Can. J. Math. Vol. **48** (1), 1996 pp. 159–174

MOORE COHOMOLOGY AND CENTRAL TWISTED CROSSED PRODUCT C^*-ALGEBRAS

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ABSTRACT. Let G be a locally compact second countable group, let X be a locally compact second countable Hausdorff space, and view $C(X,\mathbb{T})$ as a trivial G-module. For G countable discrete abelian, we construct an isomorphism between the Moore cohomology group $H^n(G,C(X,\mathbb{T}))$ and the direct sum $\mathrm{Ext}(H_{n-1}(G),\check{H}^1(\beta X,\mathbb{Z}))\oplus C(X,H^n(G,\mathbb{T}))$; here $\check{H}^1(\beta X,\mathbb{Z})$ denotes the first Čech cohomology group of the Stone-Čech compactification of X, βX, with integer coefficients. For more general locally compact second countable groups G, we discuss the relationship between the Moore group $H^2(G,C(X,\mathbb{T}))$, the set of exterior equivalence classes of element-wise inner actions of G on the stable continuous trace C^*-algebra $C_0(X)\otimes\mathcal{K}$, and the equivariant Brauer group $\mathrm{Br}_G(X)$ of Crocker, Kumjian, Raeburn, and Williams. For countable discrete abelian G acting trivially on X, we construct an isomorphism $\mathrm{Br}_G(X)\cong\check{H}^3(X,\mathbb{Z})\oplus\mathcal{HP}(X,\hat{G})\oplus C(X,H^2(G,\mathbb{T}))$; here $\mathcal{HP}(X,\hat{G})$ is the group of equivalence classes of principal $\hat{G}$ bundles over X first considered by Raeburn and Williams.

0. **Introduction.** During the past decade, there has been a great deal of research by R. Hermann, I. Raeburn, J. Rosenberg, and D. Williams, and most recently, D. Crocker, A. Kumjian, Raeburn, and Williams, ([HR], [Ro], [RR], [RW 1], [CKRW]) concerning the Moore cohomology groups $H^2\big(G,C(X,\mathbb{T})\big)$, $H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$, and $H^3\big(G,C(X,\mathbb{T})\big)$ and their relationship to operator algebras, where G is a locally compact second countable (hereafter, abbreviated l.c.s.c.) group, X is a l.c.s.c. Hausdorff space, and $C(X,\mathbb{T})$ is viewed as a trivial G-module (here, as throughout the paper, if X and Y are topological spaces, $C(X,Y)$ refers to the set of continuous functions on X taking values in Y; if Y is an abelian group, so is $C(X,Y)$, under pointwise multiplication). In their very recent work, Crocker *et al.* have established an exact sequence relating $H^n\big(G,C(X,\mathbb{T})\big)$ for $n=2$ and $n=3$ to the subgroup of the equivariant Brauer group $\mathrm{Br}_G(X)$ consisting of exterior equivalence classes of actions of G on $C_0(X)\otimes\mathcal{K}$ which act trivially on the spectrum X. When G is abelian, the group $H^2_{\mathrm{pt}}\big(G,C,(X,\mathbb{T})\big)$ has been related to the classifications of exterior equivalence classes of locally unitary and pointwise unitary G-actions on stable continuous trace C^*-algebras with spectrum X, via work of J. Phillips and Raeburn [PhR 2], and D. Olesen and Raeburn [OR], respectively; most recently, this group has been used by L. Baggett and the author in the study of group C^*-algebras of two-step nilpotent groups [BP]. Indeed, Raeburn and Williams in [RW 1] have shown that $H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$ injects as a subgroup of $\mathcal{HP}(X,\hat{G})$, the group of equivalence classes of

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Received by the editors June 7, 1994.
AMS subject classification: Primary: 46L05, 46L35; secondary: 55N91, 55R10.

https://doi.org/10.4153/CJM-1996-007-6 Published online by Cambridge University Press

principal $\hat{\mathcal{G}}$-bundles over X, which plays a key role in the classification of exterior equivalence classes of pointwise unitary G-actions on the stable continuous trace C^*-algebras with spectrum X mentioned above. For connected l.c.s.c. abelian groups G, Rosenberg in [Ro] had already shown this injection to be an isomorphism; for non-connected G, the subgroup of $\mathcal{HP}(X, \hat{\mathcal{G}})$ corresponding to $H^2_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big)$ has also been characterized by Raeburn and Williams in [RW 1, Theorem 4.4]. However, the relationship for non-connected G is not as transparent as in the connected case, so that a further investigation of this relationship, together with a more explicit description of $H^n_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big)$ and $H^n\big(G, C(X, \mathbb{T})\big)$ for all $n \in \mathbb{Z}^+$ would be useful. In this note, we initiate this project for countable discrete abelian groups G, and discuss the connection between the structure of these cohomology groups with both central twisted crossed product C^*-algebras of the form $C_0(X) \times_{\iota,\sigma} G$, where ι is the trivial action of G on $C_0(X)$ and σ is an element of the group of two-cocycles $Z^2\big(G, C(X, \mathbb{T})\big)$, and ordinary crossed products of the stable continuous trace C^*-algebra $C_0(X) \otimes \mathcal{K}$ by element-wise inner actions of G.

Obviously the discrete case avoids many of the technical difficulties inherent in the definition of the Moore cohomology groups; these difficult technical points have led to new areas of interest as well as to many open problems. (See, *e.g.*, [RW 1, proof of Proposition 3.4 and the remark after the proof of Theorem 4.5]). On the other hand, so much remains unknown about the structure of central twisted crossed product C^*-algebras by discrete groups, including a description of the topology on the primitive ideal spaces of such C^*-algebras, that any information at all about the cohomology groups involved should be useful. It turns out that, at least when G is discrete, information is readily available by means of elementary group extension theory, and this information can be interpreted in a variety of ways of interest to C^*-algebraists.

To describe our main theorem on Moore cohomology, we recall that if G is a l.c.s.c. group and $C(X, \mathbb{T})$ is viewed as a trivial G-module, there is an injection $i_*: H^n_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big) \longrightarrow H^n\big(G, C(X, \mathbb{T})\big)$. When in addition G is discrete, for each $[\sigma] \in H^n\big(G, C(X, \mathbb{T})\big)$ one can define a continuous map $\Pi_*([\sigma]): X \longrightarrow H^n(G, \mathbb{T})$ by setting $\Pi_*([\sigma])(x) = [e_x(\sigma)]$, where for each $x \in X$, $e_x: Z^n\big(G, C(X, \mathbb{T})\big) \longrightarrow Z^n(G, \mathbb{T})$ is the evaluation map. Then $\Pi_*: H^n\big(G, C(X, \mathbb{T})\big) \longrightarrow C\big(X, H^n(G, \mathbb{T})\big)$ is a homomorphism, and the main result of our first section is:

THEOREM. *Let G be a countable discrete abelian group acting trivially on the* l. c. s. c. *Hausdorff space X. Then there is a split short exact sequence*

$$0 \longrightarrow H^n_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big) \xrightarrow{i_*} H^n\big(G, C(X, \mathbb{T})\big) \xrightarrow{\Pi_*} C\big(X, H^n(G, \mathbb{T})\big) \longrightarrow 0.$$

Hence $H^n\big(G, C(X, \mathbb{T})\big) \cong H^n_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big) \oplus C\big(X, H^n(G, \mathbb{T})\big)$. ∎

We have already noted how Raeburn and Williams have identified $H^2_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big)$ with the group of equivalence classes of *characteristic* principal $\hat{G}$-bundles over X. On the other hand, as shown by H. Smith [Sm 2], it is clear that $H^2_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big)$ corresponds to the group of equivalence classes of abelian group extensions of G by the Polish group $C(X,\mathbb{T})$, $\mathcal{E}_{\mathrm{ab}}\big(G, C(X,\mathbb{T})\big)$. Smith also showed that this last group is isomorphic to $\mathcal{E}_{\mathrm{ab}}\big(G, \check{H}^1(\beta X,\mathbb{Z})\big)$, where βX represents the Stone-Čech compactification of X. Since standard results from group extension theory (*cf.* [Br, p. 127, Exercise 5]) show that $\mathcal{E}_{\mathrm{ab}}(G,A) \cong \mathrm{Ext}(G,A)$, where A is an arbitrary trivial G-module and "Ext" represents the standard Ext group familiar from algebraic topology and homological algebra, we obtain generalizations of the above results for arbitrary positive integer n in the following corollary of the theorem above:

COROLLARY. *Let G be a countable discrete abelian group, let X be a* l. c. s. c. *Hausdorff space, and view* $C(X,\mathbb{T})$ *as a trivial G module. Then*

$$H^n\big(G, C(X,\mathbb{T})\big) \cong \mathrm{Ext}\big(H_{n-1}(G), \check{H}^1(\beta X,\mathbb{Z})\big) \oplus C\big(X, H^n(G,\mathbb{T})\big). \qquad \blacksquare$$

Thus when X is a compact metric space, so that $\check{H}^1(X,\mathbb{Z})$ is countable, $H^n\big(G, C(X,\mathbb{T})\big)$ is easily computed; for example, if $G = \mathbb{Z}^m$, then $H^2_{\mathrm{pt}}\big(\mathbb{Z}^m, C(X,\mathbb{T})\big) = \{0\}$ and $H^2\big(\mathbb{Z}^m, C(X,\mathbb{T})\big) \cong \oplus_{i=1}^{m(m-1)/2}[C(X,\mathbb{T})]_i$.

In our second section, we discuss the group structure on the set of exterior equivalence classes of element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$, viewed as a subgroup of the equivariant Brauer group $\mathrm{Br}_G(X)$, where now G is allowed to be an arbitrary l.c.s.c. group, and $\mathcal{K}$ represents the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. It is folklore that the group so constructed is isomorphic to $H^2\big(G, C(X,\mathbb{T})\big)$, where $C(X,\mathbb{T})$ is again viewed as a trivial G-module (see, for example [CKRW]), but we feel it helpful to discuss this relationship in more detail than in previous references, and to point out that the crossed products of $C_0(X) \otimes \mathcal{K}$ by the element-wise inner actions of G are precisely those which are strongly Morita equivalent to central twisted crossed product C^*-algebras of the form $C_0(X) \times_{\iota,\sigma} G$. Finally, we use results from our first section to discuss the interrelationship between pointwise unitary actions of G on $C_0(X)\otimes\mathcal{K}$, element-wise inner actions of G on $C_0(X)\otimes\mathcal{K}$, and the Moore groups $H^2_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$ and $H^2\big(G, C(X,\mathbb{T})\big)$, and prove that for G countable discrete abelian, the group of exterior equivalence classes of actions of G on $C_0(X)\otimes\mathcal{K}$ which fix the spectrum pointwise is generated by the classes of element-wise inner actions of G and pointwise unitary actions of G on $C_0(X) \otimes \mathcal{K}$. This gives a short exact sequence of abelian groups involving $\mathcal{HP}(X,\hat{\mathcal{G}})$, $C\big(X, H^2(G,\mathbb{T})\big)$, and a subgroup of $\mathrm{Br}_G(X)$ which is related to the exact sequence of Crocker, Kumjian, Raeburn, and Williams involving the equivariant Brauer group mentioned earlier.

We thank Jon Berrick, Larry Baggett and Iain Raeburn for useful conversations on this topic.

1. **Moore cohomology for discrete abelian groups taking on values in a trivial module.** We first recall the necessary definitions. Let G be a l.c.s.c. group acting trivially on the l.c.s.c. Hausdorff space X, hence acting trivially on the Polish group $C(X,\mathbb{T})$, given the topology of uniform convergence on compact sets. Let $\underline{C}^n\big(G, C(X,\mathbb{T})\big)$ be

https://doi.org/10.4153/CJM-1996-007-6 Published online by Cambridge University Press

the abelian group (under pointwise multiplication) of normalized Borel n-cochains on G taking values in $C(X,\mathbb{T})$, *i.e.* $\underline{C}^n\big(G,C(X,\mathbb{T})\big) = \{\text{Borel maps } f\colon G^n \longrightarrow C(X,\mathbb{T}) \text{ s.t. } f(g_1,\ldots,g_n) = 1 \text{ if some } g_i = 1\}$. Let $\partial\colon \underline{C}^n\big(G,C(X,\mathbb{T})\big) \longrightarrow \underline{C}^{n+1}\big(G,C(X,\mathbb{T})\big)$ be the usual coboundary operator, *i.e.*

$$(\partial f)(g_1,\ldots,g_{n+1}) = f(g_2,\ldots,g_{n+1})f(g_1g_2,\ldots,g_{n+1})^{-1}\times\cdots \times f(g_1,\ldots,g_ng_{n+1})^{(-1)^n}f(g_1,\ldots,g_n)^{(-1)^{n+1}}. \tag{1.1}$$

Let $Z^n\big(G,C(X,\mathbb{T})\big)$ be the kernel of ∂, and let $B^n\big(G,C(X,\mathbb{T})\big) \subseteq Z^n\big(G,C(X,\mathbb{T})\big)$ be the image of $\partial\colon \underline{C}^{n-1}\big(G,C(X,\mathbb{T})\big) \longrightarrow \underline{C}^n\big(G,C(X,\mathbb{T})\big)$.

Let $\underline{Z}^n\big(G,C(X,\mathbb{T})\big)$ and $\underline{B}^n\big(G,C(X,\mathbb{T})\big)$ denote the equivalence classes of $Z^n\big(G,C(X,\mathbb{T})\big)$ and $B^n\big(G,C(X,\mathbb{T})\big)$ respectively obtained by identifying cocycles which are equal almost everywhere on G^n with respect to Haar measure. Then $\underline{Z}^n\big(G,C(X,\mathbb{T})\big)$ under the topology of convergence in measure is a Polish topological group, and $\underline{B}^n\big(G,C(X,\mathbb{T})\big)$ is the continuous image of a Polish group, and a result of C. Moore [M3] shows that $\underline{Z}^n\big(G,C(X,\mathbb{T})\big)/\underline{B}^n\big(G,C(X,\mathbb{T})\big) \cong Z^n\big(G,C(X,\mathbb{T})\big)/B^n\big(G,C(X,\mathbb{T})\big) = H^n\big(G,C(X,\mathbb{T})\big)$ is a topological group which is Hausdorff exactly when $\underline{B}^n\big(G,C(X,\mathbb{T})\big)$ is closed. For notational purposes, given a cocycle $\sigma \in Z^n\big(G,C(X,\mathbb{T})\big)$, let $[\sigma]$ denote its cohomology class in $H^n\big(G,C(X,\mathbb{T})\big)$. If X is a point, $C(X,\mathbb{T}) \cong \mathbb{T}$, and we obtain the usual cohomology groups for G, $H^n(G,\mathbb{T}) = Z^n(G,\mathbb{T})/B^n(G,\mathbb{T})$; for $n=2$ this gives the standard multiplier group for G, $H^2(G,\mathbb{T})$. For discrete G there is no distinction between Z^n and $\underline{Z}^n$ or B^n and $\underline{B}^n$, and the topologies involved are the topologies of pointwise convergence; in this case, $B^n\big(G,C(X,\mathbb{T})\big)$ is closed in $Z^n\big(G,C(X,\mathbb{T})\big)$ so that $H^n\big(G,C(X,\mathbb{T})\big)$ is always Hausdorff. For each $x \in X$, define $e_x\colon Z^n\big(G,C(X,\mathbb{T})\big) \longrightarrow Z^n(G,\mathbb{T})$ by $e_x(\sigma) = \sigma(.\,,\ldots,.)(x)$. Then we define

$$Z^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big) = \big\{\sigma \in Z^n\big(G,C(X,\mathbb{T})\big) : e(\sigma_x) \in B^n(G,\mathbb{T}), \forall x \in X\big\},$$

and we define $\underline{Z}^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$ to be the image of $Z^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$ in $\underline{Z}^n\big(G,C(X,\mathbb{T})\big)$ (again, there is no distinction for G discrete). It is clear that $B^n\big(G,C(X,\mathbb{T})\big) \subseteq Z^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$; this containment is proper in general. We write $H^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big) = Z^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)/B^n\big(G,C(X,\mathbb{T})\big)$; see the articles of Rosenberg [Ro] and Raeburn and Williams [RW 1] mentioned above for more details on this group in the case $n=2$. As in the introduction, let $i_*\colon H^n_{\mathrm{pt}}(G,C(X,\mathbb{T})\big) \longrightarrow H^n\big(G,C(X,\mathbb{T})\big)$ be the injection map and let $\Pi_*\colon H^n\big(G,C(X,\mathbb{T})\big) \longrightarrow C\big(X,H^n(G,\mathbb{T})\big)$ be defined by $\Pi_*([\sigma])(x) = [e_x(\sigma)]$.

THEOREM 1.1. *Let G be a countable discrete abelian group, let X be a* l. c. s. c. *Hausdorff space, and view $C(X,\mathbb{T})$ as a trivial G-module. Then there is a split short exact sequence*

$$0 \longrightarrow H^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big) \xrightarrow{i_*} H^n\big(G,C(X,\mathbb{T})\big) \xrightarrow{\Pi_*} C\big(X,H^n(G,\mathbb{T})\big) \longrightarrow 0. \tag{1.2}$$ ■

PROOF. It is clear that i_* is an injection. Therefore it is sufficient to prove that $\ker\Pi_* = \operatorname{im} i_*$ and construct a monomorphism $\theta_*: C\big(X, H^n(G,\mathbb{T})\big) \longrightarrow H^n\big(G, C(X,\mathbb{T})\big)$ which is a cross section for Π_*.

Suppose that $[\sigma] \in \operatorname{im} i_*$ so that $\sigma \in Z^n_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$. Then $\Pi_*([\sigma])(x) = [e_x(\sigma)] = [\sigma(.\,,\ldots,.)(x)]_{H^n(G,\mathbb{T})} = [1]_{H^n(G,\mathbb{T})}$, for all $x \in X$, so that $\Pi_*([\sigma]) = 1_{C(X,H^n(G,\mathbb{T}))}$, and $[\sigma] \in \ker\Pi_*$. Conversely, if $[\sigma] \in H^n\big(G, C(X,\mathbb{T})\big)$ belongs to $\ker\Pi_*$, then $[e_x(\sigma)] = [\sigma(.\,,\ldots,.)(x)]_{H^n(G,\mathbb{T})} = 1_{H^n(G,\mathbb{T})}$, $\forall x \in X$. But this means precisely that $\sigma \in Z^n_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$, so that $[\sigma] \in H^n_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$, and $\ker_* = \operatorname{im} i_*$.

We now construct θ_*. Since G is discrete abelian, by Item 2, p. 82 of [M1], the sort exact sequence below splits:

$$(1.3)\qquad 0 \longrightarrow B^n(G,\mathbb{T}) \longrightarrow Z^n(G,\mathbb{T}) \longrightarrow H^n(G,\mathbb{T}) \longrightarrow 0,$$

i.e., there is a monomorphism $\theta: H^n(G,\mathbb{T}) \longrightarrow Z^n(G,\mathbb{T})$ splitting the sequence. Fixing any such θ define $\tilde\theta: C\big(X, H^n(G,\mathbb{T})\big) \longrightarrow Z^n\big(G, C(X,\mathbb{T})\big)$ by

$$(1.4)\qquad \big(\tilde\theta(f)(g_1,\ldots,g_n)\big](x) = \theta\big(f(x)\big)(g_1,\ldots,g_n), \quad g_1,\ldots,g_n \in G,\ x \in X,$$

and define $\theta_*: C\big(X, H^n(G,\mathbb{T})\big) \longrightarrow H^n\big(G, C(X,\mathbb{T})\big)$ by $\theta_*(f) = [\tilde\theta(f)]$. One easily checks that θ_* is a monomorphism which is a cross-section to Π_*, so that Π_* is a surjective and the sequence (1.2) is split exact, as desired. ∎

We note that for non-discrete G, there would be several potential problems if one attempted to carry through the above proof: firstly, the sequence (1.3) need not split, and indeed $H^n(G,\mathbb{T})$ need not be Hausdorff for $n > 2$, and secondly, there is no clear way to map $C\big(X, \underline{Z}^n(G,\mathbb{T})\big)$ into $\underline{Z}^n\big(G, C(X,\mathbb{T})\big)$, as noted in the remarks preceding Lemma 3.5 of [RW1].

For discrete abelian G, one can alternatively calculate $H^n\big(G, C(X,\mathbb{T})\big)$ by using the universal coefficient theorem stated in [Br, p. 60, Exercise III.I.3]. Since $C(X,\mathbb{T})$ is a trivial G-module, there is a split short exact sequence

$$(1.5)\qquad 0 \longrightarrow \operatorname{Ext}\big(H_{n-1}(G), C(X,\mathbb{T})\big) \longrightarrow H^n\big(G, C(X,\mathbb{T})\big) \longrightarrow \operatorname{Hom}\big(H_n(G), C(X,\mathbb{T})\big) \longrightarrow 0$$

(here $H_n(G)$ refers to ordinary homology with integer coefficients), and since G is discrete, $\operatorname{Hom}\big(H_n(G), C(X,\mathbb{T})\big) \cong C\Big(X, \operatorname{Hom}\big(H_n(G),\mathbb{T}\big)\Big) \cong C\big(X, H^n(G,\mathbb{T})\big)$. Thus this version of the universal coefficient theorem becomes

$$(1.6)\qquad 0 \longrightarrow \operatorname{Ext}\big(H_{n-1}(G), C(X,\mathbb{T})\big) \longrightarrow H^n\big(G, C(X,\mathbb{T})\big) \longrightarrow C\big(X, H^n(G,\mathbb{T})\big) \longrightarrow 0$$

which is *almost* the same as sequence (1.2). We now want to clarify the relationship between $\operatorname{Ext}\big(H_{n-1}(G), C(X,\mathbb{T})\big)$ and $H^n_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$. Recall that $\operatorname{Ext}\big(H_{n-1}(G), C(X,\mathbb{T})\big)$ refers to the Ext group as defined in [V, p. 91], *i.e.* given a free resolution of $H_{n-1}(G)$

$$(1.7)\qquad 0 \longrightarrow R \xrightarrow{\alpha} F \xrightarrow{\beta} H_{n-1}(G) \longrightarrow 0,$$

we take the dual sequence

$$(1.8)\quad 0 \longrightarrow \mathrm{Hom}\big(H_{n-1}(G), C(X,\mathbb{T})\big) \xrightarrow{\beta_*} \mathrm{Hom}\big(F, C(X,\mathbb{T})\big) \xrightarrow{\alpha_*} \mathrm{Hom}\big(R, C(X,\mathbb{T})\big)$$

and define $\mathrm{Ext}\big(H_{n-1}(G), C(X,\mathbb{T})\big)$ to be $\mathrm{coker}\,\alpha_*$. Since $H_{n-1}(G)$, F and R are discrete abelian groups, by means of Pontryagin duality the left-exact sequence (1.8) can be written as

$$(1.9)\qquad 0 \longrightarrow C\big(X, \widehat{H_{n-1}}(G)\big) \xrightarrow{\beta_*} C(X,\hat{F}) \xrightarrow{\alpha_*} C(X,\hat{R})$$

which can alternatively be written

$$(1.10)\qquad 0 \longrightarrow C\big(X, H^{n-1}(G,\mathbb{T})\big) \xrightarrow{\beta_*} C(X,\hat{F}) \xrightarrow{\alpha_*} C(X,\hat{R}).$$

We now recall from cohomology theory

$$(1.11)\qquad B^{n-1}(G,\mathbb{T}) \subseteq Z^{n-1}(G,\mathbb{T}) \xrightarrow{i} \underline{C}^{n-1}(G,\mathbb{T}) \xrightarrow{\partial} B^n(G,\mathbb{T})$$

which gives us via elementary group theory a short exact sequence

$$(1.12)\qquad 0 \longrightarrow H^{n-1}(G,\mathbb{T}) \xrightarrow{i} \underline{C}^{n-1}(G,\mathbb{T})/B^{n-1}(G,\mathbb{T}) \xrightarrow{\partial} B^n(G,\mathbb{T}) \longrightarrow 0.$$

Since G is discrete, $\underline{C}^{n-1}(G,\mathbb{T})$ can be identified with a countable product of tori, and thus any quotient of this group also is a countable product of tori. Therefore, the dual groups of $\underline{C}^{n-1}(G,\mathbb{T})/B^{n-1}(G,\mathbb{T})$ and $B^n(G,\mathbb{T})$ are free abelian groups, and taking F and R to be these respective dual groups, the Pontryagin dual of exact sequence (1.12) gives a free resolution for $H_{n-1}(G)$. With respect to this resolution, sequence (1.10) becomes
(1.13)

$$0 \longrightarrow C\big(X, H^{n-1}(G,\mathbb{T})\big) \xrightarrow{\beta_*} C\big(X, \underline{C}^{n-1}(G,\mathbb{T})/B^{n-1}(G,\mathbb{T})\big) \xrightarrow{\alpha_*} C\big(X, B^n(G,\mathbb{T})\big),$$

where $\beta_*(f)(x) = i\big(f(x)\big)$ and $\alpha_*(f)(x) = \partial\big(f(x)\big)$, $x \in X$, i, ∂, as in (1.12). We finally reproduce here the observation of Moore [M1] that since G is discrete we have a topological splitting

$$\underline{C}^{n-1}(G,\mathbb{T}) = \underline{C}^{n-1}(G,\mathbb{T})/B^{n-1}(G,\mathbb{T}) \times B^{n-1}(G,\mathbb{T})$$

so that

$$C\big(X, \underline{C}^{n-1}(G,\mathbb{T})\big) \quad \text{has a splitting as } C(X, \underline{C}^{n-1}/B^{n-1}) \times C(X, B^{n-1})$$

(suppressing the $(G,\mathbb{T})$ for ease of notation).

Therefore any element of $\mathrm{Ext}\big(H_{n-1}(G), C(X,\mathbb{T})\big)$ can be identified with $[\omega] \in C\big(X, B^n(G,\mathbb{T})\big)/\,\mathrm{im}(\alpha_*)$ for some $\omega \in C\big(X, B^n(G,\mathbb{T})\big) = Z^n_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$ and $\mathrm{im}\,\alpha_*$ where $\alpha_*: C\big(X, \underline{C}^{n-1}(G,\mathbb{T})/B^{n-1}(G,\mathbb{T})\big) \longrightarrow B^n\big(G, C(X,\mathbb{T})\big)$. We now note that because $\partial: \underline{C}^{n-1}(G,\mathbb{T}) \longrightarrow B^n(G,\mathbb{T})$ is the zero map on the subgroup $B^{n-1}(G,\mathbb{T})$ and because of

https://doi.org/10.4153/CJM-1996-007-6 Published online by Cambridge University Press

the splitting of $C\big(X,\underline{C}^{n-1}(G,\mathbb{T})\big)$ mentioned above, im α_* can be identified with the image of the extended coboundary map $\partial_*\colon C\big(X,\underline{C}^{n-1}(G,\mathbb{T})\big)\longrightarrow C\big(X,B^n(G,\mathbb{T})\big)$, which again because G is discrete can be written as $\partial_*\colon \underline{C}^{n-1}\big(G,C(X,\mathbb{T})\big)\longrightarrow B^n\big(G,C(X,\mathbb{T})\big)$. Consequently an element of $\operatorname{Ext}\big(H_{n-1}(G),C(X,\mathbb{T})\big)$ can be identified with $[\omega]\in Z^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)/B^n\big(G,C(X,\mathbb{T})\big)=H^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$. Therefore we have constructed an explicit isomorphism between $\operatorname{Ext}\big(H_{n-1}(G),C(X,\mathbb{T})\big)$ and $Z^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)/B^n\big(G,C(X,\mathbb{T})\big)=H^n_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$, as desired.

Using this isomorphism we deduce:

COROLLARY 1.2. *If X is a* l. c. s. c. *Hausdorff space and $C(X,\mathbb{T})$ is viewed as a trivial G module for a countable discrete abelian group G, then* $H^n\big(G,C(X,\mathbb{T})\big)\cong \operatorname{Ext}\big(H_{n-1}(G),\check{H}^1(\beta X,\mathbb{Z})\big)\oplus C\big(X,H^n(G,\mathbb{T})\big)$.

PROOF. As a result of the remarks following the proof of Theorem 1.1, all that remains to show is that $\operatorname{Ext}\big(H_{n-1}(G),\check{H}^1(\beta X,\mathbb{Z})\big)\cong\operatorname{Ext}\big(H_{n-1}(G),C(X,\mathbb{T})\big)$. This follows from the isomorphisms $C(X,\mathbb{T})\cong C(\beta X,\mathbb{T})\cong\check{H}^1(\beta X,\mathbb{Z})\oplus\operatorname{Exp}\Big(2\pi i\big(C(\beta X,\mathbb{R})\big)\Big)$, where βX is the Stone-Čech compactification of X, and $\check{H}^1(\beta X,\mathbb{Z})$ is the first integral Čech cohomology of βX. The final isomorphism follows from group theory, since $\operatorname{Exp}\Big(2\pi i\big(C(\beta X,\mathbb{R})\big)\Big)$ is a divisible subgroup of the abelian group $C(\beta X,\mathbb{T})$. Basic properties of the Ext functor then show us that $\operatorname{Ext}\big(H_{n-1}(G),C(X,\mathbb{T})\big)\cong \operatorname{Ext}\big(H_{n-1}(G),\check{H}^1(\beta X,\mathbb{Z})\big)\oplus\operatorname{Ext}\Big(H_{n-1}(G),\operatorname{Exp}\Big(2\pi i\big(C(\beta X,\mathbb{R})\big)\Big)\Big)\cong\operatorname{Ext}\big(H_{n-1}(G),$ $\check{H}^1(\beta X,\mathbb{Z})\big)$, as desired. ■

REMARK 1.3. When $n=2$, $H_{n-1}(G)=H_1(G)\cong G$, and we obtain the isomorphism $H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)\cong\operatorname{Ext}\big(G,\check{H}^1(\beta X,\mathbb{Z})\big)\cong\mathcal{E}_{\mathrm{ab}}\big(G,\check{H}^1(\beta X,\mathbb{Z})\big)$ (where $\mathcal{E}_{\mathrm{ab}}(A,B)$ represents the group of equivalence classes of abelian group extensions of A by B) first obtained by H. Smith in [Sm 2, Theorem 3]. ■

COROLLARY 1.4. *If X is a* l. c. s. c. *Hausdorff space and $C(X,\mathbb{T})$ is viewed as a trivial $\mathbb{Z}^m$-module, then $H^n_{\mathrm{pt}}\big(\mathbb{Z}^m,C(X,\mathbb{T})\big)=\{0\}$ so that $H^n\big(\mathbb{Z}^m,C(X,\mathbb{T})\big)\cong C\big(X,H^n(\mathbb{Z}^m,\mathbb{T})\big)\cong C(X,\mathbb{T}^{\binom{m}{n}})=\prod_{i=1}^{\frac{m!}{(m-n)!\,n!}}[C(X,\mathbb{T})]_i$, $m\geq n$.* ■

PROOF. $\mathbb{Z}^m$ and $H_{n-1}(\mathbb{Z}^m)$ are free abelian so that $H^n_{\mathrm{pt}}\big(\mathbb{Z}^m,C(X,\mathbb{T})\big)\cong\operatorname{Ext}\big(H_{n-1}(\mathbb{Z}^m),$ $C(X,\mathbb{T})\big)\cong\{0\}$. ■

Combining the above corollaries we obtain

COROLLARY 1.5. *If either the countable discrete abelian group G is a finitely generated free abelian group or the* l. c. s. c. *Hausdorff space X is such that $\check{H}^1(\beta X,\mathbb{Z})$ is divisible, then $H^n\big(G,C(X,\mathbb{T})\big)\cong C\big(X,H^n(G,\mathbb{T})\big)$.* ■

PROOF. Under either of the above hypotheses $\operatorname{Ext}\big(H_{n-1}(G),\check{H}^1(\beta X,\mathbb{Z})\big)\cong\{0\}$. ■

https://doi.org/10.4153/CJM-1996-007-6 Published online by Cambridge University Press

2. **Applications of Moore cohomology to central twisted crossed product C^*-algebras and crossed products of continuous trace C^*-algebras.** It was shown by S. Hurder, D. Olesen, Raeburn and Rosenberg that any twisted transformation group C^*-algebra $C_0(X) \times_{\alpha,\sigma} G$ arising from a separable twisted topological dynamical system (X, α, σ, G) is stably isomorphic to an ordinary C^*-crossed product of the form $[C_0(X) \otimes \mathcal{K}] \times_\beta G$ ([HORR, Proposition 3.1]; here, as in the sequel, $\mathcal{K}$ represents the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space $\mathcal{H}$). Indeed, the systems $\big(C_0(X), \alpha, \sigma, G\big)$ and $\big(C_0(X) \otimes \mathcal{K}, \beta, 1, G\big)$ are *stably* exterior equivalent in a sense that we will make precise shortly, and thus the following converse question comes to mind: which ordinary C^*-dynamical systems $\big(C_0(X) \otimes \mathcal{K}, \beta, G\big)$ are stably exterior equivalent to twisted C^*-dynamical systems arising from twisted topological dynamical systems as above? Here we discuss this question in the case where the induced action $\tilde{\beta}$ of G on $X = \mathrm{Prim}\big(C_0(X) \otimes \mathcal{K}\big)$ (and hence the action $\tilde{\alpha}$ of G on X) is the trivial identity action, and relate this problem to the Moore group $H^2\big(G, C(X, \mathbb{T})\big)$, and the equivariant Brauer group $\mathrm{Br}_G(X)$ of [CKRW]. Some of what we discuss here is already implicit in the literature (particularly in [Ro, Section 2] and in [CKRW]), but, as in our first section, we feel it will be helpful to make these relationships explicit.

Let $\big(C_0(X) \otimes \mathcal{K}, \beta, G\big)$ be a C^*-dynamical system. If β_g acts trivially on the spectrum X of $C_0(X) \otimes \mathcal{K}$ $\forall g \in G$, then by results of C. Lance [La] and G. Elliott [El], $\beta_g \in \mathrm{Aut}_{C_b(X)}\big(C_0(X) \otimes \mathcal{K}\big)$, *i.e.* β_g is π-inner, $\forall \pi \in \mathrm{Prim}\big(C_0(X) \otimes \mathcal{K}\big)$; $\forall g \in G$. In order for such a system to be stably exterior equivalent to a system arising from a twisted topological dynamical system, even more must be true.

DEFINITION 2.1. Let $(\mathcal{A}, \alpha, \sigma, G)$ and $(\mathcal{B}, \beta, \tau, G)$ be two twisted C^*-dynamical systems as defined in [PR]. We say that the two systems are *stably exterior equivalent* if $(\mathcal{A} \otimes \mathcal{K}, \alpha \otimes \mathrm{id}, \sigma \otimes 1, G)$ is exterior equivalent in the sense of [PR, Definition 3.1] to $(\mathcal{B} \otimes \mathcal{K}, \beta \otimes \mathrm{id}, \tau \otimes 1, G)$. ■

It is easy to check that a twisted system of the form $\big(C_0(X), \iota, \sigma, G\big)$ arising from a twisted topological dynamical system with trivial G action ι is stably exterior equivalent to a system of the form $\big(C_0(X) \otimes \mathcal{K}, \beta, 1, G\big)$ with trivial twist if and only if there is a Borel map $V: G \longrightarrow \mathcal{UM}\big(C_0(X) \otimes \mathcal{K}\big) = C(X, \mathcal{U})$ (where $\mathcal{U}$ = the group of unitary operators $\mathcal{U}(\mathcal{H})$ is given the strong operator topology which coincides with the weak operator topology on $\mathcal{U}(\mathcal{H})$) such that

$$(2.1) \qquad \text{(i) } \beta_g = \mathrm{Ad}\, V_g \quad \forall g \in G$$

(so that each β_g is an inner automorphism, and not just π-inner)

$$(2.2) \qquad \text{(ii) } V_{g_1 g_2}(x) = \sigma(g_1, g_2)(x) V_{g_1}(x) V_{g_2}(x) \quad \forall g_1, g_2 \in G,\ \forall x \in X.$$

i.e. for each $x \in X$ the V's provide a projective representation of G on $\mathcal{H}$ with multiplier $\sigma(\cdot,\cdot)(x)$. ■

It is equally evident that given a C^*-dynamical system $\big(C_0(X) \otimes \mathcal{K}, \beta, G\big)$ and a Borel map $V: G \longrightarrow C(X, \mathcal{U})$ satisfying condition (i) alone, then the fact that $\beta: G \longrightarrow$

$\mathrm{Aut}\big(C_0(X) \otimes \mathcal{K}\big)$ is a homomorphism implies that there exists $\sigma \in Z^2\big(G, C(X, \mathbb{T})\big)$ such that $\big(C_0(X) \otimes \mathcal{K}, \beta, 1, G\big)$ is stably exterior equivalent to $\big(C_0(X), \iota, \sigma, G\big)$. For future reference, note that the cocycle σ associated to an element-wise inner action β of G on $C_0(X) \otimes \mathcal{K}$ (*i.e.*, an action β such that β_g is inner $\forall g \in G$) is not unique; however the class $[\sigma] \in H^2\big(G, C(X, \mathbb{T})\big)$ is unique and we denote it by $[\sigma_\beta]$. We therefore have proved:

PROPOSITION 2.2. *Let G be a* l. c. s. c. *group, let X be a* l. c. s. c. *Hausdorff space, and view $C(X, \mathbb{T})$ as a trivial G-module. A C^*-dynamical system $\big(C_0(X) \otimes \mathcal{K}, \beta, G\big)$ is stably exterior equivalent to a twisted system of the form $\big(C_0(X), \iota, \sigma, G\big)$ for some $\sigma \in Z^2\big(G, C(X, \mathbb{T})\big)$ if and only if β_g is inner $\forall g \in G$.* ∎

Thus the study of untwisted G-actions on $C_0(X) \otimes \mathcal{K}$ which are stably exterior equivalent to actions of the form $\big(C_0(X), \iota, \sigma, G\big)$ amounts to the study of element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$.

We now study the exterior equivalence classes of element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$. Most of the following result can be deduced from the literature (see in particular [CKRW, Section 6.3]); additional related results are the classifications of exterior equivalence classes of locally unitary actions and pointwise unitary actions of abelian groups on stable continuous trace C^*-algebras given in [PhR 2] and [OR], respectively. Thus we omit detailed proofs.

THEOREM 2.3 [CRKW]. *Let G be a* l. c. s. c. *group, let X be a* l. c. s. c. *Hausdorff space, and view $C(X, \mathbb{T})$ as a trivial G-module. Then the set of exterior equivalence classes of element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$ form an abelian group which is isomorphic to the Moore group $H^2\big(G, C(X, \mathbb{T})\big)$.* ∎

PROOF. Denote the set of exterior equivalence classes of element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$ by $\mathcal{EI}(X, G)$. This set is endowed with a group structure as follows (note that this construction is similar to one given in a lemma of Echterhoff and Rosenberg [ER Lemma 4]): if α and β are element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$, define a new action $\alpha \cdot \beta$ of G on $C_0(X) \otimes \mathcal{K}$ by letting the diagonal action $\alpha \otimes \beta$ of G on $[C_0(X) \otimes \mathcal{K}] \otimes [C_0(X) \otimes \mathcal{K}]$ induce an action $\alpha \cdot \beta$ of G on the balanced tensor product $[C_0(X) \otimes \mathcal{K}] \otimes_{C_0(X)} [C_0(X) \otimes \mathcal{K}]$. This last C^*-algebra is canonically isomorphic to $C_0(X) \otimes [\mathcal{K} \otimes \mathcal{K}]$ which is in turn isomorphic to $C_0(X) \otimes \mathcal{K}$; thus we can view $\alpha \cdot \beta$ as an action of G on $C_0(X) \otimes \mathcal{K}$, which one easily checks is element-wise inner. One easily checks that it makes sense to define $[\alpha] \cdot [\beta] = [\alpha \cdot \beta] = [\beta \cdot \alpha] = [\beta] \cdot [\alpha]$ as an element of $\mathcal{EI}(X, G)$. The unit of $\mathcal{EI}(X, G)$ is the equivalence class of the trivial identity action. Given an element-wise inner action β of G on $C_0(X) \otimes \mathcal{K}$, find some $\sigma_\beta \in Z^2\big(G, C(X, \mathbb{T})\big)$ as in Proposition 2.2, and define $[\beta]^{-1}$ by first finding a Borel map $\tilde{V}_\beta : G \longrightarrow C(X, \mathcal{U})$ with $\tilde{V}_\beta(g_1 g_2)(x) = \overline{\sigma_\beta(g_1, g_2)(x)} \tilde{V}_\beta(g_1)(x) \tilde{V}_\beta(g_2)(x)$, $\forall x \in X$, $\forall g_1, g_2 \in G$ (the construction given in [HORR, Proposition 3.1], which is basically just a parameterization over X of $\sigma(\cdot, \cdot)(x)$-regular representations of G, will do nicely), and then setting $[\beta]^{-1} = [\mathrm{Ad}\, \tilde{V}_\beta]$. The unit, product, and inverse in our proposed group

$\mathcal{EJ}(X,G)$ are now defined, and one easily checks that one can view $\mathcal{EJ}(X,G)$ as the subgroup $\operatorname{im}\xi$ of the equivariant Brauer group of [CKRW], $\mathrm{Br}_G(X)$, corresponding to the trivial G action on X; here ξ is the map in the exact sequence of [CRKW, 6.3], which is reproduced (in slightly modified form) in equation (2.4) for the reader's convenience. We now define $\psi\colon \mathcal{EI}(X,G)\longrightarrow H^2\big(G, C(X,\mathbb{T})\big)$ by $\psi([\beta])=[\sigma_\beta]$, $[\sigma_\beta]$ as in Proposition 2.2. Our previous computations have shown us that ψ is well-defined, and again one can easily check that ψ is a group isomorphism which is the inverse of the map ξ of [CKRW] mentioned above. ∎

We now investigate the relevance of the above results to the study of pointwise unitary actions of G on $C_0(X)\otimes\mathcal{K}$. Recall from [PhR 2] that an action β of G on $C_0(X)\otimes\mathcal{K}$ is pointwise unitary if for each $\pi\in\operatorname{Prim}\big(C_0(X)\otimes\mathcal{K}\big)=X$ there is a covariant representation (π,u) of the C^*-dynamical system $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$. Hence, for every $g\in G$ the automorphism $\beta_g\in\operatorname{Aut}_{C_b(X)}\big(C_0(X)\otimes\mathcal{K}\big)$ (it follows, of course, that the induced action of G on X is the identity action). By Proposition 2.2, in order that $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$ be stably exterior equivalent to a twisted system of the form $\big(C_0(X),\alpha,\sigma,G\big)$, it is necessary and sufficient that β_g be inner for every $g\in G$, in which case $[\sigma]=[\sigma_\beta]\in H^2\big(G,C(X,\mathbb{T})\big)$. Consequently, using the exact sequence of [PhR 1, Theorem 2.1]

$$(2.3)\qquad 0\longrightarrow \operatorname{Inn}\big(C_0(X)\otimes\mathcal{K}\big)\xrightarrow{i_*}\operatorname{Aut}_{C_b(X)}\big(C_0(X)\otimes\mathcal{K}\big)\xrightarrow{\eta}\check{H}^2(X,\mathbb{Z})\longrightarrow 0,$$

in order that $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$ be stably exterior equivalent to a twisted system as above it is necessary and sufficient that $\eta(\beta_g)=0$, $\forall g\in G$. From this we deduce:

COROLLARY 2.4. *Let $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$ be a pointwise unitary C^*-dynamical system, where G is a* l.c.s.c. *group and X is a* l.c.s.c. *Hausdorff space. In order that $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$ be stably exterior equivalent to a twisted C^*-dynamical system of the form $\big(C_0(X),\iota,\sigma,G\big)$, it is necessary and sufficient that $\eta(\beta_g)=0\in\check{H}^2(X,\mathbb{Z})$, $\forall g\in G$, η as in (2.3). In this case $\psi([\beta])\in H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$, where ψ is the isomorphism of Theorem 2.3.* ∎

PROOF. It only remains to prove the last statement. If $\eta(\beta_g)=0\in\forall g\in G$ so that β is element-wise inner, we can define $\psi([\beta])=[\sigma_\beta]$ as in Theorem 2.3, and since β is pointwise unitary, we see that $[\sigma_\beta]\in H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$ by applying [Ro], Remark 1.2].

Of course, the above results are consistent with existing results in the literature. By work of Olesen and Raeburn [OR, Theorem 1.10] generalizing result of Phillips and Raeburn for locally unitary automorphism groups [PhR 2], if G is l. c. s. c. abelian, there is a one-to-one correspondence between exterior equivalence classes of pointwise unitary C^*-dynamical systems $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$ and isomorphism classes of principal $\hat{G}$-bundles over X, denoted by $\mathcal{HP}(X,\hat{\mathcal{G}})$ in [RW 1] (if G is in addition compactly generated, all principal $\hat{G}$-bundles are automatically locally trivial, by the standard results of R. Palais [Pa] and J. Rosenberg [Ro]; consequently all pointwise unitary actions of G are locally unitary). Raeburn and Williams have shown in [RW 1, Lemma 2.24] that

https://doi.org/10.4153/CJM-1996-007-6 Published online by Cambridge University Press

$\mathcal{HP}(X,\hat{\mathcal{G}})$ forms a group, which for G compactly generated is just the Čech group $\check{H}^1(X,\hat{\mathcal{G}})$. Our Corollary 2.4 shows that if a pointwise unitary C^*-dynamical system $\big(C_0(X)\otimes\mathcal{K},\beta,G\big)$ is in addition element-wise inner, and if G is l. c. s. c. abelian, then it is stably exterior equivalent to a twisted system of the form $\big(C_0(X),\iota,\sigma,G\big)$. Moreover $[\sigma]\in H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$, and by [OR, Theorem 1.10], $\big(C_0(X)\otimes\mathcal{K}\big)\times_\beta G$ is stably isomorphic to the central twisted crossed product $C_0(X)\times_{\iota,\sigma}G$, which in turn is isomorphic to $C_0(E_{[\sigma]})$, where $E_{[\sigma]}$ is the principal $\hat{G}$ bundle over X associated to $[\sigma]$ as constructed in [Sm 2], [Ro] and [RW 1].

We mentioned in our introduction that Raeburn and Williams showed that this correspondence between $H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$ ad $\mathcal{HP}(X,\hat{\mathcal{G}})$ is a group monomorphism, and they have characterized the image of $H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$ in $\mathcal{HP}(X,\hat{\mathcal{G}})$ (as the so-called characteristic principal $\hat{G}$-bundles) in [RW 1, Theorem 4.4]. Intriguingly, the vanishing of cohomology classes $\eta(\beta_g)$ in our Corollary 2.4 can easily be interpreted in terms of the Raeburn and Williams characterization by using [RW 1, Theorem 4.4 and Corollary 4.3].

We now concentrate on the case where G is countable discrete abelian, and note how some results from Section 1 can be used in this situation to relate $H^2_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)$, $\mathcal{HP}(X,\hat{\mathcal{G}})$, $\check{H}^1(X,\hat{\mathcal{G}})$, $\mathcal{EI}(X,G)$, and the equivariant Brauer group $\mathrm{Br}_G(X)$. Of course, for G discrete, $\mathrm{Br}_G(X)$ coincides with one of Grothendieck's equivariant sheaf cohomology groups [G], and there already is a spectral sequence to aid in their calculation, but there are still additional facts we can deduce in this situation. Recall from [CKRW] that if (X,G) is a second countable transformation group then $\mathrm{Br}_G(X)$ is the set of Morita equivalence classes of C^*-dynamical systems $\{[\mathcal{A},\alpha,G]\}$ where $\mathcal{A}$ is a separable continuous trace C^*-algebra with spectrum X, α is an action of G on $\mathcal{A}$ whose induced action on $\mathrm{Prim}(\mathcal{A})=X$ coincides with the original one, and where the group multiplication is given by the balanced tensor product $[\mathcal{A},\alpha,G]\cdot[\mathcal{B},\beta,G]=[\mathcal{A}\otimes_{C_0(X)}\mathcal{B},\alpha\otimes\beta,G]$. If G acts on X trivially, the Brauer group then becomes equivalence classes of C^*-dynamical systems $[\mathcal{A},\alpha,G]$ where the induced action of G on $\mathrm{Prim}(\mathcal{A})=X$ is trivial. One deduces from [CKRW, 6.3] that in this case $\mathrm{Br}_G(X)$ can be written as the direct sum of $\ker(F)\oplus\mathrm{im}(F)$ where the forgetful map $F\colon\mathrm{Br}_G(X)\longrightarrow\check{H}^3(X,\mathbb{Z})$ is split surjective in the case of trivial G action. Thus $\mathrm{im}(F)=\check{H}^3(X,\mathbb{Z})$ and $\ker(F)=\{[C_0(X)\otimes\mathcal{K},\alpha,G]$, α induces a trivial G action on $X\}$. The subgroup $\ker(F)$ (of which, in turn, $\mathcal{EI}(X,G)$ is subgroup) is described via a 5-term exact sequence in [CKRW, 6.3]; we observe now that an easy calculation involving equations (4.4) and (4.5) of [CKRW] show that in the case where either G or $\check{H}(X,\mathbb{Z})$ is finitely generated the map d_2' between the final two groups in that sequence in fact maps into $H^3_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big)\subseteq H^3\big(G,C(X,\mathbb{T})\big)$ so that this sequence can be written

$$\begin{array}{ccccc} 0 & \longrightarrow & H^2\big(G,C(X,\mathbb{T})\big) & \xrightarrow{\ \xi\ } & \ker(F)=\{[C_0(X)\otimes\mathcal{K},\alpha,G]\} \\ & & & \swarrow_{\eta} & \\ & & \mathrm{Hom}\big(G,\check{H}^2(X,\mathbb{Z})\big) & \xrightarrow{\ d_2'\ } & H^3_{\mathrm{pt}}\big(G,C(X,\mathbb{T})\big). \end{array} \tag{2.4}$$

Our intent now is to define $\ker(F)$ by (quite different) short exact sequence in the case where G is countable discrete abelian, which will in addition show us that $\ker(F)$ is generated by the exterior equivalence classes of pointwise unitary and element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$:

THEOREM 2.5. *Let G be a countable discrete abelian group acting trivially on the* l. c. s. c. *Hausdorff space X. View* $\ker(F)$ *as the subgroup* $\{[C_0(X) \otimes \mathcal{K}, \alpha, G]\}$, *$\alpha$ induces a trivial action of G on $X\}$ of the equivariant Brauer group* $\mathrm{Br}_G(X)$. *Then there is a split short exact sequence*

$$0 \longrightarrow \mathcal{HP}(X, \hat{\mathcal{G}}) \xrightarrow{i_*} \ker(F) \xrightarrow{\partial_*} C\big(X, H^2(G, \mathbb{T})\big) \longrightarrow 0 \tag{2.5}$$

Consequently $\ker(F)$ *is generated by the pointwise unitary actions and the element-wise inner actions of G on $C_0(X) \otimes \mathcal{K}$.* ∎

PROOF. We first note that by [CKRW Lemma 3.1], Morita equivalence classes of G on $C_0(X) \otimes \mathcal{K}$ which induce the trivial action on X can be identified with exterior equivalence classes of actions of G on $C_0(X) \otimes \mathcal{K}$ which induce the trivial action on X. Thus when we write $[C_0(X) \otimes \mathcal{K}, \beta]$, without loss of generality our "$[\cdots]$" refers to exterior equivalence class. We now define the map i_*. Let $[E,p,X]$ denote an element of $\mathcal{HP}(X, \hat{\mathcal{G}})$, that is, $[E,p,X]$ denotes the isomorphism class of a $\hat{G}$-bundle over X in the sense of [RW1, Section 2]. Then by [OR, Corollary 1.11 and Proposition 1.13] we can associate a pointwise unitary C^*-dynamical system $\big(\mathcal{A}_E = C_0(X) \otimes \mathcal{K}, \alpha_E, G\big)$ with $\big(\mathrm{Prim}(\mathcal{A}_E \times_{\alpha_E} G), \mathrm{Res}, X\big)$ isomorphic to (E,p,X) as $\hat{G}$-bundles, and furthermore, $(\mathcal{A}_E, \alpha_E, G)$ is exterior equivalent to $(\mathcal{A}_{E'}, \alpha_{E'}, G)$ if and only if (E,p,X) is isomorphic to (E',p',X) as a $\hat{G}$-bundle. Therefore we can define $i_*([E,p,X]) = [\mathcal{A}_E, \alpha_E, G]$ and obtain a one-to-one map of $\mathcal{HP}(X, \hat{\mathcal{G}})$ into $\ker(F)$, which by standard calculations (as in [RW 1], [OR], [ER], and [CKRW]) is a group monomorphism; we omit details. We now define ∂_*. The group $\mathrm{Aut}_{C_b(X)}[C_0(X) \otimes \mathcal{K}]$ of all automorphisms of $C_0(X) \otimes \mathcal{K}$ which fix the spectrum X can be identified in the standard way with $C\big(X, \mathrm{Aut}(K)\big) = C\big(X, \mathcal{PU}(\mathcal{H})\big)$, where $\mathcal{PU}(\mathcal{H})$ is the quotient of the unitary group $\mathcal{U}(\mathcal{H})$ by its center $\mathbb{T}\,\mathrm{Id}$, given the quotient topology of the strong or weak operator topologies, which coincide on $\mathcal{U}(\mathcal{H})$. The exact sequence of groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z\big(\mathcal{U}(\mathcal{H})\big) & \longrightarrow & \mathcal{U}(\mathcal{H}) & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathcal{K}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{T}\,\mathrm{Id} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{PU} & \longrightarrow & 0 \end{array} \tag{2.6}$$

gives rise to the standard Bockstein sequence in low-dimensional cohomology which gives us the connecting map $\partial\colon H^1(G, \mathcal{PU}) \longrightarrow H^2(G, \mathbb{T})$. We note that $H^1(G, \mathcal{PU})$ is a set, and not a group, so that we cannot use the standard proof of [M2, Proposition 25] to show that ∂ is continuous. Though the continuity of ∂ certainly must be known, we did not see a proof in the literature so we provide the short proof here. Recall that ∂ is defined by choosing a Borel section $c\colon \mathcal{PU} \longrightarrow \mathcal{U}$ and setting $\partial(\alpha) = [\sigma_\alpha]$, where $\sigma_\alpha(g_1, g_2) = c\big(\alpha(g_1)\big)c\big(\alpha(g_2)\big)\big[c\big(\alpha(g_1g_2)\big)\big]^{-1}$, $g_1, g_2 \in G$ (here we identify $\mathbb{T}\,\mathrm{Id}$ with $\mathbb{T}$). Now suppose $\{\alpha_n\}$ is a sequence in $H^1(G, \mathcal{PU})$ converging to α_0, where $H^1(G, \mathcal{PU})$

is given the topology of pointwise convergence. Write $U_n(g) = c(\alpha_n(g))$, $g \in G$; then $\alpha_n(g)(\mathcal{k}) = \operatorname{Ad} U_n(g)(\mathcal{k})$, $\mathcal{k} \in \mathcal{K}$. Since $\mathcal{PU}$ is given the quotient topology, to say that $\operatorname{Ad} U_n(g)$ converges to $\operatorname{Ad} U_0(g)$ for fixed g in G means that there is a sequence $\{\lambda_n(g)\}$ of complex numbers of modulus one such that $\lambda_n(g)U_n(g)$ converges to $U_0(g)$ in the strong operator topology. Then standard calculations show that for fixed g and h in G, if we write $U'_n = \lambda_n U_n$, $\sigma_{\alpha_n}(g,h)\partial(\lambda_n)(g,h)U'_n(gh) = U'_n(g)U'_n(h)$ which converges in the strong operator topology to $U_0(g)U_0(h) = \sigma_{\alpha_0}(g,h)U_0(gh)$. Since we know that $U'_n(gh)$ converges to $U_0(gh)$ in the strong operator topology we must have $\sigma_{\alpha_n}(g,h)\partial(\lambda_n)(g,h)$ converging to $\sigma_{\alpha_0}(g,h)$ for fixed $g,h \in G$, that is, $[\sigma_{\alpha_n}]$ converges to $[\sigma_{\alpha_0}]$ in $H^2(G,\mathbb{T})$, so that $\partial\colon H^1(G,\mathcal{PU}) \longrightarrow H^2(G,\mathbb{T})$ is continuous. Now an action of G on $C_0(X) \otimes \mathcal{K}$ which fixes the spectrum is given by a homomorphism of G into $C(X,\mathcal{PU})$, that is, an element of $H^1\big(G, C(X,\mathcal{PU})\big)$, which, since G is discrete and acts trivially on the spectrum X, can be written as $C\big(X, H^1(G,\mathcal{PU})\big)$. We therefore define $\partial_*\colon H^1\big(G, C(X,\mathcal{PU})\big) \longrightarrow C\big(X, H^2(G,\mathbb{T})\big)$ by $\partial_*(\beta) = \partial\big(\beta(x)\big)$. It remains to show that $\operatorname{im} i_* = \ker \partial_*$ and ∂_* is surjective. Suppose that $[C_0(X) \otimes \mathcal{K}, \beta] \in \operatorname{im} i_*$ so that β is a pointwise unitary action of G on $C_0(X) \otimes \mathcal{K}$. Then the Mackey obstructions to implementing α over each $x \in X$ all vanish, that is, $[\sigma_\beta] \in H^2_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$, $[\sigma_\beta]$ as in Proposition 2.2. But this means exactly that $\partial_*\big([C_0(X) \otimes \mathcal{K}, \beta]\big)(x) = \partial\big(\beta(x)\big) = [\sigma_\beta(x)] = 1_{H^2(G,\mathbb{T})}$, for every $x \in X$. Thus $\operatorname{im} i_* \subseteq \ker \partial_*$. Conversely, if $[C_0(X) \otimes \mathcal{K}, \beta] \in \ker \partial_*$, then $[\sigma_\beta(x)] = 1_{H^2(G,\mathbb{T})}$ for all $x \in X$, so that $[\sigma_\beta] \in H^2_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$, β is a pointwise unitary action of G on $C_0(X) \otimes \mathcal{K}$, and $[C_0(X) \otimes \mathcal{K}, \beta] \in \operatorname{im} i_*$. Therefore $\operatorname{im} i_* = \ker \partial_*$. Finally, to show that ∂_* is split surjective, let $\phi_*\colon C\big(X, H^2(G,\mathbb{T})\big) \longrightarrow \ker(F)$ be the composition $\zeta \circ \theta_*$, where $\zeta\colon H^2\big(G, C(X,\mathbb{T})\big) \longrightarrow \mathcal{EI}(X,G) \subseteq \ker(F)$ is the isomorphism of Theorem 2.3 and [CKRW, 6.3] and $\theta_*\colon C\big(X, H^2(G,\mathbb{T})\big) \longrightarrow H^2\big(G, C(X,\mathbb{T})\big)$ is the splitting map of Theorem 1.1 in the case $n = 2$. It is clear that ϕ_* splits ∂_*, and we therefore obtain the desired result. The last statement in Theorem 2.5 follows from the isomorphism $\ker(F) \cong \mathcal{HP}(X,\hat{\mathcal{G}}) \oplus C\big(X, H^2(G,\mathbb{T})\big)$, the fact that the map i_* identifies $\mathcal{HP}(X,\hat{\mathcal{G}})$ with the group of exterior equivalence classes of pointwise unitary actions of G on $C_0(X) \otimes \mathcal{K}$, and the fact that $\xi \circ \theta_*$ maps $C\big(X, H^2(X,\mathbb{T})\big)$ into $\mathcal{EI}(X,G)$. ■

COROLLARY 2.6. *Let G be a countable discrete abelian group acting trivially on the* l. c. s. c. *Hausdorff space X. Then the equivariant Brauer group* $\mathrm{Br}_G(X)$ *is isomorphic to the direct sum* $\check{H}^3(X,\mathbb{Z}) \oplus \mathcal{HP}(X,\hat{\mathcal{G}}) \oplus C\big(X, H^2(G,\mathbb{T})\big)$. *If G is in addition finitely generated, this direct sum becomes* $\check{H}^3(X,\mathbb{Z}) \oplus \check{H}^1(X,\hat{\mathcal{G}}) \oplus C\big(X, H^2(G,\mathbb{T})\big)$.

PROOF. It is a consequence of [CKRW, 6.3] that $\mathrm{Br}_G(X) = \check{H}^3(X,\mathbb{Z}) \oplus \ker(F)$ (recall that $\check{H}^3(X,\mathbb{Z})$ is the image of F, and it is easy to see that $[\delta] \longrightarrow$[The stable C^*-algebra with spectrum X and Dixmier-Douady class $[\delta]$, trivial G-action] is a cross-section splitting F), and the result follows from Theorem 2.5 and the standard isomorphism $\mathcal{HP}(X,\hat{\mathcal{G}}) \cong \check{H}^1(X,\hat{\mathcal{G}})$ for finitely generated discrete abelian G. ■

REMARK 2.7. It is interesting to compare the equivariant Brauer group to the equivariant cohomology group $\check{H}^2_G(X,\mathcal{S})$ introduced by Raeburn and Williams in [RW 2], which roughly speaking, classifies up to Morita equivalence those C^*-dynamical systems $(\mathcal{A}, \alpha, G)$ where $\mathcal{A}$ is a continuous trace C^*-algebra with spectrum X and the action α of G on $\hat{\mathcal{A}} = X$ is equal to the given action) which are locally Morita equivalent to $\big(C_0(X), G\big)$. In [RW 2, Section 8.1], Raeburn and Williams prove that (under some mild technical conditions) for l. c. s. c. abelian G acting trivially on the paracompact space X, $\check{H}^2_G(X, \mathcal{S})$ is isomorphic to $\check{H}^3(X, \mathbb{Z}) \oplus \check{H}^1(X, \hat{\mathcal{G}})$. By Theorem 2.5 and Corollary 2.6, we can conclude that for G a countable discrete finitely generated abelian group acting trivially on the l. c. s. c. Hausdorff space X, the Raeburn-Williams group $\check{H}^2_G(X, \mathcal{S})$ is isomorphic to $\mathrm{Br}_G(X)/\xi \circ \theta_*\Big(C\big(X, H^2(G, \mathbb{T})\big)\Big)$, where ξ and θ_* are the maps discussed in the proof of Theorem 2.5. ∎

REMARK 2.8. In the case where G is discrete and finitely generated, one can easily use Theorem 2.5 together with the five-term exact sequence (2.4) of [CKRW, 6.3] to identify $\mathrm{Range}\,\eta = \ker d_2' \subseteq \mathrm{Hom}\big(G, \check{H}^2(X, \mathbb{Z})\big)$ in that sequence with the image of the obvious map $j: \check{H}^1(X, \hat{\mathcal{G}}) \cong \check{H}^1\big(X, \mathcal{H}om(G, \mathbb{T})\big) \longrightarrow \mathrm{Hom}\big(G, \check{H}^2(X, \mathbb{Z})\big) \cong \mathrm{Hom}\big(G, \check{H}^1(X, \mathcal{S})\big)$ given in terms of transition functions. If G is in addition torsion free (*i.e.* if $G = \mathbb{Z}^n$ for some positive integer n) or if $\check{H}^1(\beta X, \mathbb{Z})$ is divisible, Corollary 1.5 allows us to deduce that $H^3_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big) = \{0\}$, so that be the exact sequence (2.4), the map η, hence the map j, must be surjective. ∎

REMARK 2.9. One can check that by Theorem 1.1, $\mathcal{HP}(X, \hat{\mathcal{G}}) \oplus C\big(X, H^2(G, \mathbb{T})\big)$ can be defined as the quotient of $\mathcal{HP}(X, \hat{\mathcal{G}}) \oplus H^2\big(G, C(X, \mathbb{T})\big)$ by the action of $H^2_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big)$ viewed as a subgroup of both $\mathcal{HP}(X, \hat{\mathcal{G}})$ and $H^2\big(G, C(X, \mathbb{T})\big)$) given by $x \cdot (y, z) = (y + x, z - x)$. Now this last quotient is defined whether or not G is discrete, and we conjecture that for arbitrary l. c. s. c. abelian groups G, $\ker(F)$ will be equal to this quotient, thus giving a related expression for $\mathrm{Br}_G(X)$. ∎

We now use our results to consider several examples, and relate these examples to the previous literature. For connected G, J. Rosenberg has shown in [Ro, Theorem 2.5] that the image of $H^2_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big)$ is all of $\check{H}^1(X, \hat{\mathcal{G}})$, by showing that any action of such a group G on a separable continuous trace C^*-algebra which acts trivially on the spectrum X must be element-wise inner; certainly our Theorem 1.1 tells us that for discrete G, and, as shown in [RW1], for more general non-connected G, such an equality need not hold. In [RW1, Example 4.7] Raeburn and Williams had already provided a counterexample with $G = \mathbb{R} \times \mathbb{Z}$. Our results in Section 1 show that there are many examples of discrete abelian G with $H^2_{\mathrm{pt}}\big(G, C(X, \mathbb{T})\big) \neq \check{H}^1(X, \hat{\mathcal{G}})$. Taking $G = \mathbb{Z}^n$, Corollary 1.2 shows that $H^2_{\mathrm{pt}}\big(\mathbb{Z}^n, C(X, \mathbb{T})\big)$ is always trivial for any l. c. s. c. Hausdorff space X. On the other hand, the Bockstein long exact sequence in sheaf cohomology shows that (using the fact that $\mathbb{R}^n$ is a fine sheaf) $\check{H}^1(X, \hat{\mathbb{Z}}^n) = \check{H}^1(X, \mathcal{S}^n) \cong \check{H}^2(X, \mathbb{Z}^n) = \oplus_{i=1}^n [\check{H}^2(X, \mathbb{Z})]_i$. (Compare to the exact sequence (2.3) and consider Remark 2.8!!) Therefore by Corollary 1.4, Corollary 2.6, and the above, if $\mathbb{Z}^n$ acts trivially on the locally compact Hausdorff space X, then $\mathrm{Br}_{\mathbb{Z}^n}(X) = \check{H}^3(X, \mathbb{Z}) \oplus \oplus_{i=1}^n \big(\check{H}^1(X, \mathbb{Z})\big)_i \oplus \oplus_{i=1}^{n(n-1)/2} [C(X, \mathbb{T})]_i$. All of the summands in this last group will certainly be non-trivial in general (*e.g.* consider $X = \mathbb{T}^m$ where $m \geq 2$). Therefore, in particular, for the group $\mathbb{Z}^n$, $n \geq 2$, and, one would expect, for a wide variety of discrete abelian groups G containing $\mathbb{Z}^n$ as a direct summand, it should be expected that in general, not all pointwise unitary C^*-dynamical systems will be stably exterior equivalent to twisted abelian systems of the form $\big(C_0(X), \iota, \sigma, G\big)$.

Using generalizations of the ideas presented above, together with the split short exact sequence of [PhR 1, Theorem 2.2],

$$(2.7) \qquad 0 \longrightarrow \mathrm{Aut}_{C_b(X)}[C_0(X) \otimes \mathcal{K}] \longrightarrow \mathrm{Aut}[C_0(X) \otimes \mathcal{K}] \longrightarrow \mathrm{Homeo}(X) \longrightarrow 0$$

together with the equivariant Brauer group $\mathrm{Br}_G(X)$ of [CKRW], it should be possible to approach the problem of characterizing which general C^*-dynamical systems of the form $\big(C_0(X) \otimes \mathcal{K}, \beta, G\big)$ are stably exterior equivalent to systems $\big(C_0(X), \alpha, \sigma, G\big)$ arising from twisted topological dynamical systems, where now the group G in question need not act trivially on X; work on this project is now in progress.

NOTE ADDED IN REVISION. With help from a suggestion of Lawrence Baggett, we realized that the conclusion of Corollary 2.6 for finitely generated groups can more informatively be written as:

$$(2.8) \qquad \mathrm{Br}_G(X) \cong \check{H}^2\big(X, \underline{H^0(G,\mathbb{T})}\big) \oplus \check{H}^1\big(X, \underline{H^1(G,\mathbb{T})}\big) \oplus \check{H}^0\big(X, \underline{H^2(G,\mathbb{T})}\big)$$

We now have a proof of equation (2.8) in the case where G is a compactly generated abelian Lie group acting trivially on the l. c. s. c. Hausdorff space X. Iain Raeburn has suggested to us that equation (2.8) can be derived from a spectral sequence of Grothendieck, which in turn suggests that interesting results can be obtained even when G does not act trivially on X. Details will appear elsewhere.

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