Can. J. Math., Vol. XXVI, No. 6, 1974, pp. 1484–1497

GREEN'S RELATIONS FOR REGULAR ELEMENTS OF SEMIGROUPS OF ENDOMORPHISMS

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This paper is dedicated to Alfred H. Clifford on the occasion of his 65th birthday.

1. Introduction. X is a set and End X is a semigroup, under composition, of functions, which map X into X. We characterize those elements of End X which are regular and then we completely determine Green's relations for these elements. The conditions we place on End X are sufficiently mild to permit such semigroups as $S(X)$, the semigroup of all continuous selfmaps of a topological space X and $L(V)$, the semigroup of all linear transformations on a vector space V, to be regarded as special cases.

In order to give some idea of the kind of results we obtain, we discuss the situation for $S(X)$. First of all, we show that an element f in $S(X)$ is regular if and only if its range is a retract and it maps some subspace of X homeomorphically onto its range. It then follows from this that those spaces X for which the semigroup $S(X)$ is regular, are rather exceptional. For example $S(X)$ is never regular if X is completely regular, Hausdorff and contains an arc. If X is a noncompact 0-dimensional metric space, then $S(X)$ is regular if and only if X is discrete. Of course, when X is discrete, $S(X)$ is just $\mathscr{T}_X$, the full transformation semigroup on X and the fact that this semigroup is regular was discovered by C. G. Doss [**4**]. For two regular elements of $S(X)$ we show: they are $\mathscr{L}$-related if and only if the decompositions on X induced by them are identical; they are $\mathscr{R}$-related if and only if their ranges coincide; they are $\mathscr{D}$-related if and only if their ranges are homeomorphic; and they are $\mathscr{J}$-related if and only if the range of each contains a retract which is homeomorphic to the range of the other. By taking X to be discrete, we obtain as a special case, characterizations of Green's relations on the full transformation semigroup. These results were first obtained by D. D. Miller and C. G. Doss [**4**] and a nice account of them is given in [**3**].

Now, D. D. Miller and A. H. Clifford [**7**] proved some years ago that if one element in the $\mathscr{D}$-class is regular, then they are all regular. Such classes are called regular $\mathscr{D}$-classes. Our results allow us to completely determine the regular $\mathscr{D}$-classes of $S(X)$ and we find that certain spaces are characterized within a huge class of spaces by the number of regular $\mathscr{D}$-classes their semigroups have. For example, if X is any completely regular Hausdorff space

Received June 12, 1973 and in revised form, October 29, 1973.

https://doi.org/10.4153/CJM-1974-144-x Published online by Cambridge University Press

which contains an arc, then X is itself an arc if and only if $S(X)$ has precisely two regular $\mathscr{D}$-classes.

One further remark is in order. So far as we know, the only other attempt to systematically study Green's relations on $S(X)$ has been made by F. A. Cezus in his doctoral dissertation [**1**]. The approach there is quite different than it is here and consequently, his results are of a different nature than ours.

2. Some general results. The symbols $\mathrm{Dom}(f)$ and $\mathrm{Ran}(f)$ will be used to denote respectively the domain and range of a function f. Composition of functions will be denoted by simple juxtaposition. Furthermore f/A will denote the restriction of a function f to a subset A of its domain.

Definition (2.1). By a Δ-*structure* on a nonempty set X, we mean any pair $(\mathscr{A}, \mathscr{M})$ where $\mathscr{A}$ is a family of subsets of X containing X itself and

$$\mathscr{M} = \{\mathrm{Hom}(A, B) : (A, B) \in \mathscr{A} \times \mathscr{A}\}$$

where $\mathrm{Hom}(A, B)$ is a collection of functions with domains equal to A and ranges contained in B, and the following conditions are satisfied.

(2.1.1) End $X = \mathrm{Hom}(X, X)$ is a semigroup under composition which contains id_X the identity map on X.

(2.1.2) $\mathrm{Ran}(f) \in \mathscr{A}$ for each f in End X.

(2.1.3) If $f \in$ End X and $g \in \mathrm{Hom}(\mathrm{Ran}(f), B)$, then $gf \in$ End X.

(2.1.4) Suppose $f, g \in$ End X; $A, B, \in \mathscr{A}$; $f(B) \subset A$, $g(A) \subset B$ and suppose also that $fg/A = \mathrm{id}_A$ and $gf/B = \mathrm{id}_B$. Then $g/A \in \mathrm{Hom}(A, B)$ and $f/B \in \mathrm{Hom}(B, A)$.

We will refer to End X as the semigroup of the Δ-structure $(\mathscr{A}, \mathscr{M})$.

Definition (2.2). A function f in $\mathrm{Hom}(A, B)$ is a Δ-isomorphism if there exists a g in $\mathrm{Hom}(B, A)$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$. If $\mathrm{Hom}(A, B)$ contains a Δ-isomorphism, we say that A and B are Δ-isomorphic.

Definition (2.3). A Δ-retract of X is any subset which is the range of an idempotent map in End X.

We recall that an element a of a semigroup is regular if $aba = a$ for some element b of the semigroup. Our first result characterizes the regular elements of semigroup End X of an arbitrary Δ-structure $(\mathscr{A}, \mathscr{M})$ on X.

THEOREM (2.4). *Let* End X *be the semigroup of a* Δ-*structure* $(\mathscr{A}, \mathscr{M})$ *on* X. *Then the following statements about a function* f *in* End X *are equivalent.*

(2.4.1) *f is regular.*

(2.4.2) *The range of f is a Δ-retract of X and there exists a Δ-retract A of X such that f/A is a Δ-isomorphism from A onto* $\mathrm{Ran}(f)$.

(2.4.3) *The range of f is a Δ-retract of X and there exists a set $A \in \mathscr{A}$ such that f/A is a Δ-isomorphism from A onto* $\mathrm{Ran}(f)$.

Proof. We first show that (2.1.1) implies (2.1.2). Since f is regular, there exists a function g in End X such that $fgf = f$. Then fg is idempotent and one easily shows that $\mathrm{Ran}(f) = \mathrm{Ran}(fg)$. Thus, the range of f is a Δ-retract of X. Furthermore, gf is also idempotent so that $\mathrm{Ran}(gf)$ is a Δ-retract of X. Now let $y \in \mathrm{Ran}(f)$. Then $y = f(x)$ for some x in X and

$$fg(y) = fgf(x) = f(x) = y.$$

Thus the restriction of fg to $\mathrm{Ran}(f)$ is the identity on $\mathrm{Ran}(f)$. On the other hand, if $y \in \mathrm{Ran}(gf)$, then $y = gf(x)$ for some x in X and we have

$$gf(y) = gfgf(x) = gf(x) = y.$$

Thus, the restriction of gf to $\mathrm{Ran}(gf)$ is the identity on $\mathrm{Ran}(gf)$ so by (2.1.4), we have $f/\mathrm{Ran}(gf) \in \mathrm{Hom}(\mathrm{Ran}(gf), \mathrm{Ran}(f))$ and

$$g/\mathrm{Ran}(f) \in \mathrm{Hom}(\mathrm{Ran}(f), \mathrm{Ran}(gf)).$$

All this implies that $f/\mathrm{Ran}(gf)$ is a Δ-isomorphism from $\mathrm{Ran}(gf)$ onto $\mathrm{Ran}(f)$. This verifies that (2.1.1) implies (2.1.2) and since it is evident that (2.1.2) implies (2.1.3), we need only show that (2.1.3) implies (2.1.1). Since $\mathrm{Ran}(f)$ is a Δ-retract of X, there exists an idempotent map v in End X such that $\mathrm{Ran}(v) = \mathrm{Ran}(f)$. Furthermore, since f/A is a Δ-isomorphism from A onto $\mathrm{Ran}(f)$, there exists a $t \in \mathrm{Hom}(\mathrm{Ran}(f), A)$ so that $t(f/A)$ is the identity on A and $(f/A)t$ is the identity on $\mathrm{Ran}(f)$. By condition (2.1.3), the function $g = tv$ belongs to End X. One readily shows that $fgf = f$ and the proof is complete.

Now we characterize Green's relations for the regular elements of End X. These relations were introduced by J. A. Green [**5**] in 1951 and are discussed in detail in [**3**]. We recall them briefly. Two elements of an arbitrary semigroup T are $\mathscr{L}$-related if they both generate the same left ideal, $\mathscr{R}$-related if they both generate the same right ideal and $\mathscr{J}$-related if they both generate the same two-sided ideal. The relation $\mathscr{H}$ is defined to be the intersection of $\mathscr{L}$ and $\mathscr{R}$. Furthermore, $\mathscr{L}$ and $\mathscr{R}$ commute so that $\mathscr{L} \circ \mathscr{R}$ is also an equivalence relation which is denoted by $\mathscr{D}$. The five relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$ are Green's relations and we determine just what they are for regular elements of End X. Throughout the remainder, it will be assumed without specifically stating it that End X is the semigroup of some Δ-structure $(\mathscr{A}, \mathscr{M})$ on X.

Before we state our next result, we introduce some notation. For any function f in End X, we let $\pi(f)$ denote the decomposition of X induced by f; that is,

$$\pi(f) = \{f^{-1}(y) : y \in \mathrm{Ran}(f)\}.$$

THEOREM (2.5). *Two regular elements f and g of* End X *are $\mathscr{L}$-related if and only if $\pi(f) = \pi(g)$ and they are $\mathscr{R}$-related if and only if* $\text{Ran}(f) = \text{Ran}(g)$. *Consequently, they are $\mathscr{H}$-related if and only if $\pi(f) = \pi(g)$ and* $\text{Ran}(f) = \text{Ran}(g)$.

Proof. First suppose that f and g are $\mathscr{L}$-related. Then $f = kg$ and $g = hf$ for appropriate k and h in End X. The former implies that $\pi(g)$ refines $\pi(f)$ and the latter implies that $\pi(f)$ refines $\pi(g)$. Thus, $\pi(f) = \pi(g)$. Suppose, conversely, that $\pi(f) = \pi(g)$. We show that f and g are $\mathscr{L}$-related. First of all, since f and g are regular, there exist idempotents v and w such that v is $\mathscr{L}$-related to f and w is $\mathscr{L}$-related to g [**3**, p. 27]. By our previous observations, $\pi(v) = \pi(f)$ and $\pi(w) = \pi(g)$. Thus $\pi(v) = \pi(w)$ and this means that for any $x, y, \in X$, $v(x) = v(y)$ if and only if $w(x) = w(y)$. Since v is idempotent, $v(v(x)) = v(x)$ for any $x \in X$ and this implies that $w(v(x)) = w(x)$. Similarly, $v(w(x)) = v(x)$. That is, $wv = w$ and $vw = v$ which implies that v and w are $\mathscr{L}$-related. Hence f and g are $\mathscr{L}$-related.

Now suppose that f and g are $\mathscr{R}$-related. Then $f = gk$ and $g = fh$ for appropriate k and h in End X. It follows immediately from this that $\text{Ran}(f) = \text{Ran}(g)$. Suppose, conversely that $\text{Ran}(f) = \text{Ran}(g)$. Since f and g are regular, there exist idempotents v and w $\mathscr{R}$-related to f and g respectively. Then $\text{Ran}(v) = \text{Ran}(w)$ and since v and w are idempotent, we immediately get $v = wv$ and $w = vw$. Thus v and w are $\mathscr{R}$-related and it follows that f and g are $\mathscr{R}$-related.

Remarks. For arbitrary elements of End X, the conditions given above are necessary but not sufficient. We will defer discussing any specific examples until the section where we deal with semigroups of continuous functions.

THEOREM (2.6). *Two regular elements f and g of* End X *are $\mathscr{D}$-related if and only if* $\text{Ran}(f)$ *is Δ-isomorphic to* $\text{Ran}(g)$.

Proof. Suppose first that there exists a Δ-isomorphism t from $\text{Ran}(f)$ onto $\text{Ran}(g)$. We show that f and g are $\mathscr{D}$-related. Since they are regular, there exist idempotents v and w such that f is $\mathscr{R}$-related to v and g is $\mathscr{R}$-related to w. By the previous theorem, $\text{Ran}(f) = \text{Ran}(v)$ and $\text{Ran}(g) = \text{Ran}(w)$. Since $\mathscr{R} \subset \mathscr{D}$, we need only show that v and w are $\mathscr{D}$-related in order to conclude that f and g are $\mathscr{D}$-related. Since t is injective, $v(x) = v(y)$ if and only if $tv(x) = tv(y)$. Thus $\pi(v) = \pi(tv)$ which, according to the previous result, means that v and tv are $\mathscr{L}$-related. Since t maps $\text{Ran}(v)$ onto $\text{Ran}(w)$, we have $\text{Ran}(tv) = t(\text{Ran}(v)) = \text{Ran}(w)$. Thus, by Theorem (2.5), tv and w are $\mathscr{R}$-related. All this implies that v and w, and hence f and g are $\mathscr{D}$-related.

Now suppose that f and g are $\mathscr{D}$-related. We must show that $\text{Ran}(f)$ and $\text{Ran}(g)$ are Δ-isomorphic. Again, we choose idempotents v and w which are $\mathscr{R}$-related to f and g respectively. By Theorem (2.5), $\text{Ran}(f) = \text{Ran}(v)$ and $\text{Ran}(g) = \text{Ran}(w)$. Since $\mathscr{R} \subset \mathscr{D}$, v and w are $\mathscr{D}$-related and we use this fact to show that $\text{Ran}(v)$ and $\text{Ran}(w)$ are Δ-isomorphic. Since $\mathscr{D} = \mathscr{L} \circ \mathscr{R}$, there exists an element p of End X such that v and p are $\mathscr{L}$-related while p and w are $\mathscr{R}$-related. Then $p = tv$ and $v = kp$ for appropriate t, k in End X. Moreover, it follows from Theorem (2.5) that $\mathrm{Ran}(p) = \mathrm{Ran}(w)$. We show that the restriction of t to $\mathrm{Ran}(v)$ is a Δ-isomorphism onto $\mathrm{Ran}(p) = \mathrm{Ran}(w)$. We have

$$p = tv = tkp$$

which implies that the restriction of tk to $\mathrm{Ran}(w) = \mathrm{Ran}(p)$ is the identity on $\mathrm{Ran}(w)$. Moreover, for any x in $\mathrm{Ran}(v)$, we have

$$kt(x) = ktv(x) = kp(x) = v(x) = x.$$

Thus, the restriction of kt to $\mathrm{Ran}(v)$ is the identity map on $\mathrm{Ran}(v)$. Since t maps $\mathrm{Ran}(v)$ into $\mathrm{Ran}(w)$ and k maps $\mathrm{Ran}(w)$ into $\mathrm{Ran}(v)$, it follows that $t/\mathrm{Ran}(v)$ is a Δ-isomorphism from $\mathrm{Ran}(v)$ onto $\mathrm{Ran}(w)$. Since $\mathrm{Ran}(f) = \mathrm{Ran}(v)$ and $\mathrm{Ran}(g) = \mathrm{Ran}(w)$, we conclude that $\mathrm{Ran}(f)$ and $\mathrm{Ran}(g)$ are Δ-isomorphic.

As we mentioned previously, if one element in a $\mathscr{D}$-class is regular, then all of the elements in that $\mathscr{D}$-class are regular and such $\mathscr{D}$-classes are referred to as regular $\mathscr{D}$-classes. Theorems (2.4) and (2.6) together allow us to describe quite simply the regular $\mathscr{D}$-classes of End X. This is the content of the next theorem.

THEOREM (2.7). *Let A be any Δ-retract of X and let D_A consist of all those functions f in* End X *such that* $\mathrm{Ran}(f)$ *is a Δ-retract of X which is Δ-isomorphic to A and the restriction of f to some B in $\mathscr{A}$ is a Δ-isomorphism onto* $\mathrm{Ran}(f)$. *Then D_A is a regular $\mathscr{D}$-class of End X and all regular $\mathscr{D}$-classes are obtained in exactly this manner.*

Proof. We show first that D_A is a regular $\mathscr{D}$-class. Since A is a Δ-retract, there exists an idempotent v in End X such that $A = \mathrm{Ran}(v)$. Let $D(v)$ denote the $\mathscr{D}$-class to which v belongs. We show that $D_A = D(v)$. Choose any f in D_A. By Theorem (2.4), f is regular and this fact, together with Theorem (2.6) implies that f belongs to $D(v)$. Now suppose that $g \in D(v)$. Then g is regular and by Theorem (2.6), $\mathrm{Ran}(g)$ is Δ-isomorphic to $\mathrm{Ran}(v) = A$. It now follows from Theorem (2.4) that $g \in D_A$. Thus, $D_A = D(v)$ which proves that D_A is a regular $\mathscr{D}$-class.

On the other hand, let D be any regular $\mathscr{D}$-class. We must show that $D = D_A$ for an appropriate Δ-retract A. Since D is regular it contains an idempotent v so that $D = D(v)$. Let $A = \mathrm{Ran}(v)$. By previous considerations, D_A is a regular $\mathscr{D}$-class and Theorems (2.4) and (2.6) together imply that $D(v) \subset D_A$. Thus $D_A = D(v) = D$ and the proof is complete.

Remark. Suppose we define two Δ-retracts to be equivalent if they are Δ-isomorphic. Theorems (2.6) and (2.7) tell us, among other things, that there are exactly as many regular $\mathscr{D}$-classes of End X as there are equivalence classes of Δ-retracts.

THEOREM (2.8). *Two regular elements of* End X *are* $\mathscr{J}$*-related if and only if the range of each contains a* Δ*-retract which is* Δ*-isomorphic to the range of the other.*

Proof. Let f and g be regular elements of End X. Then there exist idempotents v and w which are $\mathscr{R}$-related to f and g respectively. By Theorem (2.4), $\operatorname{Ran}(v) = \operatorname{Ran}(f)$ and $\operatorname{Ran}(w) = \operatorname{Ran}(g)$ so we show that $\operatorname{Ran}(v)$ contains a Δ-retract which is Δ-isomorphic to $\operatorname{Ran}(w)$. First of all, $\mathscr{R} \subset \mathscr{J}$ so it follows that v and w are $\mathscr{J}$-related. Thus, $v = hwk$ for appropriate h and k in End X. It is immediate since v is idempotent that the restriction of $h(wk)$ to $\operatorname{Ran}(v)$ is the identity map on $\operatorname{Ran}(v)$. Now, let y be any element of $\operatorname{Ran}(wkv)$. Then $y = wk(x)$ for some x in $\operatorname{Ran}(v)$ and we get

$$(wk)h(y) = wkhwk(x) = wkv(x) = wk(x) = y.$$

Thus, the restriction of $(wk)h$ to $\operatorname{Ran}(wkv)$ is the identity on $\operatorname{Ran}(wkv)$. Let $t = (wk)/\operatorname{Ran}(v)$. Since wk maps $\operatorname{Ran}(v)$ into $\operatorname{Ran}(wkv)$ and h maps $\operatorname{Ran}(wkv)$ into $\operatorname{Ran}(v)$, it follows from condition (2.1.4) that $t \in \operatorname{Hom}(\operatorname{Ran}(v), \operatorname{Ran}(wkv))$ and, in fact, the previous observations allow us to conclude that t is a Δ-isomorphism from $\operatorname{Ran}(v)$ onto $\operatorname{Ran}(wkv)$. It is evident that $\operatorname{Ran}(wkv) \subset \operatorname{Ran}(w)$. To see that $\operatorname{Ran}(wkv)$ is a Δ-retract, consider the function tvh. It belongs to End X because of (2.1.3) and certainly, $\operatorname{Ran}(tvh) \subset \operatorname{Ran}(wkv)$. Moreover, for $y \in \operatorname{Ran}(wkv)$, we have $y = wkv(x)$ for some $x \in X$ and hence

$$y = wkv(x) = tv(x) = tvv(x) = tvh(wk(x)).$$

This means $y \in \operatorname{Ran}(tvh)$ and we conclude that $\operatorname{Ran}(tvh) = \operatorname{Ran}(wkv)$. Now we show that tvh is idempotent. Any $y \in \operatorname{Ran}(tvh)$ is of the form $wkv(x)$ for an appropriate x in X. Thus,

$$tvh(y) = tvhwkv(x) = wkvhwkv(x) = wkvvv(x) = wkv(x) = y.$$

That is, the restriction of tvh to its range is the identity map so we conclude that tvh is idempotent and hence that $\operatorname{Ran}(wkv) = \operatorname{Ran}(tvh)$ is a Δ-retract. In a similar manner, one shows that $\operatorname{Ran}(v)$ contains a Δ-retract which is Δ-isomorphic to $\operatorname{Ran}(w)$.

Now suppose f and g are regular elements such that the range of each contains a Δ-retract which is Δ-isomorphic to the range of the other. Again, we choose idempotents v and w in End X which are $\mathscr{R}$-related to f and g respectively and, again, we have $\operatorname{Ran}(v) = \operatorname{Ran}(f)$ and $\operatorname{Ran}(w) = \operatorname{Ran}(g)$. Then $\operatorname{Ran}(v)$ contains a Δ-retract A which is Δ-isomorphic to $\operatorname{Ran}(w)$. Let t be any idempotent in End X with $\operatorname{Ran}(t) = A$ and let h be any Δ-isomorphism from A onto $\operatorname{Ran}(w)$. Then there exists a k in $\operatorname{Hom}(\operatorname{Ran}(w), A)$ such that hk is the identity on $\operatorname{Ran}(w)$ and kh is the identity on A. Then, by (2.1.3), both ht and kw belong to End X. Since v, as well as t, is the identity on A, one readily verifies that $w = (ht)v(kw)$. In a similar manner, one produces two functions p and q in End X such that $v = pwq$. Thus, v and w are $\mathscr{J}$-related and this implies that f and g are $\mathscr{J}$-related.

The next several sections will be devoted to applying the results of this section. We consider first semigroups of continuous functions.

3. Green's relations for regular elements of $S(X)$**.** Let X be any topological space and define a Δ-structure on X, by taking $\mathscr{A}$ to be all nonempty subspaces of X and $\text{Hom}(A, B)(A, B \in \mathscr{A})$ to be all continuous functions from A into B. Then End X is just $S(X)$, the semigroup of all continuous selfmaps of X. Δ-retracts are, of course, just retracts in the usual topological sense and Δ-isomorphisms are just homeomorphisms. With all this in mind, the results of the previous section translate immediately into the following theorems.

THEOREM (3.1). *Let X be any topological space. Then the following statements about an element f in $S(X)$ are equivalent.*

(3.1.1) *f is regular.*

(3.1.2) *The range of f is a retract of X and it maps some retract of X homeomorphically onto its range.*

(3.1.3) *The range of f is a retract of X and it maps some subspace of X homeomorphically onto its range.*

THEOREM (3.2). *Let X be an arbitrary topological space. Then two regular elements f and g of $S(X)$ are $\mathscr{L}$-related if and only if $\pi(f) = \pi(g)$ and they are $\mathscr{R}$-related if and only if* $\text{Ran}(f) = \text{Ran}(g)$. *Consequently they are $\mathscr{H}$-related if and only if both $\pi(f) = \pi(g)$ and* $\text{Ran}(f) = \text{Ran}(g)$. *They are $\mathscr{D}$-related if and only if* $\text{Ran}(f)$ *and* $\text{Ran}(g)$ *are homeomorphic and they are $\mathscr{J}$-related if and only if the range of each contains a retract which is homeomorphic to the range of the other.*

THEOREM (3.3) *Let X be any topological space, let A be any retract of X and let D_A consist of all those functions f such that* $\text{Ran}(f)$ *is a retract of X which is homeomorphic to A, and f maps some subspace of X homeomorphically onto* $\text{Ran}(f)$. *Then D_A is a regular $\mathscr{D}$-class of $S(X)$ and all regular $\mathscr{D}$-classes of $S(X)$ are obtained in exactly this manner.*

Now we derive two corollaries from Theorem (3.1) which indicate that $S(X)$ is rarely regular.

COROLLARY (3.4). *Let X be a completely regular Hausdorff space which contains an arc. Then $S(X)$ is not regular.*

Proof. There is no loss of generality if we assume that the closed interval I is a subspace of X. Let f be any function in $S(X)$ which maps X onto I and let g be any continuous function mapping I onto I which is not injective on any nondegenerate subinterval of I. For example, any continuous nowhere differentiable function mapping I onto I will suffice. Then gf belongs to $S(X)$. Its range is I but it cannot be regular since it does not map any subspace of X homeomorphically onto I. In fact, it is not injective on any nondegenerate connected subset.

COROLLARY (3.5). *Let X be a noncompact 0-dimensional metric space. Then $S(X)$ is regular if and only if X is discrete.*

Proof. If X is discrete $S(X)$ is the full transformation semigroup and as we have noted previously, C. G. Doss [**4**] proved that it is regular. This also follows easily from Theorem (3.1). To get the subspace required in (3.1.3), simply choose a point from the pre-image of each point in the range of the function.

Now, assume X is not discrete. Since it is a noncompact 0-dimensional metric space, it is the union of a countably infinite family $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint nonempty subspaces which are both open and closed. Moreover, since X is not discrete, it has a limit point p and since it is first countable, there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points, all different from p, which converges to p. Define $f(x) = a_n$ for $x \in A_n$. Then $f \in S(X)$ but f is not regular since $\mathrm{Ran}(f)$ is not closed and hence cannot be a retract of X since S is Hausdorff.

There are examples of nondiscrete X for which $S(X)$ is regular. H. deGroot [**6**] has proven the existence of 2^c connected one-dimensional subspaces of the plane such that the only continuous maps from any one of these spaces into any other are the constant maps and the only continuous selfmaps of these spaces are the constant maps and the identity maps. Thus, for any such space X, $S(X)$ is just a left zero semigroup with identity which, of course, is regular.

Now we look a bit more at Green's relations on $S(X)$. It is well-known that the two relations $\mathscr{D}$ and $\mathscr{J}$ agree on the full transformation semigroup. The following result shows that this is more the exception than the rule for semigroups of the form $S(X)$.

THEOREM (3.6). *Let X be any completely regular Hausdorff space which contains an arc. Then the two relations $\mathscr{D}$ and $\mathscr{J}$ are distinct on $S(X)$.*

Proof. We may assume that the closed unit interval I is actually a subspace of X. Since X is completely regular and Hausdorff, there exists a continuous function v mapping X onto I whose restriction to I is the identity mapping. Let f be any continuous function mapping I onto I with the property that the restriction of f to $[0, \frac{1}{2}]$ is the identity and f maps $[\frac{1}{2}, 1]$ onto itself but is not injective on any nondegenerate subinterval of $[\frac{1}{2}, 1]$. Then put $g = fv$. Since v is idempotent, we have $gv = g$ which implies that g belongs to the principal ideal generated by v. Now let h be any element of $S(X)$ which maps $[0, 1]$ homeomorphically onto $[0, \frac{1}{2}]$ and let k be any element of $S(X)$ whose restriction to $[0, \frac{1}{2}]$ is the inverse of h. Well known extension theorems imply the existence of such functions. It readily follows that $v = kghv$ which implies that v belongs to the principal ideal generated by g. Thus, v and g are $\mathscr{J}$-related. Since v is regular, we need only show g is not regular in order to conclude that they are not $\mathscr{D}$-related. With this in mind, we let A be any nondegenerate connected subset of X. If $v(A) \subset [0, \frac{1}{2}]$, then

$$g(A) = f(v(A)) = v(A) \subset [0, \tfrac{1}{2}].$$

If $v(A) \cap (\frac{1}{2}, 1] \neq \emptyset$, then $g = fv$ is not injective on A. In either event, g does not map A homeomorphically onto I. Since $\operatorname{Ran}(g) = I$, it follows from Theorem (3.1) that g is not regular and hence cannot be $\mathscr{D}$-related to v.

It follows from the previous result that $\mathscr{D}$ and $\mathscr{J}$ are distinct on both $S(I)$ and $S(R)$ (R denotes the space of real numbers), however, in both of these cases $\mathscr{D}$ and $\mathscr{J}$ restricted to the set of regular elements do coincide. That is, if f and g are regular elements which belong to either $S(I)$ or $S(R)$, then f and g are $\mathscr{D}$-related if and only if they are $\mathscr{J}$-related. This follows easily from Theorem (3.2). Actually, I and R are members of classes of spaces with this property and we consider them now in some more detail.

Definition (3.7). An N-star is the space formed by identifying the left end-points of N copies of the half-open interval $[0, 1)$. A closed N-star is the space formed by identifying the left end-points of N copies of the closed interval $[0, 1]$.

A 1-star is homeomorphic to a half-open interval while a 2-star is, of course, homeomorphic to the space R of real numbers. Both the closed 1-star and the closed 2-star are homeomorphic to the closed unit interval. In the following discussion, V_N will denote an N-star and W_N will denote a closed N-star. In order to determine the regular $\mathscr{D}$-classes of $S(V_N)$ and $S(W_N)$, we need to examine the retracts of V_N and W_N respectively. First of all a retract of W_N is either a point or is homeomorphic to W_M for some $M \leqq N$. Since W_1 and W_2 are homeomorphic, it follows that there are exactly N mutually nonhomeomorphic retracts of W_N and hence, by Theorem (3.2), there are exactly N regular $\mathscr{D}$-classes of $S(W_N)$. Furthermore, since no two of these retracts are mutually embeddable in each other, it follows from that same theorem that any two regular functions of $S(W_N)$ which belong to the same $\mathscr{J}$-class must also belong to the same $\mathscr{D}$-class. We collect these results and state them formally as

THEOREM (3.8). *The semigroup $S(W_N)$ $(N \geqq 2)$ has exactly N regular $\mathscr{D}$-classes. Moreover, two regular elements of $S(W_N)$ are $\mathscr{D}$-related if and only if they are $\mathscr{J}$-related.*

It takes a bit more effort to count the retracts of V_N. A retract of V_N is either a point or is formed by taking m copies of $[0, 1]$ and r copies of $[0, 1)$ where $m + r \leqq N$ and identifying all left end points. One verifies, by induction, that for $N \geqq 2$, V_N has exactly

$$\frac{(n+1)(n+2)}{2} - 2$$

mutually nonhomeomorphic retracts. In this case also no two are mutually embeddable into each other *as retracts*. Thus, we appeal to Theorem (3.2) to get

THEOREM (3.9). *The semigroup $S(V_N)$ $(N \geqq 2)$ has exactly*

$$\tfrac{1}{2}(n+1)(n+2) - 2$$

regular $\mathscr{D}$-classes. Moreover, two regular elements of $S(V_N)$ are $\mathscr{D}$-related if and only if they are $\mathscr{J}$-related.

The following two corollaries indicate how various spaces are characterized, to some extent, by the number of regular $\mathscr{D}$-classes their semigroups have.

COROLLARY (3.10). *Let X be any completely regular Hausdorff space which contains an arc. Then X is itself an arc if and only if $S(X)$ has exactly two regular $\mathscr{D}$-classes.*

Proof. Take $N = 2$ in Theorem (3.8) and we get the fact that if X is an arc, then $S(X)$ has exactly two regular $\mathscr{D}$-classes. Now suppose that $S(X)$ has exactly two regular $\mathscr{D}$-classes. Take any point and arc in X. Then these two subspaces are nonhomeomorphic retracts of X. Since X is also a retract, it follows from Theorem (3.2) that X must be homeomorphic to the arc.

COROLLARY (3.11). *Let X be any space which contains a retract which is homeomorphic to the space R of real numbers. Then X is itself homeomorphic to R if and only if $S(X)$ has exactly four regular $\mathscr{D}$-classes.*

Proof. We take $N = 2$ in Theorem (3.9) and conclude that $S(R)$ has four distinct $\mathscr{D}$-classes. Suppose conversely that $S(X)$ has exactly four regular $\mathscr{D}$-classes. Since R is a retract of X, then every retract of R is also a retract of X. Now R has essentially (up to homeomorphism) four different retracts. Topologically, they are represented by a point, a closed interval, a half-open interval and R. Since X is a retract of itself, it follows from Theorem (3.2) and the fact that $S(X)$ has only four regular $\mathscr{D}$-classes that X must be homeomorphic to one of the latter and the only possibility is that X is homeomorphic to R.

Remarks. For arbitrary elements f and g of $S(X)$, it is necessary that $\pi(f) = \pi(g)$ for f and g to be $\mathscr{L}$-related but it is not sufficient and similarly, it is necessary that $\operatorname{Ran}(f) = \operatorname{Ran}(g)$ for f and g to be $\mathscr{R}$-related but it is not sufficient. For an example of the former, take any Euclidean N-space E^N and let h be a continuous injection from E^N into E^N whose range is bounded. Then $\pi(h) = \pi(i)$ where i is the identity but h is not $\mathscr{L}$-related to i. For if h and i

are $\mathscr{L}$-related, then $i = kh$ for some $k \in S(E^N)$ but this is a contradiction since $\operatorname{Ran}(kh) \subset k(\operatorname{cl}(\operatorname{Ran} h))$ which is compact. More generally, i is not $\mathscr{L}$-related or $\mathscr{R}$-related, for that matter, to any element of $S(E^N)$ which is not a unit. This follows from the fact that the composition of two functions in $S(E^N)$ is a unit if and only if each of the functions is a unit and this is a consequence of Corollary (3.8) of [**2**]. Thus, if f in $S(E^N)$ is injective but not surjective, then $\pi(f) = \pi(i)$ but f and i are not $\mathscr{L}$-related and if $g \in S(E^N)$ is surjective but not injective, then $\operatorname{Ran}(g) = \operatorname{Ran}(i)$ but g and i are not $\mathscr{R}$-related.

For still other examples where $\operatorname{Ran}(f)$ coincides with $\operatorname{Ran}(g)$ without f and g being $\mathscr{R}$-related, let X be any completely regular Hausdorff space which contains an arc. There is no harm in supposing that the closed unit interval I is actually a subspace of X. Let f be any map from X onto I whose restrictions to I is the identity map and let k be any continuous mapping from I onto I which is not injective on any nondegenerate subinterval. Let $g = kf$. Then $\operatorname{Ran}(f) = \operatorname{Ran}(g) = I$ but f and g cannot possibly be $\mathscr{R}$-related. Among other things, g is not injective on any nondegenerate connected subset and since f is, we cannot have $f = gh$ for any h in $S(X)$. Of course, g is far from being a regular element.

4. The semigroup of closed functions on a T_1 space. A function from a topological space X into a topological space Y is a *closed* function if $f(A)$ is a closed subset of Y whenever A is closed in X. The family of all closed functions mapping X into X is a semigroup under composition and is denoted by $\Gamma(X)$. *We will assume throughout this section that X is a T_1 space.* In order to apply the results of Section 2 to get information about $\Gamma(X)$, we define a Δ-structure on X as follows: $\mathscr{A}$ consists of all nonempty closed subsets of X and for $A, B \in \mathscr{A}$, $\operatorname{Hom}(A, B)$ is just the family of all closed functions mapping A into B. The semigroup End X of this particular Δ-structure is, of course, just $\Gamma(X)$. Here, too, the Δ-isomorphisms are simply homeomorphisms. The Δ-retracts are precisely the nonempty closed subsets of X. Certainly, every Δ-retract is closed. To see that a nonempty closed subset A of X is a Δ-retract, choose a point $a \in A$ and define a function f by

$$f(x) = x \text{ for } x \in A$$
$$f(x) = a \text{ for } x \in X - A.$$

Then for any closed subset B of X,

$$f(B) = f((B \cap A) \cup (B \cap (X - A))) = f(B \cap A) \cup f(B \cap (X - A)) = (B \cap A) \cup f(B \cap (X - A)).$$

Now $B \cap A$ is closed and $f(B \cap (X - A))$ is either empty or a point. In either event, $f(B)$ is closed since X is T_1. Thus, f is an idempotent element of $\Gamma(X)$ and since $\operatorname{Ran}(f) = A$, we conclude that A is a Δ-retract. In particular

Ran(g) is a Δ-retract for each $g \in \Gamma(X)$ and from Theorem (2.1), we immediately get

THEOREM (4.1). *An element f in $\Gamma(X)$ is regular if and only if it maps some closed subspace of X homeomorphically onto its range.*

As in the case for $S(X)$, these semigroups are also rarely regular. In the proof of Corollary (3.4), we produced a function with the property that it does not map *any* subspace of X homeomorphically onto its range. Now, if X is taken to be compact, that function is a closed function so we see that if X is any compact Hausdorff space which contains an arc, then $\Gamma(X)$ is not regular.

The theorems in Section 2 concerning Green's relations easily translate into the following two results.

THEOREM (4.2). *Two regular elements f and g of $\Gamma(X)$ are $\mathscr{L}$-related if and only if $\pi(f) = \pi(g)$ and they are $\mathscr{R}$-related if and only if* Ran(f) = Ran(g). *Consequently, they are $\mathscr{H}$-related if and only if both $\pi(f) = \pi(g)$ and* Ran(f) = Ran(g). *They are $\mathscr{D}$-related if and only if* Ran(f) *and* Ran(g) *are homeomorphic and they are $\mathscr{J}$-related if and only if the range of each contains a closed subset homeomorphic to the range of the other.*

THEOREM (4.3). *Let A be any closed subset of X and let D_A consist of all those functions f in $\Gamma(X)$ with the property that f maps some closed subset of X homeomorphically onto* Ran(f). *Then D_A is a regular $\mathscr{D}$-class of $\Gamma(X)$ and each regular $\mathscr{D}$-class of $\Gamma(X)$ is obtained in exactly this manner.*

Since closed subsets of a space X are, in general, more abundant than retracts, we can expect $\Gamma(X)$ to usually have more regular $\mathscr{D}$-classes than $S(X)$. For example, we have seen that $S(I)$ (I is the closed unit interval) has exactly two regular $\mathscr{D}$-classes. The situation is far different for $\Gamma(X)$. It has an infinite number of $\mathscr{D}$-classes. In particular, for each positive integer N, the set D_N of functions in $\Gamma(I)$, whose range has N elements is a regular $\mathscr{D}$-class.

$\Gamma(I)$ also differs in another respect from $S(I)$. We recall that two regular elements in $S(I)$ are $\mathscr{J}$-related if and only if they are $\mathscr{D}$-related. This is not the case for $\Gamma(I)$. Let A be any nondegenerate closed subinterval of I and let B be the union of any nondegenerate closed subinterval of I with a point not in that subinterval. Let v be any idempotent map in $\Gamma(I)$ with Ran$(v) = A$ and let w be any idempotent map in $\Gamma(I)$ with Ran$(w) = B$. Then v and w are regular and according to Theorem (4.2) they are $\mathscr{J}$-related but not $\mathscr{D}$-related.

5. The semigroup of linear transformations on a vector space. With possibly one exception, the results in this section are not new. They are stated as problems in [**3**, p. 57]. Nevertheless we thought it might be instructive to show how they can be derived from the results of Section 2.

Let V be a vector space over a division ring and let $L(V)$ denote the semigroup under composition of all linear transformations on V. Define a Δ-structure on V as follows: $\mathscr{A}$ is the collection of all subspaces of V and for A, B in $\mathscr{A}$, $\mathrm{Hom}(A, B)$ is just the collection of all linear transformations from A into B. The semigroup End V of this Δ-structure is, of course, just $L(V)$ and it is easily seen that the Δ-isomorphisms are just the linear isomorphisms between subspaces of V. The Δ-retracts coincide with the subspaces of V. It is immediate that every Δ-retract is a subspace of V. To see the converse, let A be any subspace of V with basis A^*. Extend A^* to a basis V^* of V. Choose any $a \in A^*$ and let φ denote the linear transformation on V which is determined by

$$\varphi(x) = x \text{ for } x \in A^*$$
$$\varphi(x) = a \text{ for } x \in B^* - A^*.$$

Then φ is idempotent and $\mathrm{Ran}(\varphi) = A$. Thus, A is a Δ-retract of V. Moreover, for any $\psi \in L(V)$, we choose C^* to be any basis for the subspace $\psi(V)$ of V. Then choose exactly one vector from each of the sets $\psi^{-1}(x)$, $x \in C^*$ and denote the resulting set by E^*. Then E^* is a linearly independent set and ψ maps the space E, which is generated by E^*, isomorphically onto $\psi(V)$. Thus, the following result is a consequence of Theorem (2.1).

THEOREM (5.1). *The semigroup of all linear transformations on any vector space is regular.*

Let us recall that for any two linear transformation φ, ψ in $L(V)$, $\pi(\varphi) = \pi(\psi)$ if and only if their null spaces coincide and $\mathrm{Ran}(\varphi)$ is isomorphic to $\mathrm{Ran}(\psi)$ if and only if φ and ψ have the same rank. In fact, any two vector spaces over the same division ring are isomorphic if and only if they have the same dimension. These observations, together with the results in Section 2 result in the following two theorems.

THEOREM (5.2). *Let V be any vector space over a division ring. Then two elements φ and ψ of $L(V)$ are $\mathscr{L}$-related if and only if they have the same null space and they are $\mathscr{R}$-related if and only if* $\mathrm{Ran}(\varphi) = \mathrm{Ran}(\psi)$. *Consequently, they are $\mathscr{H}$-related if and only if they have the same null space and* $\mathrm{Ran}(\varphi) = \mathrm{Ran}(\psi)$.

THEOREM (5.3). *Let V be any vector space over a division ring. Then the following statements are equivalent about two linear transformations φ and ψ from $L(V)$.*

(5.3.1) *φ and ψ are $\mathscr{D}$-related;*

(5.3.2) *φ and ψ are $\mathscr{J}$-related;*

(5.3.3) *φ and ψ have the same rank.*

As we mentioned previously, most of these results are stated as problems in [**3**, p. 57]. The only result in this section which might possibly have escaped detection is contained in Theorem (5.3) and it is the fact that the two relations $\mathscr{D}$ and $\mathscr{J}$ coincide on $L(V)$.

In conclusion, we take the opportunity to thank the referee for the helpful comments and suggestions.

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