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EVENTUAL DISCONJUGACY AND RIGHT DISFOCALITY OF LINEAR DIFFERENCE EQUATIONS

BY
P. W. ELOE

ABSTRACT. Let $a \geqq 0$, $I_a = \{a, a+1, \ldots\}$ and consider the nth order linear difference equation $Pu(m) = \sum_{j=0}^{n} \alpha_j(m)\Delta^{n-j}u(m) = 0$, $m \in I_a$, $\alpha_0(m) \equiv 1$ on I_a. Summability conditions are placed on the coefficients $\alpha_j(m)$, $1 \leqq j \leqq n$, such that the equation $Pu(m) = 0$ is eventually disconjugate. Conditions for eventual right disfocality are also given.

Introduction. Let a be a nonnegative real number and define the unbounded set $I_a = \{a, a+1, \ldots\}$. We consider the nth order linear difference equation

$$(1.1) \qquad Pu(m) = \sum_{j=0}^{n} \beta_j(m)u(m+j) = 0,$$

where $\beta_n(m) \equiv 1$, $\beta_0(m) \neq 0$ on I_a and the independent variable m ranges over I_a. Define $\Delta^0 u(m) \equiv u(m)$, $\Delta u(m) = u(m+1) - u(m)$, and $\Delta^j u(m) = \Delta(\Delta^{j-1}u(m))$, $2 \leqq j \leqq n$. We shall also consider equation (1.1) in the form

$$(1.2) \qquad Pu(m) = \sum_{j=0}^{n} \alpha_j(m)\Delta^{n-j}u(m),$$

where $\alpha_0(m) \equiv \beta_n(m) \equiv 1$ on I_a.

We now list several definitions. The first three can be found in Hartman's paper [4].

DEFINITIONS. (i) *For a sequence u: $u(a), u(a+1), \ldots, m = a$ is a node for u if $u(a) = 0$ and $m > a$ is a node for u if either $u(m) = 0$ or $u(m-1)u(m) < 0$.*

(ii) *For a sequence u: $u(a), u(a+1), \ldots, m = a$ is a generalized zero for u if $u(a) = 0$ and $m > a$ is a generalized zero for u if either $u(m) = 0$ or there is an integer k, $1 \leqq k \leqq m - a$, such that $(-1)^k u(m-k)u(m) > 0$ and, if $k > 1$, $u(m-k+1) = \ldots = u(m-1) = 0$.*

(iii) *The difference equation* (1.2) (*and thus,* (1.1)) *is disconjugate on I_a if no*

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solution $u \not\equiv 0$ *has more than* $n - 1$ *generalized zeros on* I_a.

(iv) *The difference equation* (1.2) *is right disfocal on* I_a *if the only solution* u *of* (1.2) *satisfying* $\Delta^{j-1}u$ *has a node at* m_j, $m_j \in I_a$, $1 \leqq j \leqq n$, *where* $a \leqq m_1 \leqq \ldots \leqq m_n$, *is* $u \equiv 0$.

(v) *The difference equation* (1.2) *is eventually disconjugate* (*eventually right disfocal*) *if there exists* $m_0 \geqq a$, $m_0 \in I_a$, *such that the equation* (1.2) *is disconjugate* (*right disfocal*) *on* I_{m_0}.

Willett [8] showed that the linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = 0 \tag{1.3}$$

is eventually disconjugate if

$$\int^{\infty} t^{k-1}|p_k(t)dt| < \infty, \ 1 \leqq k \leqq n.$$

Trench [6], [7] weakened these conditions for absolute integrability and showed that conditional convergence of the integrals can be allowed. To weaken the integrability conditions, Trench [6], [7] constructed a fundamental set of solutions $\{y_0, \ldots, y_{n-1}\}$ of (1.3) satisfying the Pólya criterion for disconjugacy [5],

$$W(y_0, \ldots, y_{k-1})(t) > 0, \ 1 \leqq k \leqq n,$$

on some interval $[t_0, \infty)$. Here, W denotes the usual Wronskian determinant. Eloe and Henderson [2] have obtained analogues of these integrability conditions for the eventual right disfocality of (1.3).

In this paper, we shall provide summability conditions on the coefficients $\alpha_j(m)$ analogous to the integrability conditions given by Willett [8], Trench [6], [7] and Eloe and Henderson [2] for the eventual disconjugacy and eventual right disfocality of (1.2). In section 2, we shall provide several lemmas concerning the calculus of finite differences. In section 3, we shall obtain summability conditions on the coefficients $\alpha_j(m)$ for the eventual disconjugacy of (1.2); the technique of proof presented here is analogous to that of Trench [6], [7], which is described in the above paragraph. In section 4, we shall indicate how the method of section 3 can be adapted to obtain summability conditions for the eventual right disfocality of (1.2).

2. **Preliminary lemmas.** We present four lemmas. The first three lemmas are elementary results from the calculus of finite differences; the reader is referred to Fort's text [3] for a thorough treatment of the calculus of finite differences.

LEMMA 2.1. *Let* k *be a positive integer.*

(i) *If* $k \geqq 2$, *then*

$$\Delta m(m-1)\ldots(m-(k-1)) = km(m-1)\ldots(m-(k-2)).$$

In particular, if $k \geqq 1$, *then*

$$\Delta^k(m(m-1)\ldots(m-(k-1))) = k!.$$

(ii) *If* $k \geqq 1$, *then*

$$\Delta m^{-1}\ldots(m+(k-1))^{-1} = -km^{-1}\ldots(m+k)^{-1},\ m > 0.$$

PROOF OF (i). Let $k \geqq 2$. Then

$$\begin{aligned}
&\Delta m(m-1)\ldots(m-(k-1))\\
&= (m+1)m\ldots(m+1-(k-1)) - m(m-1)\ldots(m-(k-1))\\
&= ((m+1)-(m-(k-1)))[m(m-1)\ldots(m-(k-2))]\\
&= km(m-1)\ldots(m-(k-2)).
\end{aligned}$$

Now, $\Delta^k(m(m-1)\ldots(m-(k-1))) = k!$ for $k \geqq 1$ follows by induction.

LEMMA 2.2. *Let* j *and* k *be positive integers. Then*

$$\sum_{i=1}^{k} i(i+1)\ldots(i+j-1) = k(k+1)\ldots(k+j)/(j+1).$$

PROOF. The proof follows by induction on k.

LEMMA 2.3. *Let* $m \in I_a$, *let* $\alpha(m)$ *be defined on* I_a *and assume that the following infinite sums converge.*

(i) $$\Delta\left(\sum_{s=m}^{\infty}\alpha(s)\right) = -\alpha(m),\ \Delta\left(\sum_{s=m}^{\infty}(m-1-s)\alpha(s)\right) = \sum_{s=m}^{\infty}\alpha(s),$$

and for $k \geqq 3$,

$$\Delta\left(\sum_{s=m}^{\infty}(m-1-s)\ldots(m-(k-1)-s)\alpha(s)\right)$$
$$= (k-1)\sum_{s=m}^{\infty}(m-1-s)\ldots(m-(k-2)-s)\alpha(s).$$

Thus, for $k \geqq 2$,

$$\Delta^k\left(\sum_{s=m}^{\infty}(m-1-s)\ldots(m-(k-1)-s)\alpha(s)\right) = -(k-1)!\alpha(m).$$

(ii) *For* $m-1 \geqq m_0$,

$$\Delta\left(\sum_{s=m_0}^{m-1}\alpha(s)\right) = \alpha(m).$$

For $m-2 \geqq 0$,

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$$\Delta\left(\sum_{s=m_0}^{m-2} (m-1-s)\alpha(s)\right) = \sum_{s=m_0}^{m-1} \alpha(s).$$

For $k \geqq 3$ *and* $m - k \geqq m_0$,

$$\Delta\left(\sum_{s=m_0}^{m-k} (m-1-s)\dots(m-(k-1)-s)\alpha(s)\right)$$

$$= (k-1)\sum_{s=m_0}^{m-(k-1)} (m-1-s)\dots(m-(k-2)-s)\alpha(s).$$

Thus, for $k \geqq 2$,

$$\Delta^k\left(\sum_{s=m_0}^{m-k} (m-1-s)\dots(m-(k-1)-s)\alpha(s)\right) = (k-1)!\alpha(m).$$

PROOF OF (i) Let $k \geqq 3$.

$$\Delta\left(\sum_{s=m}^{\infty} (m-1-s)\dots(m-(k-1)-s)\alpha(s)\right)$$

$$= \sum_{s=m+1}^{\infty} (m-s)\dots(m-(k-2)-s)\alpha(s)$$

$$- \sum_{s=m}^{\infty} (m-1-s)\dots(m-(k-1)-s)\alpha(s)$$

$$= (k-1)\sum_{s=m+1}^{\infty} ((m-1-s)\dots$$

$$(m-(k-2)-s)\alpha(s)) - (-1)\dots(-(k-1))\alpha(m)$$

$$= (k-1)\sum_{s=m}^{\infty} (m-1-s)\dots(m-(k-2)-s)\alpha(s).$$

Now, for $k \geqq 2$,

$$\Delta^k\left(\sum_{s=m}^{\infty} (m-1-s)\dots(m-(k-1)-s)\alpha(s)\right) = -(k-1)!\alpha(m)$$

follows by induction.

LEMMA 2.4. *Let* $\alpha(m)$ *be defined on* I_a, *let* $k \geqq 2$ *and assume*

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$$\sum_{s=a}^{\infty} (s+1)\dots(s+k-1)\alpha(s)$$

converges. Define

(2.1) $$S_0(m;\alpha) = \alpha(m),\ S_1(m;\alpha) = \sum_{s=m}^{\infty} \alpha(s),$$

and

(2.2) $$S_j(m;\alpha) = \sum_{s=m}^{\infty} ((s+1-m)\dots(s+(j-1)-m)/(j-1)!)\alpha(s),$$

$2 \leqq j \leqq k.$

Then, for $1 \leqq j \leqq k$,

(2.3) $$|S_j(m;\alpha)| \leqq 2\delta(m)m^{j-k}/(j-1)!$$

(2.4) $$\delta(m) = \sup_{s \geqq m} \left| \sum_{r=s}^{\infty} (r+1)\dots(r+k-1)\alpha(r) \right|.$$

PROOF. We first note that each sum in (2.1) and (2.2) converges by Abel's test since

$$(s+1-m)\dots(s+(j-1)-m)\alpha(s)$$
$$= (s+1)\dots(s+k-1)\alpha(s)(1-(m/(s+1)))\dots$$
$$(1-(m/s+(j-1)))(s+j)^{-1}\dots(s+(k-1))^{-1}.$$

To obtain (2.3), set

$$U(m) = \sum_{s=m}^{\infty} (s+1)\dots(s+k-1)\alpha(s).$$

Then, for $2 \leqq j \leqq k-1$,

$$\sum_{s=m}^{\infty} (s+1-m)\dots(s+(j-1)-m)\alpha(s)$$

$$= -\sum_{s=m}^{\infty} (1-(m/(s+1)))\dots(1-(m/s+(j-1)))$$
$$\times (s+j)^{-1}\dots(s+k-1)^{-1}\Delta U(s)$$
$$= \sum_{s=m}^{\infty} U(s)\Delta((1-m/s)\dots(1-(m/s+(j-2))))$$
$$\times ((s+j-1)^{-1}\dots(s+k-2)^{-1}).$$

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Thus,

$$\left|\sum_{s=m}^{\infty} (s - m + 1) \ldots (s + (j - 1) - m)\alpha(s)\right|$$

$$\leqq \delta(m)\Big(\left|\sum_{s=m}^{\infty} \Delta(s + j - 1)^{-1} \ldots (s + k - 2)^{-1}\right|$$

$$+ m^{j-k}\left|\sum_{s=m}^{\infty} \Delta(1 - m/s) \ldots (1 - m/(s + j - 2))\right|\Big)$$

$$\leqq 2\delta(m)m^{j-k}.$$

For $j = k$,

$$\sum_{s=m}^{\infty} (s + 1 - m) \ldots (s + (k - 1) - m)\alpha(s)$$

$$= \sum_{s=m}^{\infty} U(s)\Delta((1 - m/s) \ldots (1 - m/(s + (k - 2))))$$

and so,

$$\left|\sum_{s=m}^{\infty} (s + 1 - m) \ldots (s + (k - 1) - m)\alpha(s)\right| \leqq \delta(m) \leqq 2\delta(m)m^{k-k}.$$

For $j = 1$,

$$\sum_{s=m}^{\infty} \alpha(s) = -\sum_{s=m}^{\infty} (s + 1)^{-1} \ldots (s + k - 1)^{-1}\Delta U(s)$$

$$= m^{-1} \ldots (m + k - 2)^{-1}U(m)$$

$$+ \sum_{s=m}^{\infty} U(s)\Delta(s^{-1} \ldots (s + k - 2)^{-1}).$$

Thus, $|\sum_{s=m}^{\infty} \alpha(s)| \leqq 2\delta(m)m^{1-k}$. (2.3) now follows for $1 \leqq j \leqq k$.

REMARK. It can be shown using (2.3) that

$$S_j(m; \alpha) = \sum_{s=m}^{\infty} S_{j-1}(s; \alpha), \quad 1 \leqq j \leqq k. \tag{2.5}$$

3. **Eventual disconjugacy.** For the remainder of the paper, let $a = 0$ for simplicity. Hartman [4, Theorem 5.1] obtained a Pólya criterion for the disconjugacy of linear difference equations. Let m_0 be a nonnegative integer.

(1.2) is disconjugate on I_{m_0} if and only if there exists a fundamental set of solutions $\{u_0, \ldots, u_{n-1}\}$ of (1.2) on I_{m_0} such that

$$W(u_0, \ldots, u_{k-1})(m) = \det\{u_i(m + j)\} > 0, \; m \in I_{m_0}, \tag{3.1}$$

$i, j = 0, \ldots, k - 1$, $1 \leqq k \leqq n$. Note that by properties of determinants and elementary row operations,

$$W(u_0, \ldots, u_{k-1})(m) = \det\{\Delta^j u_i(m)\}, \tag{3.2}$$

$i, j = 0, \ldots, k - 1$, $1 \leqq k \leqq n$.

REMARK. The Pólya criterion can be employed to show that the equation $\Delta^n u = 0$ is disconjugate on I_0. Set

$$v_0 = 1, \; v_1 = m, \; v_k = m(m-1)\ldots(m-(k-1))/k!, \; 2 \leqq k \leqq n. \tag{3.3}$$

By Lemma 2.1 and (3.2), $W(v_0, \ldots, v_{k-1})(m) \equiv 1$ on I_0, $1 \leqq k \leqq n$.

We now follow the lead of Trench [6], [7] and show that under suitable summability conditions on the coefficients $\alpha_j(m)$ in (1.2), there exists a fundamental set of solutions $\{u_0, \ldots, u_{n-1}\}$ of (1.2) on I_0 such that

$$\Delta^j u_i(m) = \begin{cases} \Delta^j v_i(m)(1 + o(1)), & 0 \leqq j \leqq i, \\ o(m^{j-i}), & i + 1 \leqq j \leqq n - 1, \end{cases} \tag{3.4}$$

where o denotes behaviour as $m \to \infty$ and $v_i(m)$, $0 \leqq i \leqq n - 1$ are given by (3.3). It will then follow that

$$W(u_0, \ldots, u_{k-1})(m) = W(v_0, \ldots, v_{k-1})(m)(1 + o(1)), \tag{3.5}$$

$1 \leqq k \leqq n$, and hence, (1.2) is eventually disconjugate.

THEOREM 3.1. *Suppose the sums* $\sum^\infty m^{k-1}\alpha_k(m)$, $1 \leqq k \leqq n$, *are finite and* $\sum^\infty |S_{k-1}(m; \alpha_k)| < \infty$, $1 \leqq k \leqq n$. *Then* (1.2) *is eventually disconjugate.*

PROOF. First note that if $\sum^\infty m^{k-1}\alpha_k(m)$ converges, then $\sum^\infty (m + 1) \ldots (m + k + 1)\alpha_k(m)$ converges by Abel's test.

Let ℓ be a fixed integer, $0 \leqq \ell \leqq n - 1$. Let m_0 be a nonnegative integer and define the Banach space

$$B_\ell(m_0) = \{u : I_{m_0} \to IR \text{ such that } \Delta^i u(m) = 0(m^{\ell - i}), \; 0 \leqq i \leqq n - 1\},$$

with norm

$$||u||_\ell = \sup_{m \geqq m_0} \left\{ \sum_{i=0}^{n-1} m^{i-\ell} |\Delta^i u(m)| \right\}. \tag{3.6}$$

In this paper, 0 also denotes behavior as $m \to \infty$.

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Define $Q(m) = \sum_{j=1}^{n} \alpha_j(m)\Delta^{n-j}u(m)$. For $\ell = 0$, define the transformation

$$(3.7)\quad T_0u(m) = v_0(m) + \sum_{s=m}^{\infty} ((m-1-s)\dots(m-(n-1)-s)/(n-1)!)Qu(s)$$

and for $1 \leqq \ell \leqq n-1$, define the transformations

$$(3.8)\quad T_\ell u(m) = v_\ell(m) + \sum_{s=m_0}^{m-\ell} ((m-1-s)\dots(m-(\ell-1)-s)/(\ell-1)!) \times \left(\sum_{r=s}^{\infty} ((s-1-r)\dots(s-(n-\ell-1)-r)/(n-\ell-1)!\right)Qu(s).$$

It follows from Lemma 2.3 that $\Delta^n T_\ell u(m) = -Qu(m)$; thus if u_ℓ is a fixed point of T_ℓ then $u_\ell(m)$ is a solution of (1.2).

The remainder of the proof is to show that each transformation T_ℓ, $0 \leqq \ell \leqq n-1$, has a fixed point $u_\ell \in B_\ell(m_0)$ and the set$\{u_0, \dots, u_{n-1}\}$ satisfies (3.4) and thus, (3.5). We first show that for m_0 sufficiently large each T_ℓ maps $B_\ell(m_0)$ into itself and is a contraction map. Thus, each T_ℓ will have a unique fixed point u_ℓ.

For $0 \leqq \ell \leqq n-2$, define

$$J_\ell(m; u) = \sum_{s=m}^{\infty} (s+1)\dots(s+n-\ell-1)Qu(s)$$

and define

$$J_{n-1}(m; u) = \sum_{s=m}^{\infty} Qu(s).$$

Fix $\ell \in \{0, \dots, n-2\}$, let $m_0 \geqq 0$ and let $u \in B_\ell(m_0)$. Applying (2.5) and repeated summation by parts, we have that for each $2 \leqq j \leqq n$,

$$\sum_{s=m}^{\infty} (s+1)\dots(s+n-\ell-1)\alpha_j(s)\Delta^{n-j}u(s)$$

$$= \sum_{i=1}^{j-1} S_i(m+i-1; \alpha_j)\Delta^{i-1}((s+1)\dots(s+n-\ell-1)\Delta^{n-j}u(s))$$

$$+ \sum_{s=m}^{\infty} S_{j-1}(s+j-1; \alpha_j)\Delta^{j-1}((s+1)\dots(s+n-\ell-1)\Delta^{n-j}u(s)).$$

Thus,

(3.9) $J_\ell(m; u)$

$$= \sum_{j=2}^{n} \left(\sum_{i=1}^{j-1} S_i(m + i - 1; \alpha_j)\Delta^{i-1}((s + 1)\dots(s + n - \ell - 1)\Delta^{n-j}u(s)) \right)$$

$$+ \sum_{j=1}^{n} \left(\sum_{s=m}^{\infty} S_{j-1}(s + j - 1; \alpha_j) \times \Delta^{j-1}((s + 1)\dots(s + n - \ell - 1)\Delta^{n-j}u(s)) \right).$$

Similarly,

$$(3.10)\quad J_{n-1}(m; u) = \sum_{j=2}^{n} \left(\sum_{i=1}^{j-1} S_i(m + i - 1; \alpha_j)\Delta^{n-j+i-1}u(s) \right) + \sum_{j=1}^{n} \left(\sum_{s=m}^{\infty} S_{j-1}(s + j - 1; \alpha_j)\Delta^{n-1}u(s) \right).$$

Now,

$$\Delta^{i-1}((s + 1)\dots(s + n - \ell - 1)\Delta^{n-j}u(s))$$
$$= (s + 1)\dots(s + n - \ell - 1)\Delta^{n-j+i-1}u(s) + \dots$$
$$+ K_{n-\ell-2}(s + n - \ell - 1)\Delta^{-j+i+\ell+1}u(s + n - \ell - 2)$$
$$+ K_{n-\ell-1}\Delta^{-j+i+\ell}u(s + n - \ell - 1),$$

where each K_ν is constant and $K_\nu = 0$ if $\nu \geqq i$. Since $u \in B_\ell(m_0)$ with $||u||_\ell$ given by (3.6), we have that

$$(3.11)\quad |\Delta^{i-1}((s + 1)\dots(s + n - \ell - 1)\Delta^{n-j}u(s))| \leqq K_{ij}||u||_\ell m^{j-i}$$

for $1 \leqq i \leqq j$, where K_{ij} is a constant depending only on i and j. Also, from (2.3) it follows that

$$(3.12)\quad |S_i(m + i - 1; \alpha_j)| \leqq 2\delta_j(m)m^{i-j}/(i - 1)!,\ 1 \leqq i \leqq j,$$

where $\delta_j(m)$ is given by (2.4) with $\alpha = \alpha_j$. From (3.9), (3.10), (3.11) and (3.12) it follows that

$$(3.13)\quad |J_\ell(m; u)| \leqq \sigma(m)\,||u||_\ell$$

where

$$\sigma(m) = \sum_{j=2}^{n} \left(\sum_{i=1}^{j-1} 2K_{ij}\delta_i(m)/(i - 1)! \right) + \sum_{j=1}^{\infty} K_{jj}\left(\sum_{s=m}^{\infty} S_{j-1}(s + j - 1; \alpha_j) \right).$$

Note that $\sigma(m)$ is nonincreasing and $\lim_{m\to\infty} \sigma(m) = 0$.

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Define

$$\hat{u}_0(m) = \sum_{s=m}^{\infty} (m-1-s)\ldots(m-(n-1)-s)/(n-1)!Qu(s)$$

and for $1 \leqq \ell \leqq n-1$, define

$$\hat{u}_\ell(m) = \sum_{s=m_0}^{m-\ell} (m-1-s)\ldots(m-(\ell-1)-s)/(\ell-1)!$$

$$\times\left(\sum_{r=s}^{\infty}((s-1-r)\ldots(s-(n-\ell-1)-r)/(n-\ell-1)!Qu(r))\right).$$

Applying Lemma 2.3, for $\ell \leqq i \leqq n-1$,

$$\Delta^i\hat{u}_\ell(m) = \sum_{s=m}^{\infty} (m-1-s)\ldots(m-(n-i-1)-s)/(n-i-1)!Qu(s)$$

and so, applying Lemma 2.4 and (3.13), with $\alpha = Qu$,

$$|\Delta^i\hat{u}_\ell(m)| \leqq 2\sigma(m)\,||u||_\ell m^{\ell-i}/(n-i-1)!. \tag{3.14}$$

If $\ell \geqq 1$, note that

$$\hat{u}_\ell(m) = \sum_{s=m_0}^{m-\ell} (m-1-s)\ldots(m-(\ell-1)-s)/(\ell-1)!\Delta^\ell\hat{u}_\ell(s)$$

and so, for $0 \leqq i \leqq \ell-2$,

(3.15) $|\Delta^i\hat{u}_\ell(m)|$

$$= \left|\sum_{s=m_0}^{m-\ell+i} (m-1-s)\ldots(m-(\ell-i-1)-s)/(\ell-i-1)!\Delta^\ell\hat{u}_\ell(s)\right|$$

$$\leqq 2||u||_\ell/(n-\ell-1)!$$

$$\times\left(\sum_{s=m_0}^{m-\ell+i} \sigma(s)(m-1-s)\ldots(m-(\ell-i-1)-s)/(\ell-i-1)!\right)$$

$$\leqq K(2||u||_\ell\sigma(m_0)/(n-\ell-1)!(\ell-1)!)m^{\ell-i} \tag{3.16}$$

where K depends on m_0. (3.14) and (3.16) show that T_ℓ maps $B_\ell(m_0)$ into $B_\ell(m_0)$. Also, since $\sigma(m) \to 0$ as $m \to \infty$, it is readily shown using (3.14) and (3.16) that T_ℓ is a contraction map and thus, T_ℓ has a unique fixed point u_ℓ.

The proof is now complete if we show $\{u_0, \ldots, u_{n-1}\}$ satisfies (3.4), where

$$u_0(m) = v_0(m) + \sum_{s=m}^{\infty} (m - 1 - s) \dots (m - (n - 1) - s)/(n - 1)!Qu_0(s)$$

and

$$u_\ell(m) = v_\ell(m) + \sum_{s=m_0}^{m-\ell} ((m - 1 - s) \dots (m - (\ell - 1) - s)/(\ell - 1)!)$$

$$\times \left(\sum_{r=s}^{\infty} ((s - 1 - r) \dots (s - (n - \ell - 1) - r)/(n - \ell - 1)!)Qu_\ell(r) \right)$$

for $1 \leqq \ell \leqq n - 1$. For $\ell \leqq i \leqq n - 1$, (3.14) readily implies that

$$m^{i-\ell}|\Delta^i u_\ell(m)| \to 0 \text{ as } m \to \infty.$$

For $0 \leqq i \leqq \ell - 1$, (3.15) implies

$$m^{i-\ell}|\Delta^i u_\ell(m)| \leqq (2\|u_\ell\|_\ell/(n - \ell - 1)!(\ell - i - 1)!)m^{-1} \sum_{s=m_0}^{m-\ell-i} \sigma(s) \to 0$$

as $m \to \infty$ since $(\sum^m \sigma(s))/m \to 0$ as $m \to \infty$. Thus, the proof is complete.

COROLLARY 3.2. *If* $\sum^\infty m^{k-1}|\alpha_k(m)| < \infty$, $1 \leqq k \leqq n$, (1.2) *is eventually disconjugate.*

PROOF. $\sum^\infty m^{k-1}|\alpha_k(m)| < \infty$, $1 \leqq k \leqq n$, implies that $\sum^\infty |S_{k-1}(m; \alpha_k)| < \infty$, $1 \leqq k \leqq n$. We point out that the corollary can be proved directly by showing the operators T_ℓ are contraction maps. However, the inequalities employed are straight forward and the lemmas from section 2 are unnecessary.

4. **Eventual right disfocality.** The techniques of section 3 are readily adapted to obtain the following theorem.

THEOREM 4.1. *Assume the hypotheses of Theorem* 3.1. *Then* (1.2) *is eventually right disfocal.*

We outline the proof here. In [1], we obtained a Pólya type criterion for right disfocality. Let $w_0, \dots, w_{n-1}$ be sequences defined on I_{m_0}. Let $k \in \{1, \dots, n\}$ and let $1 \leqq i_0 < \dots < i_{k-1} \leqq n$. Define

$$D(i_0, \dots, i_{k-1})(m) = \det\{\Delta^{i_j - 1} w_\ell(m)\},\ j, \ell = 0, \dots, k - 1.$$

Note that $D(1, \dots, k)(m) = W(w_0, \dots, w_{k-1})(m)$. Then (1.2) is right disfocal on I_{m_0} if and only if there exists a fundamental set of solutions $\{w_0, \dots, w_{n-1}\}$ of (1.2) on I_{m_0} such that

(4.1) $$D(i, \dots, i - k - 1)(m) > 0,$$

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$$1 \leqq i \leqq n - k + 1,\ 1 \leqq k \leqq n,\ m \in I_{m_0}.$$

Let $(u_0, \ldots, u^{n-1}\}$ be the fundamental set of solutions constructed in section 3. By Theorem (5.1) [4], (3.1) implies

$$W(u_i, \ldots, u_{i+k-1})(m) > 0,\ 0 \leqq i \leqq n - k,\ 1 \leqq k \leqq n,$$

for m sufficiently large. Set $w_j = (-1)^j u_{n-j-1}$, $0 \leqq j \leqq n - 1$. Then it follows from (3.4) that $\{w_0, \ldots, w_{n-1}\}$ satisfies (4.1) and (1.2) is eventually right disfocal.

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Department of Mathematics
University of Dayton
Dayton, Ohio 45469