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Toeplitz Algebras and Extensions of Irrational Rotation Algebras

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Abstract. For a given irrational number θ, we define Toeplitz operators with symbols in the irrational rotation algebra $\mathcal{A}_\theta$, and we show that the C^*-algebra $\mathcal{T}(\mathcal{A}_\theta)$ generated by these Toeplitz operators is an extension of $\mathcal{A}_\theta$ by the algebra of compact operators. We then use these extensions to explicitly exhibit generators of the group $KK^1(\mathcal{A}_\theta, \mathbb{C})$. We also prove an index theorem for $\mathcal{T}(\mathcal{A}_\theta)$ that generalizes the standard index theorem for Toeplitz operators on the circle.

Let $\mathbb{T}$ denote the unit circle equipped with Haar measure, let $H^2(\mathbb{T})$ be the subspace of $L^2(\mathbb{T})$ consisting of functions that have a holomorphic extension to the unit disk, and let $P\colon L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal projection. The elements of $L^\infty(\mathbb{T})$ act on $L^2(\mathbb{T})$ by multiplication, and for f in $L^\infty(\mathbb{T})$, the operator $T_f = Pf\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is called the Toeplitz operator with symbol f. These operators have been extensively studied by many researchers, and Toeplitz operators give rise to interesting C^*-algebras by taking a C^*-subalgebra $\mathcal{A}$ of $L^\infty(\mathbb{T})$ and looking at the C^*-subalgebra $\mathcal{T}(\mathcal{A})$ of $\mathcal{B}(H^2(\mathbb{T}))$ generated by the set $\{T_f : f \in \mathcal{A}\}$.

Another collection of C^*-algebras that has attracted a great deal of attention are the irrational rotation algebras. Given an irrational number θ, we define $\mathcal{A}_\theta$ in the standard way: let V_θ be the unitary operator on $L^2(\mathbb{T})$ defined by $(V_\theta f)(z) = f(e^{-2\pi i\theta}z)$, and take $\mathcal{A}_\theta$ to be the C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ generated by $C(\mathbb{T})$ and V_θ.

In this paper we construct extensions of $\mathcal{A}_\theta$ by considering the C^*-algebra generated by a class of generalized Toeplitz operators. Specifically, for each X in $\mathcal{A}_\theta$, define $T_X = PX\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$, and let $\mathcal{T}(\mathcal{A}_\theta)$ denote the C^*-subalgebra of $\mathcal{B}(H^2(\mathbb{T}))$ generated by the set $\{T_X : X \in \mathcal{A}_\theta\}$. We begin our study of these Toeplitz operators by showing that the norms of T and X are equal, which generalizes a classical result about Toeplitz operators. Next, we show that $\mathcal{T}(\mathcal{A}_\theta)$ is an extension of $\mathcal{A}_\theta$ by the algebra of compact operators, and that $KK^1(\mathcal{A}_\theta, \mathbb{C})$ is generated as a group by this extension and a pullback of the corresponding extension of $\mathcal{A}_{-\theta}$ by the compacts. Finally, we consider the index theory of $\mathcal{T}(\mathcal{A}_\theta)$. It is well known that an element T in $\mathcal{T}(C(\mathbb{T}))$ is Fredholm if and only if its symbol is invertible, and in this case, the index of T equals minus the winding number of its symbol; we generalize this theorem to operators in $\mathcal{T}(\mathcal{A}_\theta)$.

Proposition 1 *For all X in $\mathcal{A}_\theta$, $\|T_X\| = \|X\|$.*

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Proof Obviously $\|T_X\| \le \|X\|$. To obtain the reverse inequality, consider $X \in \mathcal{A}_\theta$ of the form

$$X = \sum_{k=m}^{M} \sum_{l=n}^{N} a_{kl} z^k V_\theta^l.$$

Fix $\epsilon > 0$, and choose $\rho = \sum_{j=r}^{R} c_j z^j$ in $L^2(\mathbb{T})$ such that $\|\rho\|_2 = 1$ and $\|X\rho\|_2 > \|X\| - \frac{\epsilon}{2}$. Then

$$\begin{aligned} X\rho &= \Big(\sum_{k=m}^{M} \sum_{l=n}^{N} a_{kl} z^k V_\theta^l \Big) \Big(\sum_{j=r}^{R} c_j z^j \Big) \\ &= \sum_{k=m}^{M} \sum_{l=n}^{N} \sum_{j=r}^{R} a_{kl} c_j e^{-2\pi i l j \theta} z^{k+j} \\ &= \sum_{h=m+r}^{M+R} \Big(\sum_{l=n}^{N} \sum_{j=r}^{R} a_{(h-j)l} c_j e^{-2\pi i l j \theta} \Big) z^h, \end{aligned}$$

whence

$$\|X\rho\|_2^2 = \sum_{h=m+r}^{M+R} \Big| \sum_{l=n}^{N} \sum_{j=r}^{R} a_{(h-j)l} c_j e^{-2\pi i l j \theta} \Big|^2 .$$

A similar computation shows that for every natural number q,

$$\begin{aligned} \|X(\rho z^q)\|_2^2 &= \sum_{h=m+r}^{M+R} \Big| \sum_{l=n}^{N} \sum_{j=r}^{R} a_{(h-j)l} c_j e^{-2\pi i l (j+q) \theta} \Big|^2 \\ &= \sum_{h=m+r}^{M+R} \Big| \sum_{l=n}^{N} \sum_{j=r}^{R} a_{(h-j)l} c_j e^{-2\pi i l j \theta} \left(e^{-2\pi i q \theta} \right)^l \Big|^2 . \end{aligned}$$

By choosing q so that $q\theta$ is sufficiently close to an integer, we can make $(e^{-2\pi i q\theta})^l$ close to 1 for all $n \le l \le N$. Therefore, there exists a natural number q so that $\|X(\rho z^q)\|_2 > \|X\rho\|_2 - \frac{\epsilon}{2}$. Furthermore, we can choose an arbitrarily large value of q with this property. For q sufficiently large,

$$\|T_X(\rho z^q)\|_2 = \|PX(\rho z^q)\|_2 = \|X(\rho z^q)\|_2 ,$$

and because $\|\rho z^q\|_2 = \|\rho\|_2 = 1$,

$$\|T_X\| \ge \|T_X(\rho z^q)\|_2 = \|X(\rho z^q)\|_2 > \|X\rho\|_2 - \frac{\epsilon}{2} > \|X\| - \epsilon.$$

Therefore $\|T_X\| \ge \|X\|$, and the continuity of the norm implies that this inequality holds for all X in A_θ. ∎

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Theorem 2 *There is a short exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}(\mathcal{A}_\theta) \xrightarrow{\sigma} \mathcal{A}_\theta \longrightarrow 0,$$

where $\mathcal{K}$ denotes the algebra of compact operators and σ has the property that $\sigma(T_X) = X$ for all X in $\mathcal{A}_\theta$.

Proof Because the Toeplitz algebra $\mathcal{T}(C(\mathbb{T}))$ contains $\mathcal{K}$ as an ideal, so does $\mathcal{T}(\mathcal{A}_\theta)$. Define a map $\xi\colon \mathcal{A}_\theta \to \mathcal{T}(\mathcal{A}_\theta)/\mathcal{K}$ by the formula $\xi(X) = T_X + \mathcal{K}$. Clearly ξ is $*$-linear, and Proposition 1 implies that ξ is continuous. The commutator $[P, f]$ is compact for each f in $C(\mathbb{T})$ [Do, Proposition 7.12] and it is easy to check that $[P, V_\theta] = 0$. Thus $[P, X]$ is compact for all X in $\mathcal{A}_\theta$, and therefore ξ is an algebra homomorphism. The image of ξ contains all the cosets $T_X + \mathcal{K}$, whence ξ is surjective. Furthermore, $\mathcal{A}_\theta$ is simple, so ξ is injective as well. We can therefore define $\sigma\colon \mathcal{T}(\mathcal{A}_\theta) \to \mathcal{A}_\theta$ as $\xi^{-1}\pi$, where $\pi\colon \mathcal{T}(\mathcal{A}_\theta) \to \mathcal{T}(\mathcal{A}_\theta)/\mathcal{K}$ is the quotient map, and the short exact sequence follows. ∎

Corollary 3 *For each natural number n, there is a short exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathrm{M}(n, \mathcal{T}(\mathcal{A}_\theta)) \xrightarrow{\sigma} \mathrm{M}(n, \mathcal{A}_\theta) \longrightarrow 0,$$

where σ has the property that $\sigma(T_X) = X$ for all X in $\mathrm{M}(n, \mathcal{A}_\theta)$.

Proof Tensoring through by $\mathrm{M}(n, \mathbb{C})$ preserves exact sequences, and

$$\mathrm{M}(n, \mathcal{K}) \cong \mathcal{K}.$$ ∎

Corollary 4 *An operator T in $\mathrm{M}(n, \mathcal{T}(\mathcal{A}_\theta))$ is Fredholm if and only if $\sigma(T)$ is invertible.*

Proposition 5 *For each irrational number θ, $K_0(\mathcal{T}(\mathcal{A}_\theta)) \cong K_0(\mathcal{A}_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ and $K_1(\mathcal{T}(\mathcal{A}_\theta)) \cong \mathbb{Z}$.*

Proof Apply the K-theory six-term exact sequence to the short exact sequence from Theorem 2. The operator T_z is in $\mathcal{T}(\mathcal{A}_\theta)$ and has index minus one, so the index map $\partial\colon K_1(\mathcal{A}_\theta) \to K_0(\mathcal{K})$ is surjective. The desired results follow from the facts $K_0(\mathcal{A}_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ and $K_1(\mathcal{A}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$ [Da, Example VII.5.2]. ∎

The short exact sequence in the statement of Theorem 2 defines an element $[\mathcal{T}(\mathcal{A}_\theta)]$ of $KK^1(\mathcal{A}_\theta, \mathbb{C})$, and it is natural to ask about other elements of this group. By the universal coefficient theorem [RS, Theorem 1.17], there is an isomorphism $\gamma\colon KK^1(\mathcal{A}_\theta, \mathbb{C}) \to \mathrm{Hom}(K_1(\mathcal{A}_\theta), \mathbb{Z})$, and $\mathrm{Hom}(K_1(\mathcal{A}_\theta), \mathbb{Z})$ is in turn isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let $\delta\colon KK^1(\mathcal{A}_\theta, \mathbb{C}) \to \mathbb{Z} \oplus \mathbb{Z}$ be the composition of these two isomorphisms. Then $\delta[\mathcal{T}(\mathcal{A}_\theta)] = (\operatorname{index} T_z, \operatorname{index} T_{V_\theta}) = (-1, 0)$.

We construct another element of $KK^1(\mathcal{A}_\theta, \mathbb{C})$ in the following way. From [Da, Corollary VI.5.3 and Theorem VI.1.4] we know that for each irrational number θ,

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there is a C^*-algebra isomorphism $\mu\colon \mathcal{A}_\theta \to \mathcal{A}_{-\theta}$ such that $\mu(V_\theta) = z$ and $\mu(z) = V_{-\theta}$; obviously these two facts completely determine μ. From Theorem 2 we have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}(\mathcal{A}_{-\theta}) \xrightarrow{\ \sigma\ } \mathcal{A}_{-\theta} \longrightarrow 0$$

that defines an element $[\mathcal{T}(\mathcal{A}_{-\theta})]$ in $KK^1(\mathcal{A}_{-\theta}, \mathbb{C})$. Let $\mu^*\colon KK^1(\mathcal{A}_{-\theta}, \mathbb{C}) \to KK^1(\mathcal{A}_\theta, \mathbb{C})$ be the map that μ induces on KK-theory. Then $\mu^*[\mathcal{T}(\mathcal{A}_{-\theta})]$ is in $KK^1(\mathcal{A}_\theta, \mathbb{C})$, and an easy computation shows that $\delta(\mu^*[\mathcal{T}(\mathcal{A}_{-\theta})]) = (0, -1)$. Therefore $[\mathcal{T}(\mathcal{A}_\theta)]$ and $\mu^*[\mathcal{T}(\mathcal{A}_{-\theta})]$ generate the group $KK^1(\mathcal{A}_\theta, \mathbb{C})$.

We next consider index theory. The index of Fredholm operators in the algebra $M_n(\mathcal{T}(C(\mathbb{T})))$ can be computed in terms of the winding number of the determinant of the symbol, and we can extend this result to Fredholm operators in $M_n(\mathcal{T}(\mathcal{A}_\theta))$. Define

$$\mathcal{A}_\theta^\infty = \Big\{ \sum_{k\in\mathbb{Z}} f_k V_\theta^k : f_k \in C^\infty(\mathbb{T}), \{\|f_k\|\}_{k\in\mathbb{Z}} \text{ rapidly decreasing} \Big\}$$

and

$$\Omega^1(\mathcal{A}_\theta^\infty) = \Big\{ \sum_{k\in\mathbb{Z}} \omega_k V_\theta^k : \omega_k \in \Omega^1(\mathbb{T}), \{\|\omega_k\|\}_{k\in\mathbb{Z}} \text{ rapidly decreasing} \Big\}.$$

It is straightforward to check that $\mathcal{A}_\theta^\infty$ is a dense subalgebra of $\mathcal{A}_\theta$ that is closed under the holomorphic functional calculus. The vector space $\Omega^1(\mathcal{A}_\theta^\infty)$ can be given the structure of a left $\mathcal{A}_\theta^\infty$-module in the following way [C, Example 2(b), pp. 183–184]: let $\phi_\theta\colon \mathbb{T} \to \mathbb{T}$ be the diffeomorphism $\phi_\theta(z) = e^{-2\pi i\theta} z$. Then given $\sum_{k\in\mathbb{Z}} f_k V_\theta^k$ in $\mathcal{A}_\theta^\infty$ and $\sum_{l\in\mathbb{Z}} \omega_l V_\theta^l$ in $\Omega^1(\mathcal{A}_\theta^\infty)$, define

$$\Big(\sum_{k\in\mathbb{Z}} f_k V_\theta^k\Big) \cdot \Big(\sum_{l\in\mathbb{Z}} \omega_l V_\theta^l\Big) = \sum_{k,l\in\mathbb{Z}} f_k \left((\phi_\theta^*)^k \omega_l\right) V_\theta^{k+l}.$$

We also have an exterior derivative map $d\colon \mathcal{A}_\theta^\infty \to \Omega^1(\mathcal{A}_\theta^\infty)$ given by the formula

$$d\Big(\sum_{k\in\mathbb{Z}} f_k V_\theta^k\Big) = \sum_{k\in\mathbb{Z}} (d f_k) V_\theta^k,$$

where the d on the right-hand side is the ordinary exterior derivative. We then extend d to map from $M_n(\mathcal{A}_\theta^\infty)$ to $\mathrm{M}(n, \Omega^1(\mathcal{A}_\theta^\infty))$ by applying d entrywise. We impose the relation $(dz)V_\theta = e^{2\pi i\theta} V_\theta(dz)$; a straightforward computation shows that $d(XY) = (dX)Y + X(dY)$ for all X and Y in $\mathrm{M}(n, \mathcal{A}_\theta^\infty)$.

For each natural number n, define $\nu\colon M_n(\Omega^1(\mathcal{A}_\theta^\infty)) \to \Omega^1(\mathbb{T})$ as

$$\nu\Big(\sum_{k\in\mathbb{Z}} \omega_k V_\theta^k\Big) = \operatorname{Tr}\omega_0,$$

where Tr is the ordinary matrix trace. Then define $\widetilde{\mathrm{Ch}}\colon \mathrm{GL}(n, \mathcal{A}_\theta^\infty) \to \Omega^1(\mathbb{T})$ by the formula

$$\widetilde{\mathrm{Ch}}(X) = -\frac{1}{2\pi i}\nu(X^{-1}dX).$$

Lemma 6 *Let $\{X_t\}_{t\in[0,1]}$ be a smooth path in $\mathrm{GL}(n,\mathcal{A}_\theta^\infty)$. Then*

$$\frac{\partial}{\partial t}\widetilde{\mathrm{Ch}}(X_t)=d\Big(-\frac{1}{2\pi i}\nu\Big(X_t^{-1}\frac{\partial X_t}{\partial t}\Big)\Big).$$

Proof A simple computation shows that $\frac{\partial}{\partial t}dX_t = d\Big(\frac{\partial X_t}{\partial t}\Big)$. Therefore, using the cyclic property of the trace, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\widetilde{\mathrm{Ch}}(X_t) &= \frac{\partial}{\partial t}\Big(-\frac{1}{2\pi i}\nu(X_t^{-1}dX_t)\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(\frac{\partial}{\partial t}(X_t^{-1}dX_t)\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(\frac{\partial}{\partial t}\Big(X_t^{-1}\Big)dX_t + X_t^{-1}\frac{\partial}{\partial t}(dX_t)\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(-X_t^{-1}\frac{\partial X_t}{\partial t}X_t^{-1}dX_t + X_t^{-1}d\Big(\frac{\partial X_t}{\partial t}\Big)\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(-X_t^{-1}dX_tX_t^{-1}\frac{\partial X_t}{\partial t} + X_t^{-1}d\Big(\frac{\partial X_t}{\partial t}\Big)\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(d(X_t^{-1})\frac{\partial X_t}{\partial t} + X_t^{-1}d\Big(\frac{\partial X_t}{\partial t}\Big)\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(d\Big(X_t^{-1}\frac{\partial X_t}{\partial t}\Big)\Big)\\
&= d\Big(-\frac{1}{2\pi i}\nu\Big(X_t^{-1}\frac{\partial X_t}{\partial t}\Big)\Big).
\end{aligned}$$

■

Proposition 7 *The map $\widetilde{\mathrm{Ch}}$ induces a group homomorphism*

$$\mathrm{Ch}\colon K_1(\mathcal{A}_\theta^\infty)\longrightarrow H^1(\mathbb{T}).$$

Proof Let π denote the quotient map from $\Omega^1(\mathbb{T})$ to $H^1(\mathbb{T})$. We see from Lemma 6 that given a path $\{X_t\}$ in $\mathrm{GL}(n,\mathcal{A}_\theta^\infty)$, $\frac{\partial}{\partial t}\widetilde{\mathrm{Ch}}(X_t)$ is identically zero in $H^1(\mathbb{T})$, whence $\pi\circ\widetilde{\mathrm{Ch}}$ is homotopy invariant. Next, take X in $\mathrm{GL}(n,\mathcal{A}_\theta^\infty)$, and consider the matrix $\left(\begin{smallmatrix}X&0\\0&1\end{smallmatrix}\right)$ in $\mathrm{GL}(n+1,\mathcal{A}_\theta^\infty)$. Then

$$\begin{aligned}
\widetilde{\mathrm{Ch}}\begin{pmatrix}X&0\\0&1\end{pmatrix} &= -\frac{1}{2\pi i}\nu\Big(\begin{pmatrix}X&0\\0&1\end{pmatrix}^{-1}d\begin{pmatrix}X&0\\0&1\end{pmatrix}\Big)\\
&= -\frac{1}{2\pi i}\nu\Big(\begin{pmatrix}X^{-1}&0\\0&1\end{pmatrix}\begin{pmatrix}dX&0\\0&0\end{pmatrix}\Big)\\
&= -\frac{1}{2\pi i}\nu\begin{pmatrix}X^{-1}dX&0\\0&0\end{pmatrix}\\
&= -\frac{1}{2\pi i}\nu\Big(X^{-1}dX\Big)\\
&= \widetilde{\mathrm{Ch}}(X).
\end{aligned}$$

Thus $\widetilde{\mathrm{Ch}}$ commutes with the usual inclusion of $\mathrm{GL}(n, \mathcal{A}_\theta^\infty)$ into $\mathrm{GL}(n+1, \mathcal{A}_\theta^\infty)$. Therefore $\pi \circ \widetilde{\mathrm{Ch}}$ induces a map Ch from $K_1(\mathcal{A}_\theta^\infty)$ to $H^1(\mathbb{T})$. Finally, to show that Ch is a homomorphism, it suffices to show that $\widetilde{\mathrm{Ch}}$ is a homomorphism:

$$\begin{aligned}
\widetilde{\mathrm{Ch}}(XY) &= \widetilde{\mathrm{Ch}}\left(\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}\right) \\
&= -\frac{1}{2\pi i}\nu\left(\left(\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}\right)^{-1} d\left(\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}\right)\right) \\
&= -\frac{1}{2\pi i}\nu\left(\begin{pmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix}\begin{pmatrix} dX & 0 \\ 0 & dY \end{pmatrix}\right) \\
&= -\frac{1}{2\pi i}\nu\begin{pmatrix} X^{-1}dX & 0 \\ 0 & Y^{-1}dY \end{pmatrix} \\
&= -\frac{1}{2\pi i}\nu\left(X^{-1}dX\right) - \frac{1}{2\pi i}\nu\left(Y^{-1}dY\right) \\
&= \widetilde{\mathrm{Ch}}(X) + \widetilde{\mathrm{Ch}}(Y).
\end{aligned}$$

■

We can now state the index theorem.

Theorem 8 *Take X in $\mathrm{GL}(n, \mathcal{A}_\theta^\infty)$. Then*

$$\text{index } T_X = \int_{\mathbb{T}} \mathrm{Ch}(X).$$

Proof We have two homomorphisms from $K_1(\mathcal{A}_\theta^\infty)$ to $\mathbb{C}$: the index homomorphism, and the map $\int_{\mathbb{T}} \mathrm{Ch}(-)$. By [Da, Example VIII.5.2], we know that $K_1(\mathcal{A}_\theta^\infty) \cong K_1(\mathcal{A}_\theta)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, with generators z and V_θ. Thus to prove the theorem, it suffices to show that these homomorphisms agree on z and V_θ. First,

$$\int_{\mathbb{T}} \mathrm{Ch}(z) = -\frac{1}{2\pi i}\int_{\mathbb{T}} \nu(z^{-1}dz) = -\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{dz}{z} = -1 = \text{index } T_z.$$

Second,

$$\int_{\mathbb{T}} \mathrm{Ch}(V_\theta) = -\frac{1}{2\pi i}\int_{\mathbb{T}} \nu(V_\theta^{-1}(d1\, V_\theta)) = 0 = \text{index } T_{V_\theta},$$

because V_θ commutes with P. ■

Example 9 Let

$$X = \begin{pmatrix} zV_\theta^2 & z^2V_\theta \\ zV_\theta & 2z^2 \end{pmatrix}.$$

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Then

$$X^{-1} = \frac{1}{2 - e^{2\pi i\theta}} \begin{pmatrix} 2e^{-4\pi i\theta} z^{-1} V_\theta^{-2} & -z^{-1} V_\theta^{-1} \\ -e^{-2\pi i\theta} z^{-2} V_\theta^{-1} & z^{-2} \end{pmatrix}$$

and

$$dX = \begin{pmatrix} (dz) V_\theta^2 & (2z\, dz) V_\theta \\ (dz) V_\theta & 4z\, dz \end{pmatrix}.$$

Thus

$$X^{-1} dX = \begin{pmatrix} z^{-1} dz & \star \\ \star & 2z^{-1} dz \end{pmatrix},$$

and so

$$\text{index } T_X = \int_{\mathbb{T}} \mathrm{Ch}(X) = -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{3}{z}\, dz = -3.$$

We close with a question. A well-known fact about Toeplitz operators on the circle is that for all g, either $\ker T_g = 0$ or $\ker T_g^* = 0$ [Do, Proposition 7.4]; this immediately implies that $T \in \mathcal{T}(C(\mathbb{T}))$ is invertible if and only if index $T = 0$. The analogous statement about the kernels of Toeplitz operators with symbols in $\mathcal{A}_\theta$ is not true. For example, if we take $X = e^{-2\pi i\theta} I - V_\theta$, then $f(z) = z$ is in the kernel of both T_X and T_X^*. This leaves the following open question:

Question 10 Are there Fredholm operators in $\mathcal{T}(\mathcal{A}_\theta)$ that have index 0 and are not invertible?

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