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AN ALMOST EVERYWHERE VERSION OF SMÍTAL'S ORDER–CHAOS DICHOTOMY FOR INTERVAL MAPS

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Abstract

We prove that if $f : I = [0, 1] \to I$ is a C^3-map with negative Schwarzian derivative, nonflat critical points and without wild attractors, then exactly one of the following alternatives must occur: (i) $R(f)$ has full Lebesgue measure λ; (ii) both $S(f)$ and $\mathrm{Scramb}(f)$ have positive measure. Here $R(f)$, $S(f)$, and $\mathrm{Scramb}(f)$ respectively stand for the set of approximately periodic points of f, the set of sensitive points to the initial conditions of f, and the two-dimensional set of points (x, y) such that $\{x, y\}$ is a scrambled set for f. Also, we show that if f is piecewise monotone and has no wandering intervals, then either $\lambda(R(f)) = 1$ or $\lambda(S(f)) > 0$, and provide examples of maps g, h of this type satisfying $S(g) = S(h) = I$ such that, on the one hand, $\lambda(R(g)) = 0$ and $\lambda^2(\mathrm{Scramb}(g)) = 0$, and, on the other hand, $\lambda(R(h)) = 1$.

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1. Introduction and main results

This paper deals with the dynamics of continuous maps f from a compact interval I into itself (written $f \in C(I)$). For simplicity we always assume $I = [0, 1]$. A relevant issue in this setting is to find clear-cut and informative criteria allowing us to decompose $C(I)$ into 'dynamically well-behaved' (regular) and 'dynamically badly behaved' (chaotic) maps. Needless to say, no such decomposition can be meaningful unless it is *complete* (every map must be either regular or chaotic) and *consistent* (no map can be both regular and chaotic). In the 1980s J. Smítal, in collaboration with several authors, solved this problem in a very elegant way.

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In order to discuss Smítal's result we must recall a number of notions naturally related to the ideas of regular and chaotic dynamics. To begin with, the simplest dynamical behaviour of the *orbit* $(f^n(x))_{n=0}^{\infty}$ of a point $x \in I$ is periodicity: we say that x is *periodic (for f)* when $f^r(x) = x$ for some $r \geq 1$ (the *period* of x being the minimal number r with this property). Since only long-time behaviour is really important to us, if x is *asymptotically periodic*, that is, there is a periodic point p such that

$$\lim_{n\to\infty} |f^n(x) - f^n(p)| = 0$$

(or, equivalently, the limit set $\omega_f(x)$ of its orbit is finite), then its dynamics are still pretty trivial. Even allowing small round-off errors should not be of particular significance. In this way we arrive at the notion of approximate periodicity.

DEFINITION 1.1. Let $f \in C(I)$ and $x \in I$. We say that x is *approximately periodic* if for every $\epsilon > 0$ there is a periodic point p such that

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \epsilon.$$

We denote by $R(f)$ the set of approximately periodic points of f.

While we can agree that $R(f)$ collects all points of I having reasonably simple dynamics for f, as Akin and Kolyada wrote in [1], 'the definitions associated with the term chaos have proliferated so much that the word threatens to introduce the sort of confusion it is intended to describe'. Anyway, sensitivity to initial conditions, as introduced by Guckenheimer in [18], and the definition inspired by the famous paper by Li and Yorke [26], are by far the most popular approaches.

DEFINITION 1.2. Let $f \in C(I)$, $x \in I$, and $\delta > 0$. We say that x is *δ-sensitive* (or just *sensitive* if we do not need to put an emphasis on δ) if for every neighbourhood U of x there is some k such that the diameter of $f^k(U)$ is at least δ.

We denote by $S(f)$ the set of sensitive points of f.

DEFINITION 1.3. Let $f \in C(I)$, let $A \subset I$ contain at least two elements, and let $\delta \geq 0$. We say that A is *δ-scrambled* (or simply *scrambled* if $\delta = 0$) if for every $x, y \in A$, $x \neq y$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta,$$
$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

We denote by $\mathrm{Scramb}(f)$ the set of pairs $(x, y) \in I^2$ such that $\{x, y\}$ is a scrambled set.

In Li–Yorke's paper a map is said to be chaotic if it possesses an uncountable scrambled set (with the additional redundant condition that neither of its points is asymptotically periodic, see Proposition 1.5 below); in Guckenheimer's paper chaos is linked to the idea of $S(f)$ having positive Lebesgue measure.

REMARK. We emphasize that $R(f)$, $S(f)$, and $\mathrm{Scramb}(f)$ are Borel, hence Lebesgue measurable sets, for every $f \in C(I)$: see Section 5.

Let $f \in C(I)$. We say that $x \in I$ is *nonwandering* if for every neighbourhood U of x there is some k such that $U \cap f^k(U) \neq \emptyset$. We denote by $\Omega(f)$ the set of nonwandering points of f. Clearly, $\Omega(f)$ is compact and $f(\Omega(f)) \subset \Omega(f)$, so it makes sense to consider the set $S(f|_{\Omega(f)})$ of sensitive points of the restricted map $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ (if X is a compact metric space and $g : X \to X$ is continuous, then we can define $S(g)$ in exactly the same way as in Definition 1.2). Now we are ready to state Smítal's theorem.

THEOREM 1.4 (Smítal *et al.* [15, 16, 19, 38]). *Let $f \in C(I)$. Then exactly one of the following alternatives must occur:*
(i) *all points of I are approximately periodic;*
(ii) *f has a Cantor δ-scrambled set for some $\delta > 0$ and $S(f|_{\Omega(f)}) \neq \emptyset$.*

It is worth emphasizing that the original statement of the theorem does not include the word 'exactly'. That (i) and (ii) are incompatible properties follows from the fact that no scrambled set can have more than one approximately periodic point, which in turn is an immediate consequence of the following useful result, first noticed in [20, pp. 117–118] (a proof can also be found in [4, pp. 144–145]). Observe in passing that Theorem 1.4 together with Proposition 1.5 immediately imply that if $f \in C(I)$ has a two-point scrambled set, then it is Li–Yorke chaotic, see also [25].

PROPOSITION 1.5. *Let $f \in C(I)$ and let $x, y \in I$ be approximately periodic. If*

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0,$$

then

$$\lim_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

Theorem 1.4 is very much related to the famous Sharkovsky theorem [35] which, among other things, decomposes $C(I)$ into maps of *type less than* 2^∞ (those having periodic orbits of periods $1, 2, \ldots, 2^k$ for some $k \geq 0$ and no other periods), maps of *type* 2^∞ (those having periodic orbits of periods 2^s for every $s \geq 0$ and no other periods), and maps of *type greater than* 2^∞ (those having a periodic orbit of period not a power of 2). It turns out that if f is of type less than 2^∞, then all points are asymptotically periodic so it satisfies (i) (see, for example, [4, Proposition 1, p. 121]). On the other hand, all maps of type greater than 2^∞ are Li–Yorke chaotic (this is an easy consequence of the main result in [26]), so they satisfy (ii). There are examples of maps of type 2^∞ such that all points are asymptotically periodic, and also maps of this type that are chaotic in the sense of Li and Yorke: see [38].

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According to Theorem 1.4, either $R(f)=I$ or f features both types of chaos (although concerning sensitivity this chaos is 'qualitative' rather than 'quantitative'). Actually, from the measure-theoretical point of view, things may not be that clear. For instance, if $\alpha=3.83187\ldots$, then the point $1/2$ is periodic of period three for the map $f(x)=\alpha x(1-x)$ so (ii) occurs for f. On the other hand, in [18] is proved that almost all points of I are asymptotically periodic, hence every scrambled set S of f must have measure zero.

Proposition 1.5 implies that the family $AP(I)$ of maps f for which $R(f)$ has full measure, and that of maps possessing scrambled sets of positive measure, have empty intersection. If the union of both families were the whole class $C(I)$, then we would obtain a nice 'almost everywhere' version of Theorem 1.4, which is the main aim of the present paper. Unfortunately this is far from true. A particularly enlightening example is the full quadratic map $f(x)=4x(1-x)$. It can be proved that $R(f)$ has zero measure. (A possible way to do this is proceeding in two steps. First we show $S(f)=I$ using the analogous result for the tent map $T(y)=1-|2y-1|$ and taking into account that f and T are conjugated via the map $\varphi(y)=\sin^2(\pi y/2)$; then we apply Corollary C below.) On the other hand, f cannot have *measurable* scrambled sets of positive measure, see [21] or use Theorem D below. It is worth emphasizing that f possesses a nonmeasurable scrambled set of *exterior* full Lebesgue measure (to prove it use [37], where a similar result is stated for the tent map, and the above-mentioned smooth conjugacy φ). We see that scrambled sets may be rather problematic from the point of view of measure, which suggests that the set $\mathrm{Scramb}(f)$ should be considered instead. Actually, it is known that $\lambda^2(\mathrm{Scramb}(f))=1$ (see [21]); throughout the paper λ and λ^2 respectively denote the one-dimensional and two-dimensional Lebesgue measure. Moreover, if $LY(I)$ denotes the family of maps f for which $\lambda^2(\mathrm{Scramb}(f))>0$, then Proposition 1.5 still ensures $AP(I)\cap LY(I)=\emptyset$. Unfortunately, as we will see later, $AP(I)\cup LY(I)$ is also strictly included in $C(I)$.

The alternative way of formulating an almost everywhere version of Smítal's theorem is using sensitivity. For instance, recall that for the full quadratic map f we have $S(f)=I$. Thus, we could consider the class $ST(I)$ of maps f for which $\lambda(S(f))>0$ and investigate how $AP(I)$ and $ST(I)$ relate to each other. The bad news is that neither $AP(I)\cap ST(I)=\emptyset$ (in [3] a map f was constructed simultaneously satisfying $R(f)=I$ and $\lambda(S(f))=1$), nor $AP(I)\cup ST(I)=C(I)$ (after blowing up a dense orbit for the tent map so that the corresponding orbit of intervals have full measure, we receive a map f such that $\lambda(R(f))=0$ and $\lambda(S(f))=0$). The good news is that these are fairly sophisticated counterexamples, far from being 'natural' maps.

At this point, in order to proceed further, the nature of the sets $R(f)$ and $S(f)$ must be understood better. Here the notion of an *adding machine* plays a pivotal role.

For a sequence of integers $\alpha=(p_m)_{m=1}^\infty$ such that $p_m\geq 2$ for every m, the *α-adic adding machine* Δ_α is the set of sequences (x_m) such that $x_i\in\{0,1,\ldots,p_m-1\}$ for every m. We use the product topology in Δ_α and define the *adding machine map* $f_\alpha:\Delta_\alpha\to\Delta_\alpha$ by writing

$$f_\alpha((x_m)) = \begin{cases} (0, \ldots, 0, x_m + 1, x_{m+1}, \ldots) & \text{if } x_m < p_m - 1 \text{ and } x_j = p_j - 1 \\ & \quad \text{for every } j < m; \\ (0, 0, \ldots) & \text{if } x_m = p_m - 1 \text{ for every } m. \end{cases}$$

Let X be an infinite compact metric space and $g \in C(X)$. It is well known that g is topologically conjugate to an adding machine map if and only if, for every $\epsilon > 0$, X can be decomposed into finitely many pairwise disjoint compact sets with diameters less than ϵ which are cyclically permuted by g (see [5]). If $f \in C(I)$, $X \subset I$ is an invariant set for f, and such is the case for the restricted map $g = f|_X$, then we simply refer to X as an *adding machine (for f)*. In this particular situation it may even happen that the convex hulls of the sets from the above-mentioned partitions are still pairwise disjoint and cyclically permuted by f. If we are in such a case, then we call X a *solenoid*. More exactly we have the following definition.

DEFINITION 1.6. Let $f \in C(I)$ and $X \subset I$. We say that X is a *solenoid* if there are a decreasing sequence of compact intervals $(I_m)_{m=1}^\infty$ and a strictly increasing sequence of positive integers $(r_m)_{m=1}^\infty$ such that $f^{r_m}(I_m) \subset I_m$ for every m, the intervals $\{f^i(I_m)\}_{i=0}^{r_m-1}$ are pairwise disjoint and their lengths tend uniformly to zero as m goes to ∞, and

$$X = \bigcap_{m=1}^{\infty} \bigcup_{i=0}^{r_m-1} f^i(I_m).$$

If $f \in C(I)$ and $x \in I$, then $x \in R(f)$ if and only if $\omega_f(x)$ is either finite or an adding machine (Proposition 5.1). Concerning $S(f)$ things are not that clear-cut. For instance, if $\omega_f(x)$ is a solenoid, then $x \in I \setminus S(f)$ (because if with the notation of Definition 1.6 m is given, then there is a small neighbourhood U of x that is mapped by some iterate of f into I_m, hence the lengths of the iterates of U are bounded by those of the intervals $f^i(I_m)$, $0 \le i < r_m$).

Nevertheless an asymptotically periodic point very well may be sensitive: just think of a repelling fixed point. The first main result of this paper (Theorem A) shows that the last possibility is rather exceptional for the family $P(I)$ of piecewise monotone maps without wandering intervals: in this setting, sensitivity and having an ω-limit set different from a periodic orbit and a solenoid essentially amount to the same thing. In particular, this implies $P(I) \subset AP(I) \cup ST(I)$. (We say that f in $C(I)$ is *piecewise monotone* if there is a partition $0 = a_0 < a_1 < \cdots < a_k = 1$ of I such that $f|_{[a_i,a_{i+1}]}$ is—not necessarily strictly—monotone for each i. We say that an interval J is *wandering* for a map $f \in C(I)$ if all iterates $f^n(J)$, $n \ge 0$, are pairwise disjoint and J contains no asymptotically periodic point.)

THEOREM A. *Let f be in the class $P(I)$ of piecewise monotone continuous maps of I without wandering intervals. Then the following is true except for countably many points $x \in I$: x is not sensitive if and only if $\omega_f(x)$ is either a periodic orbit or a solenoid.*

In particular, if $f \in P(I)$, then either $\lambda(R(f)) = 1$ or $\lambda(S(f)) > 0$.

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It must be stressed that $P(I)$ is a pretty natural choice. Actually it provides an optimal work setting from the purely topological point of view. For instance, maps of type 2^∞ in $P(I)$ always satisfy (i) in Theorem 1.4 and possess some solenoid [23, 38], thus becoming a kind of boundary between maps of types less than and greater than 2^∞ in $P(I)$.

Further, 'relevant' ω-limit sets for maps from $P(I)$ can be classified as follows. Let $f \in C(I)$. We say that f is *transitive* if there is a point $x \in I$ such that $\omega_f(x) = I$. We say that f is *totally transitive* if f^n is transitive for all $n \geq 1$. If $f \in C(I)$ is totally transitive, then if features a rather strong type of sensitivity. Namely, for every subinterval J of I and every $\epsilon > 0$ there is a number k such that $f^n(K) \supset [\epsilon, 1-\epsilon]$ whenever $n \geq k$ (see [2, 10]). Note that in the particular case when $f \in P(I)$, this implies that f is *topologically exact*, that is, for every subinterval J of I there is a number k such that $f^k(J) = I$.

If $\{K_i\}_{i=0}^{r-1}$ is a periodic orbit of intervals (that is, the intervals K_i are pairwise disjoint, $f(K_i) = K_{i+1}$ for any i with $K_r := K_0$) with the property that $f^r|_{K_0}$ is totally transitive when seen as a map from $C(K_0)$, then we call each of the intervals K_i and the whole orbit $\{K_i\}_{i=0}^{r-1}$ *totally transitive* (or *topologically exact* in the piecewise monotone setting) as well. We call a set $A \subset I$ a *topological attractor* if the set $\{x \in I : \omega_f(x) = A\}$ is of the second Baire category and A is minimal with this property. It turns out that if $f \in P(I)$ (or, more generally, if $f \in C(I)$ has no wandering intervals), then for a residual set of points x we have that $\omega_f(x)$ is finite, a solenoid, or a totally transitive periodic orbit of intervals (a *totally transitive interval orbit*—or a *topologically exact interval orbit* in the piecewise monotone setting—for short). This is the well-known Blokh's spectral theorem [7, Theorem 6.2] (see also [34, Theorem 2.4, p. 25]). In particular, it implies that there are no other topological attractors than those of these three types.

We have seen that $P(I) \subset AP(I) \cup ST(I)$ and we already know that $AP(I)$ and $LY(I)$ do not intersect. Unfortunately, as our next theorem shows, neither $AP(I)$ and $ST(I)$ are disjoint in $P(I)$, nor the union of $AP(I)$ and $LY(I)$ covers $P(I)$. Both counterexamples are provided by topologically exact maps, hence satisfying that their sets of sensitive points are the whole interval I.

THEOREM B. *There are topologically exact unimodal maps g, h in the class $P(I)$ of piecewise monotone continuous maps of* $[0, 1]$ *without wandering intervals such that:*

(a) $\lambda(R(g)) = 0$ *and* $\lambda^2(\mathrm{Scramb}(g)) = 0$;

(b) $\lambda(R(h)) = 1$.

We see that to obtain the analogous result to Theorem 1.4 we are looking for, the family of maps under consideration must be restricted further. We say that c is a *critical point* of a map f if f is differentiable at c and $f'(c) = 0$. We say that a critical point c of a map f is *nonflat* if it has a neighbourhood U such that $f|_U$ is of class C^{n+1} for some $n \geq 2$ and $f^{(n)}(c) \neq 0$. We denote by $N^2(I)$ the family of C^2-maps having only nonflat critical points.

Note that if $f \in N^2(I)$, then it has finitely many critical points, so f is piecewise (strictly) monotone. A celebrated result, following from the sequence of papers [8, 18, 27, 31, 32], states that maps from $N^2(I)$ have no wandering intervals, that is, $N^2(I) \subset P(I)$. Actually, the smoothness properties of maps from $N^2(I)$ allow us to obtain a lot of information about their almost everywhere dynamics that is not available for general maps from $P(I)$, particularly as far as their ω-limit sets are concerned.

We say that a set $A \subset I$ is a *metric attractor* if $\{x \in I : \omega_f(x) = A\}$ has positive measure and A is minimal with this property. Combining results from [30] and [31] one can prove that solenoids are both topological and metric attractors for maps from $N^2(I)$. Further, it is a simple exercise to show that, even for just continuous maps, every totally transitive interval orbit contains a residual set of points whose orbits are dense in the intervals from the family, hence a totally transitive interval orbit is always a topological attractor. Surprisingly, in [12] a topologically exact polynomial map was constructed for which there is a Cantor set attracting the orbit of almost every point of I. Such a set is usually called a *wild attractor*. More precisely we define it as follows.

DEFINITION 1.7. Let $f \in C(I)$ and let $A \subset I$ be a Cantor set. We say that A is a *wild attractor* if it is a metric attractor, but not a solenoid, for f.

In [28] is shown that if $f \in N^2(I)$, then $\omega_f(x)$ is finite, a solenoid, a wild attractor, or a *metrically exact* interval orbit (with this we refer to a topologically exact interval orbit such that the union of the intervals from the family is the ω-limit set of almost all of its points) for almost every $x \in I$.

Wild attractors may represent un unsurmountable obstacle for our purposes even in the optimal setting $N^2(I)$. In fact, we conjecture that there are polynomial maps with similar properties to those of g and h in Theorem B. For such maps appropriate wild attractors should play the leading role instead of the Cantor sets C and D from the proof of the theorem. A feasible candidate for g may be the map from [13, Theorem 7], whose wild attractor, similarly to C, is semiconjugate to the dyadic adding machine. Note that here things are further complicated by the fact that wild attractors have zero measure [39]. In the second case the situation should be simpler: we just need a map h possessing a wild attractor which is also an adding machine for h. In this regard [11] may be helpful.

If there are maps in $N^2(I)$ with similar properties to those of h in Theorem B, then they have an intriguing, kind of 'boundary' behaviour. On the one hand almost all points are approximately periodic, being attracted by a wild attractor consisting also of approximately periodic points; on the other hand, they are topologically exact. Thus, such maps would become, in some sense, the measure-theoretic counterpart to maps of type 2^∞ in the topological setting.

Then we are bound to assume that $f \in N^2(I)$ has no wild attractors. While admittedly this is a strong restriction, it is nevertheless satisfied in the important case when f has exactly one critical point c and $f''(c) \neq 0$, see [17, 29, 36]. Now the nature of $R(f)$ and $S(f)$ can be completely unravelled. In fact, if $x \in R(f)$, then $\omega_f(x)$ is

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either finite or an adding machine and, hence, except for a zero measure set of points x, $\omega_f(x)$ must be either a periodic orbit or a solenoid and x cannot belong to $S(f)$ (due to Theorem A). Thus, we have shown the following result.

COROLLARY C. *Let $f \in N^2(I)$ and assume that it has no wild attractors. Then the following statements are equivalent except for a zero measure set of points $x \in I$:*
(a) *x is not sensitive;*
(b) *$\omega_f(x)$ is either a periodic orbit or a solenoid;*
(c) *x is approximately periodic.*
In particular, for such a map f, exactly one of the following alternatives must occur: either $\lambda(R(f)) = 1$ or $\lambda(S(f)) > 0$.

If, in addition, f has negative Schwarzian derivative, then we can successfully deal with the set $\mathrm{Scramb}(f)$ (Theorem D below). This last theorem, together with Corollary C, provide the almost everywhere version of Theorem 1.4 we are looking for. Moreover, as a byproduct we get that neither of these maps can possess a scrambled set of positive measure. We recall that if f is a C^3-map and x is not a critical point of f, then the *Schwarzian derivative* of f at x, $Sf(x)$, is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

We denote by $S^3(I)$ the family of maps from $N^2(I)$ having negative Schwarzian derivative outside its critical points.

THEOREM D. *Let $f \in S^3(I)$ and assume that it has no wild attractors. Then it has no scrambled sets of positive measure. Moreover, exactly one of the following alternatives must occur:*
(i) $\lambda(R(f)) = 1$;
(ii) $\lambda^2(\mathrm{Scramb}(f)) > 0$.

It is fortunate that a lot of relevant maps in applied dynamics have negative Schwarzian derivative: the typical example is the quadratic family $f_\alpha(x) = \alpha x(1 - x)$, $\alpha \in [0, 4]$. Moreover, in view of a number of recently published and remarkable papers, notably [17, 24, 39], we conjecture that this hypothesis is not essential in Theorem D so could be disposed of. Indeed Graczyk and Sand have recently announced that every C^3 map with nonflat critical points and whose periodic points are hyperbolic repelling is analytically conjugate to a map from $S^3(I)$. (A periodic point p of f is said to be *hyperbolic* (respectively, *hyperbolic repelling*) if $|(f^r)'(p)| \neq 1$ (respectively, $|(f^r)'(p)| > 1$), with r the period of p.) Using this result it is easy to see that one can drop the negative Schwarzian condition from Theorem D at the cost of adding the restriction that all periodic points are hyperbolic. Note that this alternative formulation, while apparently much stronger, has a fundamental drawback: the hyperbolicity condition cannot be checked in advance for a given, concrete map.

The structure of the paper is pretty simple. The next three sections provide proofs for, respectively, Theorems A, B, and D. We have added a last complementary section

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providing further information on approximate periodicity and where is shown that $R(f)$, $S(f)$, and $\operatorname{Scramb}(f)$ are Borel sets for every continuous map f.

2. Proof of Theorem A

Let $f \in C(I)$ and $x \in I$. Recall that if $\omega_f(x)$ is a solenoid, then x cannot be sensitive. If x is asymptotically periodic, then it is still possible that $x \in S(f)$, but if, in addition, f is piecewise monotone, then some strong restrictions arise. Indeed, if the periodic orbit attracting that of x has period r, then $f^k(x)$ must belong to the periodic orbit for some k and, moreover, $f^k(x)$ must be a one-sided or two-sided isolated fixed point of f^r. Further, neither of the iterates of x can fall into an open interval of constancy of f. Hence, $S(f)$ can contain at most countably many asymptotically periodic points.

Thus, in order to complete the proof of Theorem A, it suffices to show that the following is true.

LEMMA 2.1. *Let $f \in C(I)$ have no wandering intervals. If $x \in I$ is not sensitive, then $\omega_f(x)$ is either a periodic orbit or a solenoid.*

The next simple lemma will be used in the proof.

LEMMA 2.2. *Let $f \in C(I)$, let J be a compact subinterval of I, and assume that $f^k(J) = J$ for some (minimal) positive integer k. Then either J is periodic of period k or k is even and $J \cup f^{k/2}(J)$ is periodic of period $k/2$.*

PROOF. It clearly suffices to show that if $1 \leq r < k$ satisfies $f^r(J) \cap J \neq \emptyset$, then $r = k/2$. Suppose not. Then we can assume that $r < k/2$, which in view of the minimality of k implies (after rewriting $g = f^r$) that the intervals J, $g(J)$ and $g^2(J)$ are pairwise different. As neither $g(J)$ can be strictly contained in J nor it can strictly contain J (because $g^k(J) = J$) we can, for example, assume that $g(J)$ is to the right of J, that is, there are points of $g(J)$ to the right of J, and J is to the left of $g(J)$, that is, there are points of J to the left of $g(J)$.

We claim that $g^2(J)$ is then to the right of $g(J)$. Actually, if $g^2(J)$ is to the left of $g(J)$, then either $g^2(J)$ is to the right of J and so $g^k(J \cup g(J))$ is strictly contained in $J \cup g(J)$, or $g^2(J)$ is to the left of J and then $g^k(J \cup g(J))$ strictly contains $J \cup g(J)$; in both cases we arrive at a contradiction.

Repeating the argument we find that $g^{j+1}(J)$ is to the right of $g^j(J)$ for every j, which is impossible because $g^k(J) = J$. □

PROOF OF LEMMA 2.1. Fix $\epsilon > 0$. Since x is not sensitive, there is an interval K neighbouring x such that $\sup_{n \geq 0} \lambda(f^n(K)) < \epsilon$. Moreover, there are numbers $k \geq 0$, $r \geq 1$ such that $f^k(K) \cap f^{r+k}(K) \neq \emptyset$ (otherwise we have that either K is a wandering interval, which is impossible, or x is asymptotically periodic and we are done). We can also exclude the trivial case when some of the sets $f^n(K)$ degenerates to a point. Then every pair of intervals $f^{ri+k}(K)$, $f^{r(i+1)+k}(K)$ intersect and the closure L of $\bigcup_{i=0}^{\infty} f^{ri+k}(K)$ is also an interval.

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We claim that $\lambda(f^{ri}(L)) < 11\epsilon$ for every i large enough. Namely, let y be an accumulation point of the middle points of the intervals $f^{ri+k}(K)$ and find some i_0 such that $\mathrm{dist}(f^{ri_0+k}(K), y) < \epsilon$. It suffices to show $\lambda(f^{ri_0}(L)) < 11\epsilon$.

Suppose not. Then

$$\mathrm{dist}(f^{ri_0+k}(K), f^{rs+k}(K)) > 4\epsilon \quad \text{for some } s > i_0.$$

Let $i_0 < t < s$ be the last number satisfying $\mathrm{dist}(f^{ri_0+k}(K), f^{rt+k}(K)) < \epsilon$. Use the definition of y and the property $\mathrm{dist}(f^{ri_0+k}(K), y) < \epsilon$ to find a minimal number $u > s$ such that

$$0 < \mathrm{dist}(f^{ri_0+k}(K), f^{ru+k}(K)) < 2\epsilon$$

(thus, in fact, $u > s+1$). Then the interval

$$M = \bigcup_{i=t+1}^{u-1} f^{ri+k}(K)$$

satisfies $\mathrm{dist}(M, y) > 0$. Note that

$$\mathrm{dist}(f^{ri_0+k}(K), f^{r(t+1)+k}(K)) < 2\epsilon \quad \text{and}$$
$$2\epsilon \le \mathrm{dist}(f^{ri_0+k}(K), f^{r(u-1)+k}(K)) < 3\epsilon.$$

This means that $f^{r(u-1)+k}(K)$ is contained in the smallest interval including $f^{r(t+1)+k}(K)$ and $f^{rs+k}(K)$, that is,

$$M = \bigcup_{i=t+1}^{u-2} f^{ri+k}(K).$$

Hence, $f^r(M) \subset M$, which contradicts the definition of y. Note that the same argument, applied to each of the intervals $f^j(L)$, $1 \le j < r$, proves in fact that $\lambda(f^n(L)) < 11\epsilon$ for every n large enough.

Write $N_j = \bigcap_{i=0}^{\infty} f^{ri+j}(L)$ for every $0 \le j < r$. Note that $\lambda(N_j) < 11\epsilon$ for every j and $N_j \neq \emptyset$ because it is a decreasing intersection of compact intervals. It is easy to check that $f(N_j) = N_{j+1}$ for every $j \pmod r$. Then we either have that each set N_j degenerates to one point, or each set N_j is a nondegenerate interval. In the first case we have $\lambda(f^n(L)) \to 0$ as $n \to \infty$. Since $f^k(x) \in L$, $\omega_f(x) = \bigcup_j N_j$ is finite and x is asymptotically periodic, which finishes the proof.

Thus, we can assume that the intervals N_j are nondegenerate. Now, in view of Lemma 2.2, there is a periodic interval J_ϵ with all intervals $f^n(J_\epsilon)$ having length less than 22ϵ and with the property that $\mathrm{dist}(f^n(x), f^n(J_\epsilon)) \to 0$ as $n \to \infty$.

We have shown that either Lemma 2.1 holds or for every $\epsilon > 0$ there is a periodic interval J_ϵ with the properties described above. In the latter case there is clearly a sequence $\epsilon_m \to 0$ as $m \to \infty$ so that, after rewriting $J_m = J_{\epsilon_m}$ and denoting by r_m the period of J_m, one of the following alternatives holds:

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(a) there is an r such that $r_m = r$ for every m; moreover, for every $0 \le j < r$ there is a point p_j such that $\operatorname{dist}(f^j(J_m), p_j) \to 0$ as $m \to \infty$;

(b) $r_{m+1} > 2r_m$ for every m.

If (a) occurs, then we have $\omega_f(x) = \{p_0, \ldots, p_{r-1}\}$ and x is asymptotically periodic. Assume now that (b) occurs. Fix m and let $z_j \in \omega_f(x)$ be an accumulation point of $(f^{r_{m+1}i + r_m j}(x))_i$, $j = 0, 1, 2$. Hence, $z_j \in J_m \cap f^{r_m j}(J_{m+1})$ and, since the intervals $f^{r_m j}(J_{m+1})$ have pairwise disjoint interiors (because $r_{m+1} > 2r_m$), one of them must be contained in the interior of J_m. This implies that r_{m+1} divides r_m so we also have $J_{m+1} \subset J_m$. We see that either J_{m+1} is a periodic interval of period r_{m+1} contained in J_m, or $J_{m+1} \cup f^{r_{m+1}/2}(J_{m+1})$ is a periodic interval of period $r_{m+1}/2$ contained in J_m (use Lemma 2.2). Then

$$\omega_f(x) = \bigcap_{m=1}^{\infty} \bigcup_{i=0}^{r_m - 1} f^i(J_m)$$

is a solenoid. □

3. Proof of Theorem B

To construct the desired maps g and h we just have to devise topologically exact unimodal maps $\tilde{g}, \tilde{h} \in P(I)$ for which there are corresponding Cantor sets C and D having the following properties:

$$\tilde{g}(C) \subset C \subset I \setminus R(\tilde{g}) \quad \text{and} \quad C^2 \subset I^2 \setminus \operatorname{Scramb}(\tilde{g}); \tag{1}$$

$$D \subset R(\tilde{h}). \tag{2}$$

Indeed, since $\tilde{g}$ is topologically exact, we can easily find a family $\{C_j\}$ of pairwise disjoint Cantor sets such that $\tilde{A} = \bigcup_j C_j$ is dense in I and, for every j, there is some k_j such that $f^{k_j}(C_j) \subset C$. Note that $\tilde{A}$ also satisfies $\tilde{A} \subset I \setminus R(\tilde{g})$ and $\tilde{A}^2 \subset I^2 \setminus \operatorname{Scramb}(\tilde{g})$. Moreover, there is a homeomorphism $\varphi : I \to I$ mapping $\tilde{A}$ onto a set A of full measure. Now, for the conjugated map $g = \varphi \circ \tilde{g} \circ \varphi^{-1}$, we have that $A \subset I \setminus R(g)$ and $A^2 \subset I^2 \setminus \operatorname{Scramb}(g)$. Then $\lambda(R(g)) = 0$ y $\lambda^2(\operatorname{Scramb}(g)) = 0$, so g is one of the maps we are looking for. Likewise, using (2), we can construct a dense union of pairwise disjoint Cantor sets $\tilde{B}$ contained in $R(\tilde{h})$. Thus, if ψ is a homeomorphism mapping $\tilde{B}$ to a full measure set B, then $B \subset R(h)$ for the conjugated map $h = \psi \circ \tilde{h} \circ \psi^{-1}$, which thus becomes the other map we are looking for. Note in passing that to guarantee the existence of a set D as in (2) it suffices to produce an approximately (but not asymptotically) periodic point, see Proposition 5.1.

The map $\tilde{g}$ we need is the tent map $\tilde{g}(x) = 1 - |2x - 1|$, but to find C satisfying (1) we also use a truncated map $f(x) = \max\{\kappa, \tilde{g}(x)\}$. It turns out that if κ is appropriately chosen (its approximate value being $\kappa = 0.824\,908\ldots$), then f becomes a map of type 2^∞ whose dynamics were extensively analyzed in [33] and [22]. What we need to know about f is the following. There is a decreasing sequence of compact

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intervals $(J_m)_{m=1}^{\infty}$, intersecting exactly at the interval J of constancy of f, such that, for every m, the intervals $\{f^i(J_m)\}_{i=0}^{2^m-1}$ form a periodic orbit of intervals of period 2^m. The intersection $K = \bigcap_{m=1}^{\infty} \bigcup_{i=0}^{2^m-1} f^i(J_m)$ is not a Cantor set, but each of the nondegenerate connected components of K is monotonically mapped onto J by some iterate of f. Now, after taking off K the interiors of its nondegenerate components, we receive an invariant Cantor set C. The set C has the property that if $x, y \in C$, then either $f^k(x) = f^k(y)$ for some k (which happens when x and y are the endpoints of some nondegenerate component of K) or $\liminf_{n\to\infty} |f^n(x) - f^n(y)| > 0$; moreover, we have $\omega_f(x) = C$ for every $x \in C$. Since the orbits of points from C never visit the interior of J, we see that C still has these properties when f is replaced by $\tilde{g}$. In particular, $\omega_{\tilde{g}}(x) = C$ for every $x \in C$ and the fact that $\tilde{g}|_C$ is not one-to-one imply that no point of C is approximately periodic for $\tilde{g}$ (use Proposition 5.1). We see that C has the required properties in (1).

The map $\tilde{h}$ we need is just a topologically exact unimodal map having an adding machine. After a first version of this paper was written we learned of [6], where it was proved that the family of quadratic maps attaining 1 as its maximum value and mapping 1 to 0 contains such maps. The approach of [6] is mainly topological in nature. Thus, we have decided to include our combinatorially-based proof.

Since we desire $\tilde{h}$ to belong to the family $\{f_\mu\}_{\mu\in(1,4]}$ given by

$$f_\mu(x) = -\mu x^2 + 2(\mu - \mu^{1/2})x + 2\mu^{1/2} - \mu$$

we must explain how to choose μ. To do this some standard facts from kneading theory will be used. Proofs of the statements below can be found, for example, in [14].

Let $\{0, c, 1\}^\infty$ denote the set of infinite sequences $\alpha = \alpha_0\alpha_1\cdots$ of symbols 0, c and 1. We put $0 < c < 1$ and introduce a total ordering '$<$' in $\{0, c, 1\}^\infty$ by writing $\alpha < \beta$ whenever $\alpha \neq \beta$, $\alpha_k < \beta_k$ (respectively, $\beta_k < \alpha_k$) and $\alpha_0 \cdots \alpha_{k-1} = \beta_0 \cdots \beta_{k-1}$ contains an even number of 1's (respectively, an odd number of 1's). The *shift map* $\sigma : \{0, c, 1\}^\infty \to \{0, c, 1\}^\infty$ is defined by $\sigma(\alpha) = \beta$ with $\beta_n = \alpha_{n+1}$ for every nonnegative integer n.

Let $f \in \{f_\mu\}$. We assign to every point x its *itinerary*

$$\alpha = K(x) = K_f(x) \in \{0, c, 1\}^\infty$$

by putting $\alpha_n = 0$, $\alpha_n = c$ or $\alpha_n = 1$ according to whether $f^n(x) < c$, $f^n(x) = c$ or $f^n(x) > c$, where we also use c to denote the turning point of f. Note that $K(f^n(x)) = \sigma^n(K(x))$ for every $n \geq 0$. The itinerary $\theta = \theta_f = K(1)$ of the right endpoint of the interval I is called the *kneading sequence* of the map f. It can be proved that if $\sigma^n(\theta) < \theta$ for every $n \geq 1$ for some sequence θ, then there is a map f in $\{f_\mu\}$ having θ as its kneading sequence.

In what follows we assume $\theta \in \{0, 1\}^\infty$, which is known to have a number of important consequences. For instance, if $x, y \in I$, then $x < y$ if and only if $K(x) < K(y)$. Moreover, if $\alpha \in \{0, 1\}^\infty$ satisfies $\sigma(\theta) \leq \alpha$ and $\sigma^n(\alpha) < \theta$ for every $n \geq 0$, then there is exactly one point $x \in [0, 1)$ such that $K(x) = \alpha$.

The reason behind the properties described above is that, in this case, f has no homtervals. Here, by a *homterval*, we mean an interval J with that property that $f^n|_J$ is monotone for every n. Then, for every $\epsilon > 0$, there are a positive integer l and a number $\delta > 0$ such that, if $x, y \in I$ and $\alpha = K(x)$, $\beta = K(y)$, then the following statements hold:

- if $\alpha_n = \beta_n$ for every $n \le l$, then $|x - y| < \epsilon$;
- if there is a number m such that $|f^m(x) - f^m(y)| < \delta$ and $\alpha_n = \beta_n$ for every $n \le m$, then $|x - y| < \epsilon$.

As a consequence of these two facts we get that if ϵ is given, then there is an integer k such that, if $x, y \in I$, $\alpha = K(x)$, $\beta = K(y)$, and

$$\alpha_{n_i+j} = \beta_{n_i+j} = \theta_{j-1}, \quad 1 \le j \le k,$$

for the sequence (n_i) of indexes n with the property $\alpha_n \neq \beta_n$, then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| < \epsilon.$$

Another consequence of the absence of homtervals for f is that every subinterval of I has some iterate containing c, and then 1. Hence to prove that f is topologically exact it would suffice to find a sequence $(l_j)_{j=1}^{\infty}$ of positive integers with $l_1 = 1$ and $(f^{l_j}(1))$ converging to 1, such that every interval $[f^{l_j}(1), 1]$ $(j \ge 2)$ has some iterate covering $[f^{l_{j-1}}(1), 1]$. A possible way to guarantee that a sequence (l_j) has these properties is by finding sequences $(r_j)_{j=2}^{\infty}$ and $(m_j)_{j=2}^{\infty}$ of positive integers satisfying $r_j \to \infty$ as $j \to \infty$, $\theta_{l_j+r} = \theta_r$ whenever $0 \le r < r_j$, and

$$\theta_{l_j+m_j} \neq \theta_{m_j}, \quad \theta_{l_j+m_j+l_{j-1}} \neq \theta_{m_j+l_{j-1}}$$

(because then both $f^{m_j}([f^{l_j}(1), 1])$ and $f^{m_j+l_{j-1}}([f^{l_j}(1), 1])$ contain c, hence $f^{m_j+l_{j-1}+1}([f^{l_j}(x), 1])$ contains both $f^{l_{j-1}}(1)$ and 1). Note that in this case c cannot be asymptotically periodic, as this would imply that c is in fact periodic and this is impossible because $\theta \in \{0, 1\}^{\infty}$.

In short, to obtain a topologically exact map $\tilde{h} = f \in \{f_\mu\}$ whose turning point is approximately but not asymptotically periodic, it is sufficient to find a sequence $\theta \in \{0, 1\}^{\infty}$ having the following properties:

(i) $\sigma^n(\theta) < \theta$ for every $n \ge 1$;

(ii) for every positive integer k there is a periodic sequence $\pi \in \{0, 1\}^{\infty}$ satisfying $\sigma(\theta) < \pi$, $\sigma^n(\pi) < \sigma$ for every n, and such that $\pi_{n+j} = \theta_{n+j} = \theta_{j-1}$ for every $1 \le j \le k$ and every n with the property $\theta_n \neq \pi_n$;

(iii) there are sequences $(l_j)_{j=1}^{\infty}$, $(r_j)_{j=2}^{\infty}$ and $(m_j)_{j=2}^{\infty}$ of positive integers such that $l_1 = 1$, $r_j \to \infty$ as $j \to \infty$, and satisfying
  (a) $\theta_{l_j+r} = \theta_r$ whenever $0 \le r < r_j$,
  (b) $\theta_{l_j+m_j} \neq \theta_{m_j}$, $\theta_{l_j+m_j+l_{j-1}} \neq \theta_{m_j+l_{j-1}}$,
  for every $j \ge 2$.

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To find such a sequence we proceed constructively. We start from an 'almost periodic' sequence with some gaps and then fill them in an appropriate way. Namely, let B_k and S_k be the (finite) sequences defined by

$$\begin{aligned} B_1 &= 1001010101101010110111\,00101 \\ S_1 &= 1001010101101 \end{aligned}$$

and

$$\begin{aligned} B_{k+1} &= B_k u S_k v B_k v S_k v B_k v S_k u B_k u S_k \\ S_{k+1} &= B_k u S_k v B_k v S_k \end{aligned}$$

where $u + v = 1$ and $u = 0$ (respectively, $u = 1$) if k is odd (respectively, k is even). Then we define θ as the sequence in $\{0, 1\}^\infty$ matching all of the patterns $B_k ? S_k ? B_k ? S_k ? \ldots$ (where ? indistinctly means 0 or 1). Clearly θ is unambiguously defined.

In what follows, b_k and s_k denote respectively the lengths of the sequences $B_k?$ and $S_k?$. We also write $d_k = b_k + s_k$. Note that if k is odd (respectively, k is even) then both B_k and S_k have an odd (respectively, even) number of 1's.

Before proving that θ satisfies the properties (i), (ii) and (iii) listed above we need a preliminary result: if $\alpha, \beta \in \{0, 1\}^\infty$ admit decompositions $B_k ? S_k ? B_k ? S_k ? \ldots$ for the same number k, then

$$\sigma^n(\alpha) < \beta \quad \text{whenever } n \text{ does not divide } d_k. \tag{3}$$

We prove (3) inductively. The statement follows for $k = 1$ by direct inspection. Next assume that the statement is true for a given number $k = l$, and let α, β admit decompositions $B_{l+1} ? S_{l+1} ? B_{l+1} ? S_{l+1} ? \ldots$. Assume that n does not divide d_{l+1}. If n does not divide d_l, then the induction hypothesis applies and $\sigma^n(\alpha) < \beta$. Since $d_{l+1} = 6 d_l$, we are left to analyze the cases $n = r d_{l+1} + s d_l$, where r is a nonnegative integer and $s \in \{1, 2, 3, 4, 5\}$. We have

$$\begin{aligned} \beta &= B_l u S_l v B_l v S_l v B_l v S_l u B_l u S_l ? B_l u S_l v B_l v S_l ? \ldots \\ \sigma^n(\alpha) &= B_l v S_l v B_l v S_l u B_l u S_l ? B_l u S_l v B_l v S_l ? B_l u S_l v \ldots && (s = 1) \\ \sigma^n(\alpha) &= B_l v S_l u B_l u S_l ? B_l u S_l v B_l v S_l ? B_l u S_l v \ldots && (s = 2) \\ \sigma^n(\alpha) &= B_l u S_l ? B_l u S_l v B_l v S_l ? B_l u S_l v \ldots && (s = 3) \\ \sigma^n(\alpha) &= B_l u S_l v B_l v S_l ? B_l u S_l v \ldots && (s = 4) \\ \sigma^n(\alpha) &= B_l v S_l ? B_l u S_l v \ldots && (s = 5) \end{aligned}$$

and it is easy again to check that $\sigma^n(\alpha) < \beta$ in all cases. For instance, assume that l is even and say $s = 4$. Since both B_l and S_l have an even number of 1's and $u = 1$, $v = 0$, the sequence $B_l u S_l v B_l v S_l$ has an odd number of $1's$, so $B_l u S_l v B_l v S_l u < B_l u S_l v B_l v S_l v$. Similarly, $B_l u S_l v B_l v S_l v B_l$ has an odd number

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of $1's$, so $B_l u S_l v B_l v S_l v B_l u < B_l u S_l v B_l v S_l v B_l v$. We see that $\sigma^n(\alpha) < \beta$ regardless of what the symbol ? represents in $\sigma^n(\alpha)$. We have shown that (3) is true for $k = l + 1$.

From (3) property (i) follows immediately. Concerning (ii), fix k and put $\pi = B_k v S_k v B_k v S_k v \dots$. The inequalities $\sigma(\theta) < \pi < \theta$ are obvious, and $\sigma^n(\pi) < \theta$ for every $1 \leq n < d_k$ due to (3). Since $\sigma^{d_k}(\pi) = \pi$, we obtain $\sigma^n(\pi) < \theta$ for every n. The last statement in (ii) is clear as well: as a matter of fact, if n is such that $\pi_n \neq \theta_n$, then $\pi_{n+j} = \theta_{n+j} = \theta_{j-1}$ for every $1 \leq j < s_k$.

It only remains to prove (iii). To this aim we define $l_2 = 3$, $l_3 = 14$ and $l_k = s_{k-2}$ for every $k \geq 3$, and also $m_2 = r_2 = r_3 = 2$, $m_3 = 6$, and $m_k = r_k = b_{k-3} - 1$ for every $k \geq 4$. Then

$$
\begin{aligned}
\theta &= 10\boxed{0}\boxed{1}01010110101011011100101\mathbf{0}100101010101101\mathbf{1}\dots \\
\sigma^{l_2}(\theta) &= 10\boxed{1}\boxed{0}10110101011011100101\mathbf{0}100101010101101\mathbf{1}\dots, \\
\theta &= 100101\boxed{0}10\boxed{1}10101011011100101\mathbf{0}100101010101101\mathbf{1}\dots \\
\sigma^{l_3}(\theta) &= 101101\boxed{1}10\boxed{0}101\mathbf{0}100101010101101\mathbf{1}\dots
\end{aligned}
$$

(we have used bold type to mark the first coefficients after the blocks B_1 and S_1 of θ and boxed the key coefficients), and

$$
\begin{aligned}
\theta &= B_{k-3}\boxed{u}S_{k-3}\boxed{v}B_{k-3}vS_{k-3}vB_{k-3}vS_{k-3}uB_{k-3}uS_{k-3}\boldsymbol{v} \\
&\qquad B_{k-3}uS_{k-3}vB_{k-3}vS_{k-3}\boldsymbol{u}\dots \\
\sigma^{l_k}(\theta) &= B_{k-3}\boxed{v}S_{k-3}\boxed{u}B_{k-3}uS_{k-3}\boldsymbol{v}B_{k-3}uS_{k-3}vB_{k-3}vS_{k-3}\boldsymbol{u}\dots
\end{aligned}
$$

for every $k \geq 4$ (now bold type indicates the first coefficients after the blocks B_{k-2} and S_{k-2} of θ). It is easy to check that (iii) holds. We are done.

4. Proof of Theorem D

The proof of Theorem D virtually follows after the next sequence of lemmas. The first two of them only require f to be a C^1-map.

LEMMA 4.1. *Assume that $f \in C^1(I)$ is not monotone. Then there exists $\epsilon > 0$ such that if $A \subset I$ is a measurable set and $\lambda(A) > 1 - \epsilon$, then f is not one-to-one in A.*

PROOF. Since f is not monotone, we can easily find two points a, b interior to I such that $f(a) = f(b)$ and $f'(a) \neq 0 \neq f'(b)$. Then there are closed intervals $a \in U, b \in V$ such that $f|_U$ and $f|_V$ are diffeomorphisms onto their common image $f(U) = f(V)$. Let

$$m_U = \min_{x \in U} |f'(x)| \quad \text{and} \quad M_U = \max_{x \in U} |f'(x)|$$

and define m_V and M_V accordingly. We assume that the intervals U and V are small enough so that $m_U > M_U/2$ and $m_V > M_V/2$.

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Let

$$\epsilon = \min\left\{\frac{\lambda(U)}{m_U}\left(m_U - \frac{M_U}{2}\right), \frac{\lambda(V)}{m_V}\left(m_V - \frac{M_V}{2}\right)\right\}$$

and assume $\lambda(A) > 1 - \epsilon$ for some measurable set A. Then

$$\begin{aligned}\lambda(f(A \cap U)) &\geq m_U \lambda(A \cap U) = m_U(\lambda(U) + \lambda(A) - \lambda(A \cup U)) \\ &> m_U(\lambda(U) - \epsilon) \geq m_U\left(\lambda(U) - \frac{1}{m_U}\left(m_U - \frac{M_U}{2}\right)\lambda(U)\right) \\ &= \frac{M_U}{2}\lambda(U)\end{aligned}$$

and similarly $\lambda(f(A \cap V)) > M_V \lambda(V)/2$. Since

$$\lambda(f(U)) \leq \min\{M_U \lambda(U), M_V \lambda(V)\},$$

we have

$$\begin{aligned}\lambda(f(A \cap U) \cap f(A \cap V)) &= \lambda(f(A \cap U)) + \lambda(f(A \cap V)) \\ &\quad - \lambda(f(A \cap U) \cup f(A \cap V)) \\ &> \tfrac{1}{2} M_U \lambda(U) + \tfrac{1}{2} M_V \lambda(V) - \lambda(f(U)) \\ &\geq 0,\end{aligned}$$

so f is not one-to-one in A. □

LEMMA 4.2. *Let $f \in C^1(I)$ and $J \subset I$, and assume that $f^k(J) = I$ for some k. Let $\epsilon > 0$. Then there exists $\delta = \delta_\epsilon > 0$ such that if $A \subset J$ and $\lambda(A)/\lambda(J) > 1 - \delta$, then $\lambda(f^k(A)) > 1 - \epsilon$.*

PROOF. Take $M = \max_{x \in I} |(f^k)'(x)|$ and define $\delta = \epsilon/(M\lambda(J))$. For every subset A of J satisfying $\lambda(A)/\lambda(J) > 1 - \delta$

$$\begin{aligned}\lambda(I \setminus f^k(A)) &\leq \lambda(f^k(J \setminus A)) \leq M\lambda(J \setminus A) = M(\lambda(J) - \lambda(A)) \\ &< M(\lambda(J) - (1 - \delta)\lambda(J)) = \delta M \lambda(J) = \epsilon,\end{aligned}$$

and hence $\lambda(f^k(A)) > 1 - \epsilon$. □

In the ensuing lemma the following notion will be used. If $\delta > 0$ and K is a subinterval of an interval J with the property that both components of $J \setminus K$ have at least length $\delta\lambda(J)$, then we call J a *δ-scaled neighbourhood* of K.

LEMMA 4.3. *Assume that $f \in S^3(I)$ is topologically exact and has no wild attractors. Then there exists an interval J such that, for every subset A of I of positive measure, every $\epsilon > 0$ and almost every $x \in A$, there are an arbitrarily large integer l and a subset B of A containing x such that $f^l(B) \subset J$, $f^l|_B$ is one-to-one, and $\lambda(f^l(B))/\lambda(J) > 1 - \epsilon$.*

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PROOF. Since f is topologically exact and has no wild attractors, its only metric attractor is the whole interval I. Now we can use [9, Theorem 5.3] to find a finite set $Y \subset I$ with the property that if J and K are subintervals of I not intersecting Y, and J is contained in the interior of K, then for almost every $x \in I$ there are an arbitrarily large number l and intervals $W' \subset W''$ neighbouring x such that $f^l(W') = J$, $f^l(W'') = K$, and $f^l|_{W''}$ is diffeomorphic. In what follows we fix J and K just additionally assuming that K is a $1/2$-scaled neighbourhood of J.

Let $\epsilon > 0$ and assume that $A \subset I$ has positive measure. Then for almost every density point x of A (hence for almost every $x \in A$) we can find an arbitrarily large number l and intervals W', W'' with the properties described above. Actually, if l is large enough, then W'' must be very small, otherwise one could find arbitrarily large numbers l_m and corresponding intervals W_m such that $f^{l_m}(W_m) = K \neq I$, in contradiction with the total transitivity of f. Since x is a density point of A, we can then assume that $\lambda(A \cap W')/\lambda(W') > 1 - \epsilon/9$.

Recall that K is a $1/2$-scaled neighbourhood of J. Then we can use the Koebe inequality as stated in [9, Lemma 3.4] to obtain

$$\lambda(C)/\lambda(W') \geq \lambda(f^l(C))/(9\lambda(J))$$

for every measurable subset C of W'. In particular,

$$\frac{\lambda(f^l(W' \setminus A))}{\lambda(J)} \leq 9\frac{\lambda(W' \setminus A)}{\lambda(W')} = 9\left(1 - \frac{\lambda(A \cap W')}{\lambda(W')}\right) < \epsilon.$$

Since $f^l|_{W'}$ is a homeomorphism, $B = A \cap W'$ does the job. □

Combining Lemmas 4.2 and 4.3 we immediately obtain the following.

LEMMA 4.4. *Assume that $f \in S^3(I)$ is topologically exact and has no wild attractors. Let $A \subset I$ have positive measure and let $\epsilon > 0$. Then there is an arbitrarily large integer m such that $\lambda(f^m(A)) > 1 - \epsilon$.*

Theorem D will follow easily from the next two lemmas.

LEMMA 4.5. *Assume that $f \in S^3(I)$ is topologically exact and has no wild attractors. Let $A \subset I$ be a measurable set such that $f^n|_A$ is one-to-one for every nonnegative integer n. Then $\lambda(A) = 0$.*

PROOF. Otherwise we use Lemmas 4.1 and 4.4 to arrive at a contradiction. □

LEMMA 4.6. *If $f \in S^3(I)$ is topologically exact and has no wild attractors, then $\lambda^2(\mathrm{Scramb}(f)) = 1$.*

PROOF. Let $x \in I$ and consider the sets

$$A_x = \left\{y \in I : \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0\right\},$$

$$B_x = \left\{y \in I : \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0\right\}.$$

It suffices to show that $\lambda(A_x) = \lambda(B_x) = 1$.

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Suppose $\lambda(A_x) < 1$. Then the set

$$\left\{ y \in I : \lim_{n\to\infty} |f^n(x) - f^n(y)| = 0 \right\}$$

has positive measure and there are $0 < \epsilon < 1/3$ and $k \in \mathbb{N}$ such that

$$C = \{y \in I : |f^n(x) - f^n(y)| \leq \epsilon \text{ for every } n \geq k\}$$

has positive measure as well. By Lemma 4.4, there exists $m > k$ such that

$$\lambda(f^m(C)) > 1 - \epsilon.$$

In particular, C contains two points y, z such that

$$|f^m(y) - f^m(z)| > 1 - \epsilon > 2/3$$

but also satisfying, due to the definition of C,

$$|f^m(y) - f^m(z)| \leq |f^m(y) - f^m(x)| + |f^m(x) - f^m(z)| \leq 2\epsilon < 2/3,$$

which is impossible.

Similarly, suppose now $\lambda(B_x) < 1$ to find $\epsilon > 0$ and k such that

$$D = \{y \in I : |f^n(x) - f^n(y)| \geq \epsilon \text{ for every } n \geq k\}$$

has positive measure, and take $m > k$ with the property $\lambda(f^m(D)) > 1 - \epsilon$, which also implies that the interval $\{z \in I : |f^m(x) - z| < \epsilon\}$ has measure less than ϵ, a contradiction. $\square$

Proof of Theorem D. Let A be a measurable scrambled set of f. Then $f^n(A)$ is scrambled for f and A is scrambled for f^n for every $n \geq 1$. In addition, A can contain at most one approximately periodic point by Proposition 1.5, hence almost every point of A must be attracted by some metrically exact interval orbit. Note also that the number of topologically exact intervals of f is finite because each pair of them have pairwise disjoint interiors and the orbit of every topologically exact interval must contain some critical point of f.

Thus, to prove that $\lambda(A) = 0$, we can assume without loss of generality that f is topologically exact. Since A is scrambled, $f^n|_A$ is one-to-one for every n and $\lambda(A) = 0$ follows from Lemma 4.5.

It only remains to shows that if f has some metrically exact interval orbit, then $\lambda^2(\mathrm{Scramb}(f)) > 0$, which immediately follows from Lemma 4.6. $\square$

5. Complements on approximate periodicity and measurability

First we provide a simple characterization of approximate periodicity.

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PROPOSITION 5.1. *Let $f \in C(I)$ and $x \in I$. Then x is approximately periodic if and only if $\omega_f(x)$ is either finite or an adding machine.*

PROOF. We can assume that $\omega_f(x)$ is infinite.

Proving the 'if' part of the statement is easy, because if $\epsilon > 0$ is given, $\{C, f(C), \dots, f^{r-1}(C)\}$ is a partition of $\omega_f(x)$ such that $f^r(C) = C$ and all sets $f^i(C)$ have diameters less than ϵ, and we denote by J_i the smallest connected sets containing $f^i(C)$, then we have $f(J_i) \supset J_{i+1}$ for every i. Hence, there is a fixed point p of f^r such that $f^i(p) \subset J_i$, $0 \le i < r$. Moreover, the properties of C force $C = \omega_{f^r}(f^i(x))$ for some $0 \le i < r$, and it is not restrictive to assume $i = 0$. Therefore,

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \epsilon.$$

We prove the 'only if' part of the statement. Assume that $x \in I$ is approximately periodic. Let $A = \omega_f(x)$ and let $(P_j)_{j=1}^{\infty}$ be a sequence of periodic orbits of respective periods r_j satisfying

$$\limsup_{n\to\infty} |f^n(x) - f^n(p_j)| \le \frac{1}{j}$$

for some $p_j \in P_j$ and every j. Clearly, A is the limit set of the union $P = \bigcup_j P_j$.

Let $y \in A$ and fix j. Then there is $0 \le k < r_j$ such that y is a limit point of the sequence $(f^{mr_j+k}(x))_m$. Hence, if we put $q_j = f^k(p_j)$, we see that $|f^n(y) - f^n(q_j)| \le 1/j$ for every n. Now it is clear that $\omega_f(y)$ is also the limit set of P, that is, $\omega_f(y) = A$ for every $y \in A$. We have shown that $f|_A$ is minimal.

Let $\epsilon > 0$, fix $j > 2/\epsilon$, rename $s = r_j$, and let $C_i = \omega_{f^s}(f^i(x))$, $0 \le i < s$. Then $f(C_i) = C_{i+1}$ for every i (we mean $C_s = C_0$). Since each $f^i(x)$ is approximately periodic for f^s, we apply the previous reasoning to conclude that every map $f^s|_{C_i}$ is minimal. Let $0 \le l, m < s$, $l \ne m$, and assume that $C_l \cap C_m \ne \emptyset$. Then $f^s(C_l \cap C_m) \subset C_l \cap C_m$, so the minimality of $f^s|_{C_l}$ and $f^s|_{C_m}$ forces $C_l = C_m$. Further, observe that if $y \in C_i$, then $|y - f^i(p_j)| \le 1/j$ and therefore the diameter of C_i is at most $2/j$. If we define $C = C_0$ and r is the minimal positive number satisfying $f^r(C) = C$, then the partition $\{C, f(C), \dots, f^{r-1}(C)\}$ satisfies the required properties in the definition of an adding machine for the number ϵ. □

We conclude the section and the paper proving that $R(f)$, $S(f)$ and $\mathrm{Scramb}(f)$ are Borel sets. In the case of $S(f)$ this is clear because if $S_\delta(f)$ denotes the set of δ-sensitive points of f, then $S_\delta(f)$ is closed and $S(f) = \bigcup_{j=1}^{\infty} S_{1/j}(f)$. Also, $\mathrm{Scramb}(f)$ is a Borel set in the square I^2 because

$$\Big\{(x, y) \in I^2 : \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0\Big\} = \bigcup_{j=1}^{\infty} \bigcap_{l=0}^{\infty} \bigcup_{n=l}^{\infty} \{(x, y) \in I^2 : |f^n(x) - f^n(y)| > 1/j\}$$

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and

$$\Big\{(x, y) \in I^2 : \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0\Big\}$$
$$= \bigcap_{j=1}^{\infty} \bigcap_{l=0}^{\infty} \bigcup_{n=l}^{\infty} \{(x, y) \in I^2 : |f^n(x) - f^n(y)| < 1/j\}.$$

To prove that $R(f)$ is a Borel set it suffices to show that

$$R(f) = \bigcap_{j=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \bigcap_{l=1}^{\infty} \{x \in I : |f^{m+rl}(x) - f^m(x)| < 1/j\}.$$

Clearly, $R(f)$ is contained in the right-hand side set. Conversely, suppose that x belongs to the right-hand side set. We can assume that x is not asymptotically periodic, since otherwise it trivially belongs to $R(f)$. Fix j. Then there are numbers r and n such that $|f^{m+rl}(x) - f^m(x)| < 1/j$ whenever $m \geq n$ and $l \geq 1$. For every such m, let I_m denote the smallest closed interval containing the points $\{f^{m+rl}(x)\}_{l=0}^{\infty}$. We claim that $I_t = I_{t+r}$ for some t. Suppose not, write $y = f^n(x)$ and $g = f^r$ and consider the sequence

$$(f^{n+rl}(x))_{l=0}^{\infty} = (g^l(y))_{l=0}^{\infty}.$$

Then $g^{k+1}(y) < g^k(y)$ (respectively, $g^{k+1}(y) > g^k(y)$) implies $g^l(y) < g^k(y)$ (respectively, $g^l(y) > g^k(y)$) for every $l > k$. The sequence $(g^l(y))$ cannot be monotone because x is not asymptotically periodic (so neither is y). If, say, $g(y) > y$, then

$$\begin{aligned} y &< g(y) < \cdots < g^{l_1-1}(y) < g^{l_2}(y) < g^{l_2+1}(y) < \cdots < g^{l_3-1}(y) < g^{l_4}(y) \\ &< \cdots < \cdots < g^{l_4-1}(y) < \cdots < g^{l_3+1}(y) < g^{l_3}(y) < g^{l_2-1}(y) < \cdots < g^{l_1+1}(y) \\ &< g^{l_1}(y). \end{aligned}$$

Now it is clear that $(g^l(y))$ accumulates at most at two points, which implies that y is asymptotically periodic, a contradiction.

We have shown that $I_t = I_{t+r}$ for some $t \geq n$. Since $f(I_m) \supset I_{m+1}$ for every m, we obtain that $f^r(I_t)$ covers I_t and then there is a fixed point q of f^r such that $f^i(q) \in I_{t+i}$ for every $0 \leq i < r$. Since all intervals I_m have diameter at most $1/j$, we obtain that

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| \leq 1/j$$

for the point p of the orbit of q such that $f^t(p) = q$. Since j was arbitrarily fixed, we have shown that $x \in R(f)$.

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