$ has attracted much attention. A natural problem is to determine the 'size' of the set of multiplicatively ψ-approximable points.

Throughout the paper, f denotes a dimension function, that is, a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(r) \to 0$ as $r \to 0$, and $\mathcal{H}^f$ denotes the f-dimensional Hausdorff measure. When $f(r) = r^m$ for an integer m, then we use the notation $\mathcal{H}^m$ to denote the normalised Lebesgue measure such that $\mathcal{H}^m(\mathbb{I}^m) = 1$.

In the homogeneous multiplicative case, that is, $\mathbf{y} = \mathbf{0}$, Gallagher [5] proved the following result for the Lebesgue measure.

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THEOREM 1.1 (Gallagher [5]). *Let $\psi : \mathbb{N} \to [0, \infty)$ be an approximating function. Then, for any $m \geq 1$,*

$$\mathcal{H}^m(\mathcal{M}_m^{\mathbf{0}}(\psi) \cap \mathbb{I}^m) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) \log(q)^{m-1} < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) \log(q)^{m-1} = \infty. \end{cases}$$

For the inhomogeneous setup, there is the following result for the convergence part:

$$\mathcal{H}^m(\mathcal{M}_m^{\mathbf{y}}(\psi) \cap \mathbb{I}^m) = 0 \quad \text{if } \sum_{q=1}^{\infty} \psi(q) \log(q)^{m-1} < \infty.$$

This is an easy consequence of the Borel–Cantelli lemma. However, the result for the divergence part is still open. Partial results can be found in Beresnevich *et al.* [2] and Chow [4].

For the s-Hausdorff measure, with s not an integer, Beresnevich and Velani [3] for $m = 2$ and Hussain and Simmons [6] for $m \geq 2$ proved a 0–∞ law depending upon the convergence or divergence of a certain series. In [6], the authors also considered the case of linear forms where ψ is replaced by a multivariable function $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \to [0, \infty)$. More precisely, they considered the set

$$\mathcal{M}_{n,m}^{\mathbf{y}}(\Psi) = \Big\{ \mathbf{x} \in \mathbb{R}^{nm} : \prod_{i=1}^{m} \|\mathbf{q}\mathbf{x}^{(i)} - y_i\| < \Psi(\mathbf{q}) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \Big\},$$

where 'i.m.' is the abbreviation for 'infinitely many'. They presented a convergence result by showing that

$$\mathcal{H}^f(\mathcal{M}_{n,m}^{\mathbf{y}}(\Psi) \cap \mathbb{I}^{nm}) = 0 \quad \text{if } \sum_{\mathbf{q} \in \mathbb{Z}^n} |\mathbf{q}|^{nm} \Psi(\mathbf{q})^{-nm+1} f\Big(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\Big) < \infty, \tag{1.1}$$

where f is a dimension function satisfying $f(y) \leq C(x/y)^s \cdot f(x)$ for $0 < x < y$ and $x^{-nm+1} f(x)$ is monotonically increasing. For the divergence part, they asked whether the divergence of the series in (1.1) will yield the full $\mathcal{H}^f$ measure.

CONJECTURE 1.2 (Hussain and Simmons [6]). *Let $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$, where $\psi : \mathbb{N} \to [0, \infty)$ is a monotonically decreasing function. Let f be a dimension function such that $x \mapsto x^{-nm+1} f(x)$ is monotonically increasing. Then*

$$\mathcal{H}^f(\mathcal{M}_{n,m}^{\mathbf{y}}(\Psi) \cap \mathbb{I}^{nm}) = \infty \quad \text{if } \sum_{\mathbf{q} \in \mathbb{Z}^n} |\mathbf{q}|^{nm} \Psi(\mathbf{q})^{-nm+1} f\Big(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\Big) = \infty,$$

where $|\cdot|$ denotes the sup norm.

We prove this conjecture, which completes the Hausdorff measure theory for $\mathcal{M}_{n,m}^{\mathbf{y}}(\Psi)$.

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THEOREM 1.3. *Under the conditions given in the above conjecture,*

$$\mathcal{H}^f(\mathcal{M}^{\mathbf{y}}_{n,m}(\Psi)\cap\mathbb{I}^{nm})=\infty\quad\text{if}\ \sum_{\mathbf{q}\in\mathbb{Z}^n}|\mathbf{q}|^{nm}\Psi(\mathbf{q})^{-nm+1}f\Big(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\Big)=\infty.$$

2. Proof of Theorem 1.3

2.1. Preliminaries. The proof is a combination of the mass transference principle for linear forms [1] and the slicing lemma [7, Proposition 7.9].

Let k and l be two nonnegative integers with $k\geq l$. Let $\mathcal{R}=(R_j)_{j\in\mathbb{N}}$ be a family of planes in $\mathbb{R}^k$ of common dimension l. Let $d=k-l$ be the codimension of R_j. For every $j\in\mathbb{N}$ and $\delta\geq 0$, define

$$\nabla(R_j,\delta)=\{x\in\mathbb{R}^k:\operatorname{dist}(x,R_j)<\delta\},$$

where $\operatorname{dist}(x,R_j)=\inf\{\|x-y\|:y\in R_j\}$. Let $\Upsilon=\{\gamma_j\}$ be a countable sequence of nonnegative real numbers such that $\gamma_j\to 0$ as $j\to\infty$. Consider

$$\nabla(\Upsilon)=\{x\in\mathbb{R}^k:x\in\nabla(R_j,\gamma_j)\text{ for infinitely many } j\in\mathbb{N}\}.$$

THEOREM 2.1 (Mass transference principle for systems of linear forms [1]). *Let $\mathcal{R}$ and Υ be as defined above. Let h and $g:r\mapsto g(r)=r^{-l}h(r)$ be dimension functions such that $r^{-k}h(r)$ is monotonic and let Ω be a ball in $\mathbb{R}^k$. Suppose that, for any ball B in Ω,*

$$\mathcal{H}^k(B\cap\nabla(g(\Upsilon)^{1/d}))=\mathcal{H}^k(B).$$

Then, for any ball B in Ω,

$$\mathcal{H}^h(B\cap\nabla(\Upsilon))=\mathcal{H}^h(B).$$

LEMMA 2.2 (Slicing lemma [7]). *Fix $k,l\in\mathbb{N}$ with $l<k$. Let g be a dimension function and $f(r)=r^lg(r)$ (so that f is necessarily a dimension function). Let A be a Borel subset of $\mathbb{R}^k$ and suppose that the set*

$$\{\mathbf{x}\in\mathbb{R}^l:\mathcal{H}^g(\{\mathbf{y}\in\mathbb{R}^{k-l}:(x,y)\in A\})=\infty\}$$

has positive $\mathcal{H}^l$-measure. Then $\mathcal{H}^f(A)=\infty$.

2.2. Proofs. The convergence part can be proved by exactly the same methods as in [6]. We focus on the divergence part.

It is clear that $\mathcal{M}^{\mathbf{y}}_{n,m}(\Psi)\cap\mathbb{I}^{nm}$ contains certain slices of the form $\mathcal{M}^{y}_{n,1}(\Psi)\times\mathbb{I}^{n(m-1)}$. By the slicing lemma, we only need to prove the following result.

PROPOSITION 2.3. *Let $\Psi(\mathbf{q})=\psi(|\mathbf{q}|)$, where $\psi:\mathbb{N}\to[0,\infty)$ is a monotonically decreasing function. Let h and $g:r\to g(r)=r^{-n+1}h(r)$ be two dimension functions such that $r^{-n}h(r)$ is monotonic. Then*

$$\mathcal{H}^h(\mathcal{M}^{\mathbf{y}}_{n,1}(\Psi)\cap\mathbb{I}^{n})=\infty\quad\text{if}\ \sum_{\mathbf{q}\in\mathbb{Z}^n}|\mathbf{q}|^{n}\Psi(\mathbf{q})^{-n+1}h\Big(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\Big)=\infty.$$

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First we introduce some notation. For any $(p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and $y \in \mathbb{I}$, let

$$R_{p,\mathbf{q}} = \{\mathbf{x} \in \mathbb{I}^n : q_1x_1 + q_2x_2 + \cdots + q_nx_n - y - p = 0\},$$

which is a plane of dimension $l = n - 1$ and codimension $d = 1$. For $\delta \geq 0$, define

$$\triangledown(R_{p,\mathbf{q}}, \delta) = \{\mathbf{x} \in \mathbb{I}^n : \mathrm{dist}(x, R_{p,\mathbf{q}}) < \delta\},$$

where

$$\mathrm{dist}(\mathbf{x}, R_{p,\mathbf{q}}) = \inf_{\mathbf{z}\in R_{p,\mathbf{q}}} \|\mathbf{x} - \mathbf{z}\| = \sqrt{n}\,\frac{|q_1x_1 + \cdots + q_nx_n - y - p|}{|\mathbf{q}|_2}.$$

Note that if $\Psi(\mathbf{q}) \geq 1$ for infinitely many $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, then $\mathcal{M}^{\mathbf{y}}_{n,1}(\Psi) = \mathbb{I}^n$ and the divergence part of Theorem 1.3 is trivial. Hence, without loss of generality, we may assume that $\Psi(\mathbf{q}) < 1$ for all $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

Proof of Proposition 2.3. Define

$$\mathcal{R} = \{(p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^n \setminus \{\mathbf{0}\} : |p| \leq C|\mathbf{q}|\}, \quad \Upsilon = \left\{r_{p,\mathbf{q}} = \frac{\Psi(\mathbf{q})}{|\mathbf{q}|} : (p, \mathbf{q}) \in \mathcal{R}\right\},$$

where

$$C = \max\left\{2n, \sup_{\mathbf{q}\in\mathbb{Z}^n} h\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)\right\}.$$

Note that, since h is increasing and $\Psi(\mathbf{q}) \leq 1$, the constant C is finite. Now, for each $(p, \mathbf{q}) \in \mathcal{R}$,

$$\begin{aligned}\triangledown(R_{p,\mathbf{q}}, r_{p,\mathbf{q}}) \cap \mathbb{I}^n &= \left\{\mathbf{x} \in \mathbb{I}^n : \sqrt{n}\,\frac{|\mathbf{q}\mathbf{x} - y - p|}{|\mathbf{q}|_2} < \frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right\} \\ &= \left\{\mathbf{x} \in \mathbb{I}^n : |\mathbf{q}\mathbf{x} - y - p| < \frac{|\mathbf{q}|_2\Psi(\mathbf{q})}{|\mathbf{q}|}\right\} \\ &\subset \{\mathbf{x} \in \mathbb{I}^n : |\mathbf{q}\mathbf{x} - y - p| < \Psi(\mathbf{q})\}\end{aligned}$$

since $|\mathbf{q}|_2 \leq \sqrt{n}|\mathbf{q}|$. It follows that

$$\triangledown(\Upsilon) \cap \mathbb{I}^n \subset \mathcal{M}^{\mathbf{y}}_{n,1}(\Psi) \cap \mathbb{I}^n \subset \mathbb{I}^n,$$

where

$$\triangledown(\Upsilon) = \limsup_{|\mathbf{q}|\to\infty,\,(p,\mathbf{q})\in\mathcal{R}} \triangledown(R_{p,\mathbf{q}}, r_{p,\mathbf{q}}).$$

Therefore, it suffices to show that $\mathcal{H}^h(\triangledown(\Upsilon) \cap \mathbb{I}^n) = \mathcal{H}^h(\mathbb{I}^n)$.

Next, let $\theta : \mathbb{N} \to \mathbb{R}^+$ be defined by

$$\theta(|\mathbf{q}|) = \frac{|\mathbf{q}|}{\sqrt{n}} \cdot g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right).$$

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Note that

$$\begin{aligned}\triangledown(R_{p,\mathbf{q}}, g(r_{p,\mathbf{q}})) \cap \mathbb{I}^n &= \left\{\mathbf{x} \in \mathbb{I}^n : \frac{\sqrt{n}|\mathbf{q}\mathbf{x} - y - p|}{|\mathbf{q}|_2} < g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)\right\} \\ &= \left\{\mathbf{x} \in \mathbb{I}^n : |\mathbf{q}\mathbf{x} - y - p| < \frac{|\mathbf{q}|_2}{\sqrt{n}} \cdot g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)\right\} \\ &\supset \left\{\mathbf{x} \in \mathbb{I}^n : |\mathbf{q}\mathbf{x} - y - p| < \frac{|\mathbf{q}|}{\sqrt{n}} \cdot g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)\right\},\end{aligned}$$

where the last inclusion follows from the fact that $|\mathbf{q}| \le |\mathbf{q}|_2$.

Observe that if $\{\mathbf{x} \in \mathbb{I}^n : |\mathbf{q}\mathbf{x} - y - p| < \theta(|\mathbf{q}|)\} \neq \emptyset$, then $|p| \le C|\mathbf{q}|$. This is why we define $\mathcal{R}$ as above. It follows that

$$\triangledown\,(g(\Upsilon)) \cap \mathbb{I}^n \supset \mathcal{M}^{\mathbf{y}}_{n,1}(\theta) \cap \mathbb{I}^n. \tag{2.1}$$

The required divergence condition ensures that

$$\sum_{|\mathbf{q}|=1}^{\infty} |\mathbf{q}|^{n-1} \cdot \theta(|\mathbf{q}|) = \sum_{|\mathbf{q}|=1}^{\infty} \frac{|\mathbf{q}|^{2n-1}}{\sqrt{n}} \cdot g\left(\frac{\psi(\mathbf{q})}{|\mathbf{q}|}\right) = \sum_{\mathbf{q}\in\mathbb{Z}^n} \frac{|\mathbf{q}|^n}{\sqrt{n}} \cdot \Psi(\mathbf{q})^{-n+1} h\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right) = \infty.$$

Thus, by the inhomogeneous Khintchine–Groshev theorem [8],

$$\mathcal{H}^n(\mathcal{M}^{\mathbf{y}}_{n,1}(\theta) \cap \mathbb{I}^n) = 1 \quad \text{and so} \quad \mathcal{H}^n(\triangledown(g(\Upsilon)) \cap \mathbb{I}^n) = 1.$$

Finally, we apply the mass transference principle for systems of linear forms (Theorem 2.1). For any ball $B \subset \mathbb{I}^n$,

$$\mathcal{H}^h(B \cap \triangledown(\Upsilon)) = \mathcal{H}^h(B).$$

In particular,

$$\mathcal{H}^h(\triangledown(\Upsilon) \cap \mathbb{I}^n) = \infty \quad \text{and so} \quad \mathcal{H}^h(\mathcal{M}^{\mathbf{y}}_{n,1}(\Psi)) = \infty,$$

by the inclusion (2.1). This proves Proposition 2.3. □

Let $f(r) = r^{n(m-1)}h(r)$, which is clearly a dimension function. Note that

$$\sum_{\mathbf{q}\in\mathbb{Z}^n} |\mathbf{q}|^{nm} \cdot \Psi(\mathbf{q})^{-nm+1} \cdot f\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right) = \sum_{\mathbf{q}\in\mathbb{Z}^n} |\mathbf{q}|^n \cdot \Psi(\mathbf{q})^{-n+1} \cdot h\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right) = \infty.$$

So, by Proposition 2.3,

$$\mathcal{H}^h(\mathcal{M}^{y}_{n,1}(\Psi)) = \infty.$$

On the other hand, by using the slicing lemma (Lemma 2.2) and the fact that

$$\mathcal{M}^{y}_{n,1}(\Psi) \times \mathbb{I}^{n(m-1)} \subset \mathcal{M}^{\mathbf{y}}_{n,m}(\Psi) \cap \mathbb{I}^{nm},$$

$$\mathcal{H}^f(\mathcal{M}^{\mathbf{y}}_{n,m}(\Psi) \cap \mathbb{I}^{mn}) \ge \mathcal{H}^f(\mathcal{M}^{y}_{n,1}(\Psi) \times \mathbb{I}^{n(m-1)}) = \mathcal{H}^h(\mathcal{M}^{y}_{n,1}(\Psi)) = \infty.$$

Thus, the proof of Theorem 1.3 is complete.

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