Glasgow Math. J. **48** (2006) 65–73. © 2006 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002879. Printed in the United Kingdom

GENERALIZED DEGREE THEORY FOR SEMILINEAR OPERATOR EQUATIONS

YUQING CHEN
Department of Mathematics, Foshan University, Foshan, Guangdong 528000, P. R. China
e-mail: yqchen@foshan.net

and DONAL O'REGAN
Department of Mathematics, National University of Ireland, Galway, Ireland
e-mail: donal.oregan@nuigalway.ie

(Received 17 May, 2005; accepted 14 October, 2005)

Abstract. In this paper, we construct a generalized degree theory of Browder-Petryshyn or Petryshyn type for a class of semilinear operator equations involving a Fredholm type mapping with infinite dimensional kernel.

2000 *Mathematics Subject Classification.* Primary 47H05, 47H11; Secondary 35L05, 55M25.

1. Introduction and Preliminaries. In this paper, we study the following semilinear operator equation

$$Lx - Nx = f, x \in X, f \in Y,$$

where X, Y are Banach spaces, and $L : D(L) \subset X \to Y$ is a linear Fredholm type mapping with $\dim(\mathrm{Ker(L)}) = +\infty$, and $N : \overline{\Omega} \cap D(L) \to Y$ is a nonlinear mapping. This type of map equation has been extensively studied by Mawhin, Petryshyn and others for the case when $\dim(\mathrm{Ker(L)}) < +\infty$, see **[8]**, **[12]** for references. By imposing some suitable conditions on X, L and Y, we can apply Browder-Petryshyn's degree and Petryshyn's generalized degree theory to study such an equation. A generalized degree theory for $L - N$ is defined in three ways by following Browder-Petryshyn and Petryshyn's method or combining them with Mawhin's method. First we recall some definitions.

Definition 1.1. **[12]** Let X be a real separable Banach space, $(X_n)_{n=1}^{\infty}$ a sequence of finite dimensional subspaces of X, and $P_n : X \to X_n$ a projecton for $n = 1, 2, \dots$. If $P_n x \to x$ as $n \to \infty$, for all $x \in X$, then $\{X_n, P_n\}$ is called a *projectionally complete scheme for* X.

Definition 1.2. **[12]** Let X, Y be two real separable Banach spaces, $(X_n \subset X)_{n=1}^{\infty}, (Y_n \subset Y)_{n=1}^{\infty}$ two sequences of oriented finite dimensional subspaces such that $\dim(X_n) = \dim(Y_n)$, and let $Q_n : Y \to Y_n$ be a linear mapping of Y onto Y_n for $n = 1, 2 \dots$. If $\lim_{n\to\infty} d(x, X_n) = 0$, and (Q_n) is uniformly bounded, then we call $\Gamma_A = \{X_n, Y_n, Q_n\}$ an admissible scheme for (X, Y); if Q_n is the projection such that $Q_n y \to y$ for all $y \in Y$, then we say $\Gamma_0 = \{X_n, Y_n, Q_n\}$ is *a projectionally complete scheme for* (X, Y).

DEFINITION 1.3. **[12]** Let X, Y be real separable Banach spaces with a projectionally complete scheme $\Gamma_0 = \{X_n, Y_n, Q_n\}$, $D \subset X$, and $T : D \to Y$. Suppose that the following conditions are satisfied:

(1) $Q_n T : D \cap X_n \to Y_n$ is continuous for $n = 1, 2 \ldots$;

(2) for any bounded sequence $(x_{n_j} \in X_{n_j} \cap D)_{j=1}^{\infty}$ such that $Q_{n_j} T x_{n_j} \to y$, there exists a subsequence (x'_{n_j}) such that $x'_{n_j} \to x \in D$ and $Tx = y$;

then T is said to be *A-proper with respect to* Γ_0; if (2) is replaced by the following

(3) for any bounded sequence $(x_{n_j} \in X_{n_j} \cap D)_{j=1}^{\infty}$ such that $Q_{n_j} T x_{n_j} \to y$, there exists $x \in D$ such that $Tx = y$

then T is said to be *pseudo A-proper with respect to* Γ_0.

DEFINITION 1.4. Let X, Y be two real Banach spaces, $L : D(L) \subseteq X \to Y$ is a linear mapping, and we say L is a *Fredholm mapping of index zero type* if

(1) $\mathrm{Ker}(L) = \{x \in X : Lx = 0\}$, $\mathrm{Im}(L) = \{Lx : x \in D(L)\}$ are closed in H;

(2) $X = \mathrm{Ker}(L) \oplus X_1$ for some subspace X_1 of X, $Y = Y_1 \oplus \mathrm{Im}(L)$ for some subspace Y_1 of Y;

(3) $\mathrm{Ker}(L)$ is linearly homeomorphic to $\mathrm{Coker}(L) = Y/\mathrm{Im}(L)$.

REMARK 1. Obviously, if X is linearly homeomorphic to Y, $L = 0$ is a Fredholm mapping of index zero type, but not a Fredholm mapping of index zero. If L is a Fredholm mapping of index zero, then $\dim(\mathrm{Ker}(L)) = \dim(\mathrm{Coker}(L)) < +\infty$, and so $\mathrm{Ker}(L)$ is linearly homeomorphic to $\mathrm{Coker}(L)$; thus L is a Fredholm mapping of index zero type.

Now, assume that $L : D(L) \subset X \to Y$ is a Fredholm mapping of index zero type. Then there exist linear projections $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$ and $\mathrm{Im}(Q) = Y_1$.

Obviously, the restriction of L_P of L to $D(L) \cap \mathrm{Ker}(P)$ is one to one and onto $\mathrm{Im}(L)$, so its inverse $K_P : \mathrm{Im}(L) \to D(L) \cap \mathrm{Ker}(P)$ is defined. Let $J : \mathrm{Ker}(L) \to Y_1$ be a linear homeomorphism, and set $K_{PQ} = K_P(I - Q)$.

PROPOSITION 1.5. *$L + \lambda JP : X \to Y$ is a bijective mapping for each $\lambda \neq 0$.*

Proof. For each $\lambda \neq 0$, if $Lx + \lambda JPx = 0$, then $JPx = 0$, $Lx = 0$, so $x \in \mathrm{Ker}(L)$, thus $x = 0$. On the other hand, for $y = y_1 + y_2 \in Y$, $y_1 \in Y_1, y_2 \in \mathrm{Im}(L)$, put $x = \lambda^{-1} J^{-1} y_1 + K_P y_2$, then $Lx + \lambda JPx = y$. Therefore $L + \lambda JP$ is bijective. □

PROPOSITION 1.6. *Let X, Y be real separable Banach spaces, and (Y_n, Q_n) a projectionally complete scheme for Y, and let $L : D(L) \subset X \to Y$ be a Fredholm mapping of zero index type. Then for each $\lambda \neq 0$, there exists a projectionally complete scheme $\Gamma_{\lambda,L}$ for (X, Y).*

Proof. For each $\lambda \neq 0$, put $K_\lambda = L + \lambda JP$. By Proposition 1.5, K_λ is bijective. Set $X_n = K_\lambda^{-1} Y_n$ for $n = 1, 2 \ldots$. Obviously, we have $\dim(X_n) = \dim(Y_n)$, and $X = \overline{\cup_{n=1}^{\infty} X_n}$. Thus $\Gamma_L = \{X_n, Y_n, Q_n\}$ is a projectionally complete scheme for (X, Y). □

Petryshyn showed that if L is a Fredhom mapping of index zero, then L is A-proper with respect to $\Gamma_{1,L}$, see **[12]**. Here we have a similar result.

PROPOSITION 1.7. *Let $L : D(L) \subset X \to Y$ be a Fredholm mapping of zero index type, and assume that X is reflexive. If $G \subset X$ is bounded closed convex, then $L : G \cap D(L) \to Y$ is pseudo A-proper with respect to $\Gamma_{\lambda,L}$ for each $\lambda \neq 0$.*

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

Proof. For any sequence $x_{n_k} \in G \cap D(L) \cap X_{n_k}$ with $Q_{n_k} L x_{n_k} \to y$, we may assume that $x_{n_k} \rightharpoonup x_0 \in G$ by taking a subsequence.

Notice that $Q_{n_k}(Lx_{n_k} + \lambda JPx_{n_k}) = Lx_{n_k} + \lambda JPx_{n_k}$, and $JPx_{n_k} \rightharpoonup JPx_0$, so we have

$$x_{n_k} = (L + JP)^{-1}(Q_{n_k}(Lx_{n_k} + JPx_{n_k})) \rightharpoonup (L + JP)^{-1}(y + JPx_0) = x_0.$$

Thus $x_0 \in D(L)$, and $Lx_0 = y$, so therefore L is pseudo A-proper with respect to $\Gamma_{\lambda,L}$. □

DEFINITION 1.8. Let X be a real separable Banach space and $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for X, Y a real Banach space, $L : D(L) \subset X \to Y$ a Fredholm mapping of zero index type, and let $N : D \subset X \to Y$ be a mapping.

(1) If $I - P - (J^{-1}Q + K_{PQ})N$ is A-proper with respect to Γ_0, then we say N is *L-A-proper with respect to* Γ_0;

(2) If $I - P - (J^{-1}Q + K_{PQ})N$ is pseudo A-proper with respect to Γ_0, then we say N is *pseudo L-A-proper with respect to* Γ_0;

(3) A family of mappings $H(t, x) : [0, 1] \times D \to Y$ is called a *homotopy of L-A-proper mappings with respect to* Γ_0 if $H(t, \cdot)$ is an L-A-proper mapping with respect to Γ_0 for each $t \in [0, 1]$.

PROPOSITION 1.9. *Let $L : D(L) \subseteq X \to Y$ be a linear mapping with* $\mathrm{Ker}(L) = \{0\}$, *and* $\mathrm{Im}(L) = Y$. *Then the following conclusions hold*

(1) *if $\Gamma_0 = (X_n, P_n)$ is a projectionally complete scheme for X, then* 0 *is L-A-proper with respect to* Γ_0;

(2) *if (Y_n, Q_n) is a projectionally complete scheme for Y, and L^{-1} is continuous, then L is A-proper with respect to $\Gamma_{1,L}$, where $\Gamma_{1,L}$ is constructed as in Proposition* 1.6.

Proof. (1) We have $P = 0$, and $Q = 0$, and the identity mapping $I : X \to X$ is obviously A-proper with respect to Γ_0. Thus 0 is L-A-proper with respect to Γ_0.

(2) Since $\mathrm{Ker}(L) = \{0\}$, the mapping K in the proof of Proposition 1.6 is just the mapping L, so $X_n = L^{-1}Y_n$. If $x_{n_k} \in X_{n_k}$ such that $Q_{n_k} L x_{n_k} \to y$, then $Lx_{n_k} = Q_{n_k} L x_{n_k} \to y$. Therefore we have $x_{n_k} \to L^{-1}y$. The conclusion holds. □

PROPOSITION 1.10. *Let $L : D(L) \subset X \to Y$ be a Fredholm mapping of zero index type, $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for X, $G \subset X$ a bounded closed convex subset, and $T : G \to Y$ a weakly continuous mapping, with X reflexive. Then T is L-pseudo A-proper with respect to* Γ_0.

Proof. For any subsequence $x_{n_k} \in X_{n_k}$ such that $P_{n_k}(I - P - J^{-1}QT - K_{PQ}T)x_{n_k} \to y$, we may assume that $x_{n_k} \rightharpoonup x_0 \in G$ by taking a subsequence, and so $(I - P)x_{n_k} \rightharpoonup x_0$, $J^{-1}QTx_{n_k} \rightharpoonup J^{-1}QTx_0$, and $K_{PQ}Tx_{n_k} \rightharpoonup K_{PQ}Tx_0$. Consequently, $(I - P - J^{-1}QT - K_{PQ}T)x_0 = y$, so T is L-pseudo A-proper with respect to Γ_0. □

PROPOSITION 1.11. *Let X, Y be real separable Banach spaces, and (Y_n, Q_n) a projectionally complete scheme for Y. Let $L : D(L) \subset X \to Y$ be a Fredholm mapping of zero index type, $G \subset X$ a bounded closed subset, and $N : G \to Y$ a continuous compact mapping. Then $L + \lambda JP - N$ is A-proper with respect to $\Gamma_{\lambda,L}$ for each $\lambda > 0$.*

Proof. For any sequence $x_{n_k} \in G \cap D(L) \cap X_{n_k}$ with $Q_{n_k}(L + \lambda JP - N)x_{n_k} \to y$, in view of the compactness of N, we may assume that $Nx_{n_k} \to y_0 \in Y$ by taking a subsequence.

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

Notice that $Q_{n_k}(Lx_{n_k} + \lambda JPx_{n_k}) = Lx_{n_k} + \lambda JPx_{n_k}$, so we have

$$x_{n_k} = (L + JP)^{-1}[Q_{n_k}(L + \lambda JP - N)x_{n_k} + Q_{n_k}Nx_{n_k}] \to (L + \lambda JP)^{-1}(y + y_0) = x_0.$$

Thus $x_0 \in D(L)$, and $Nx_0 = y_0$, $(L + \lambda JP - N)x_0 = y$, and therefore L is A-proper with respect to $\Gamma_{\lambda,L}$. □

2. Generalized degree theory for L-N**.** In this section, X, Y are real separable Banach spaces, $L : D(L) \subseteq X \to Y$ is a Fredholm mapping of index zero type with $D(L)$ dense in X, and $N : \overline{\Omega} \subset X \to Y$ is a nonlinear mapping, and we consider the semilinear operator equation $Lx - Nx = 0$. We will apply Browder-Petryshyn and Petryshyn's generalized degree theory to study such an equation in three different ways.

LEMMA 2.1. *Let* $L : D(L) \subseteq X \to Y$ *be a Fredholm mapping of index zero type, and* $\Omega \subset X$ *an open bounded subset, and let* $N : \overline{\Omega} \to Y$ *be a mapping. If* $0 \notin (L - N)(\partial\Omega \cap D(L))$*, then* $0 \notin [I - P - (J^{-1}Q + K_{PQ})N](\partial\Omega)$.

Proof. Suppose the contrary i.e. suppose there exists $x_0 \in \partial\Omega$ such that $0 \in x_0 - Px_0 - (J^{-1}Q + K_{PQ})Nx_0$. Since $J^{-1}QTx_0 \in \mathrm{Ker}(L) = \mathrm{Im}(P)$, $x_0 - Px_0 \in \mathrm{Ker}(P)$, and $K_{pQ}Tx_0 \in D(L) \cap \mathrm{Ker}(P)$, we must have

$$J^{-1}QNx_0 = 0, \quad x_0 - Px_0 - K_{PQ}Nx_0 = 0.$$

Therefore we have

$$QNx_0 = 0, \quad x_0 - Px_0 - K_PNx_0 = 0, \text{ i.e. } Lx_0 - Nx_0 = 0,$$

which is a contradiction to $0 \notin (L - N)(\partial\Omega \cap D(L))$. □

Now, let $L : D(L) \subseteq X \to Y$ be a Fredholm mapping of index zero type, $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for X and $\Omega \subset X$ an open bounded subset, and let $N : \overline{\Omega} \to Y$ be an L-A-proper mapping with respect to Γ_0. Suppose $0 \notin (L - T)(\partial\Omega \cap D(L))$. By Lemma 2.1, $0 \notin [I - P - (J^{-1}Q + K_{PQ})N](\partial\Omega)$. Since $I - P - (J^{-1}Q + K_{PQ})N$ is an A-proper mapping with respect to Γ_0, the generalized degree $\deg(I - P - (J^{-1}Q + K_{PQ})N, \Omega, 0)$ is well defined, see [**3**], and we define

$$\deg_{\Gamma_0,J}(L - N, \Omega, 0) = \deg(I - P - (J^{-1}Q + K_{PQ})N, \Omega, 0), \tag{2.1}$$

which is called the generalized coincidence degree of L and N on Ω.

THEOREM 2.2. *The generalized coincidence degree of L and N defined by (2.1) on* Ω *has the following properties.*

(1) *If* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *such that* 0 *does not belong to* $(L - N)(D(L) \cap \overline{\Omega \setminus (\Omega_1 \cup \Omega_2)})$*, then*

$$\deg_{\Gamma_0,J}(L - N, \Omega, 0) \subseteq \deg_{\Gamma_0,J}(L - N, \Omega_1) + \deg_{\Gamma_0,J}(L - N, \Omega_2, 0).$$

(2) *If* $H(t, x) : [0, 1] \times \overline{\Omega} \to Y$ *is a homotopy of L-A-proper mappings with respect to* Γ_0*, and if* $0 \neq Lx - H(t, x)$ *for all* $(t, x) \in [0, 1] \times \partial\Omega \cap D(L)$*, then* $\deg_{\Gamma_0,J}(L - H(t, \cdot), \Omega, 0)$ *does not depend on* $t \in [0, 1]$.

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

(3) *If* $\deg_{\Gamma_0}(L-N,\Omega,0)\neq\{0\}$, *then* $0\in(L-N)(D(L)\cap\Omega)$.

(4) *If* $L:D(L)\subseteq X\to Y$ *is a linear mapping such that* $L^{-1}:Y\to D(L)$ *is continuous, then* $\deg_{\Gamma_0,J}(L,\Omega,0)=\{1\}$ *if* $0\in\Omega$.

(5) *If* Ω *is a symmetric neighbourhood of* 0, *and* $N:\overline{\Omega}\to Y$ *is an odd L-A-proper mapping with respect to* Γ_0 *with* $0\notin(L-N)(\partial\Omega\cap D(L))$, *then* $\deg_{\Gamma_0,J}(L-N,\Omega,0)$ *does not contain even numbers.*

Proof. (1)–(3) follow directly from the definition and the properties of generalized degree.

(4) Since $\mathrm{Ker}(L)=\{0\}$, $P=0$, $Q=0$, the zero mapping is L-A-proper with respect to Γ_0. Thus $\deg_{\Gamma_0,J}(L,\Omega,0)=\deg(I,\Omega,0)=\{1\}$.

(5) Since N is odd, the mapping $I-P-(J^{-1}Q+K_{PQ})N$ is odd. Thus $\deg(I-P-(J^{-1}Q+K_{PQ})N,\Omega,0)$ does not contain even numbers, and the conclusion follows by definition. □

Corollary 2.3. *Let* $L:D(L)\subseteq X\to Y$ *be a linear mapping such that* $L^{-1}:Y\to D(L)$ *is continuous,* $\Omega\subset X$ *an open bounded subset with* $0\in\Omega$, *and* $N:\overline{\Omega}\to Y$ *a mapping such that* $\{L-tN\}_{t\in[0,1]}$ *is a homotopy of L-A-proper mappings with respect to* Γ_0. *If* $Lx\notin tNx$ *for all* $(t,x)\in[0,1]\times\partial\Omega\cap D(L)$, *then* $\deg(L-N,\Omega,0)=1$.

In the following, let $L:D(L)\subset X\to Y$ be a densely defined Fredholm mapping of zero index type. We assume that $\Gamma_0=(Y_n,Q_n)$ is a projectionally complete scheme for Y, $\Gamma_{\lambda,L}$ is as defined in Proposition 1.6, and $L+\lambda JP-N$ is an A-proper map with respect to $\Gamma_{\lambda,L}$ for $\lambda\in(0,\lambda_0)$, where $\lambda_0>0$ is a constant. Suppose that $0\notin\overline{(L-N)(D(L)\cap\partial\Omega)}$. Then there exists $\lambda_1<\lambda_0$ such that $0\notin(L+\lambda JP-N)(D(L)\cap\partial\Omega)$ for all $\lambda\in(0,\lambda_1)$. We define a generalized degree

$$\deg(L-N,\Omega,0)=\cap_{0<\lambda<\lambda_1}\cup_{0<\epsilon\leq\lambda}\deg(L+\epsilon JP-N,\Omega,0),\tag{2.2}$$

where $\deg(L+\epsilon JP-N,\Omega,0)$ is the generalized degree for A-proper maps with respect to $\Gamma_{\lambda,L}$, see **[12]**.

Notice that if $0\notin(L+\lambda JP-N)(D(L)\cap\partial\Omega)$ for all $\lambda\in(0,\lambda_2)$, then it is easy to check that

$$\cap_{0<\lambda<\lambda_1}\cup_{0<\epsilon\leq\lambda}\deg(L+\epsilon JP-N,\Omega,0)=\cap_{0<\lambda<\lambda_2}\cup_{0<\epsilon\leq\lambda}\deg(L+\epsilon JP-N,\Omega,0).$$

Thus (2.2) is well defined.

Remark. A degree theory for uniform limits of A-proper maps has been defined by P. M. Fitzpatrick **[5]**. Since $\Gamma_{\lambda,L}$ depends on λ, and $L+\lambda JP-N$ is an A-proper map with respect to $\Gamma_{\lambda,L}$, $L-N$ is slightly different to the uniform limits of A-proper maps. Of course, a slight generalization of the ideas in **[5]** could be applied here also.

Theorem 2.4. *The generalized degree defined by (2.2) has the following properties.*

(1) *If* Ω_1 *and* Ω_2 *are two open subsets of* Ω *such that* $\Omega_1\cap\Omega_2=\emptyset$, *and* $0\notin\overline{(L-N)(D(L)\cap\overline{\Omega\setminus(\Omega_1\cup\Omega_2)})}$, *then*

$$\deg(L-N,\Omega,0)\subseteq\deg(L-N,\Omega_1)+\deg(L-N,\Omega_2,0).$$

(2) *If* $H(t,x):[0,1]\times\overline{\Omega}\to Y$ *satisfies* $0\notin\overline{\cup_{t\in[0,1]}(L-H(t,\cdot))(D(L)\cap\partial\Omega)}$, *and* $\{L+\lambda JP-H(t,\cdot)\}_{t\in[0,1]}$ *is a homotopy of A-proper maps with respect to* $\Gamma_{\lambda,L}$ *for each*

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

$\lambda \in (0, \lambda_0)$, *where* $\lambda_0 > 0$ *is a constant, then* $\deg(L - H(t, \cdot), \Omega, 0)$ *does not depend on* $t \in [0, 1]$.

(3) *If* $\deg_{\Gamma_0}(L - N, \Omega, 0) \neq \{0\}$, *then* $0 \in \overline{(L - N)(D(L) \cap \Omega)}$.

(4) *If* Ω *is a symmetric neighbourhood of* 0, *and* $N : \overline{\Omega} \to Y$ *is an odd mapping such that* $L + \lambda JP - N$ *is A-proper with respect to* $\Gamma_{\lambda,L}$ *for each* $\lambda \in (0, \lambda_0)$, *where* $\lambda_0 > 0$ *is a constant, and* $0 \notin \overline{(L - N)(\partial\Omega \cap D(L))}$, *then* $\deg(L - N, \Omega, 0)$ *does not contain even numbers.*

(5) $\deg(L, \Omega, 0) \subseteq \{\pm 1\}$ *if* $0 \in \Omega$.

Proof. (1). By assumption, there exists $\lambda_0 > 0$ such that

$$0 \notin (L + \lambda JP - N)(D(L) \cap \overline{\Omega \setminus (\Omega_1 \cup \Omega_2)})$$

for all $\lambda \in (0, \lambda_0)$. If $m \in \deg(L - N, \Omega, 0)$, then there exist $\lambda_j \to 0^+$, $\lambda_j < \lambda_0$, $j = 1, 2, \ldots$, such that $m \in \deg(L + \lambda_j JP - N, \Omega, 0)$. By Theorem 2.1 of [**11**], we have

$$\deg(L + \lambda_j JP - N, \Omega, 0) \subseteq \deg(L + \lambda_j JP - N, \Omega_1, 0) + \deg(L + \lambda_j JP - N, \Omega_2, 0)$$

for $j = 1, 2, \ldots$. Thus (1) follows from (2.2).

(2). Since $0 \notin \overline{\cup_{t\in[0,1]}(L - H(t, \cdot))(D(L) \cap \partial\Omega)}$, there exists $\lambda_1 > 0$ such that $0 \notin \cup_{t\in[0,1]}(L + \lambda JP - H(t, \cdot))(\partial\Omega \cap D(L))$ for $\lambda \in (0, \lambda_1)$. By Theorem 2.1 of [**11**], $\deg(L + \lambda JP - H(t, \cdot), \Omega, 0)$ does not depend on $t \in [0, 1]$ for $\lambda \in (0, \min\{\lambda_0, \lambda_1\})$. So (2) follows from (2.2).

(3). If $\deg_{\Gamma_0}(L - N, \Omega, 0) \neq \{0\}$, then there exists $0 \neq m \in \deg_{\Gamma_0}(L - N, \Omega, 0)$, so there exists $\lambda_j \to 0^+$ such that $m \in \deg(L + \lambda_j JP - N, \Omega, 0)$. Therefore $(L + \lambda_j JP - N)x$ has a solution in $\Omega \cap D(L)$, $j = 1, 2, \ldots$. By letting $j \to \infty$, we obtain $0 \in \overline{(L - N)(D(L) \cap \Omega)}$.

(4). We leave the proof to the reader.

(5). $L + \lambda JP$ is A-proper with respect to $\Gamma_{\lambda,L}$, and $0 \notin (L + \lambda JP)(\partial\Omega \cap D(L))$ for all $\lambda > 0$. Since $L + \lambda JP$ is bijective, $\deg(L + \lambda JP, \Omega, 0) \subseteq \{\pm 1\}$ for all $\lambda > 0$. Thus we have

$$\deg(L - N, \Omega, 0) \subseteq \{\pm 1\}. \qquad \square$$

THEOREM 2.5. *Let* X, Y *be real separable Banach spaces, and* (Y_n, Q_n) *a projectionally complete scheme for* Y, *and let* $L : D(L) \subset X \to Y$ *be a Fredholm mapping of zero index type,* $0 \in \Omega \subset X$ *a bounded subset, and* $N : \overline{\Omega} \to Y$ *a continuous compact mapping. Suppose the following conditions are satisfied*

(1) $0 \notin \overline{(L - N)(\partial\Omega \cap D(L))}$;

(2) $0 \notin \overline{QN(\partial\Omega \cap D(L))}$.

Then $\deg(L - N, \Omega, 0) = \deg(L - QN, \Omega, 0)$.

Proof. For each $\lambda \in (0, \lambda_0)$, a similar proof to Proposition 1.11 shows that $\{L + \lambda JP - tN - (1 - t)QN\}_{t\in[0,1]}$ is a homotopy of A-proper maps with respect to $\Gamma_{\lambda,L}$.

Now we claim that $0 \notin \overline{\cup_{t\in[0,1]}(L - tN - (1 - t)QN)(D(L) \cap \partial\Omega)}$.

If this is not true, then there exist $t_j \in [0, 1]$ with $t_j \to t_0$, $x_j \in \partial\Omega \cap D(L)$, such that $Lx_j - t_jNx_j - (1 - t_j)QNx_j \to 0$.

Case (1): if $t_0 = 1$, then $Lx_j - Nx_j \to 0$, which is a contradiction to assumption (1).

Case (2): if $t_0 \neq 1$, then $QLx_j - QNx_j \to 0$, thus we have $QNx_j \to 0$ and $x_j \in D(L)$, which is a contradiction to assumption (2).

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

By (2) of Theorem 2.4, we obtain $\deg(L-N,\Omega,0)=\deg(L-QN,\Omega,0)$. □

Finally, let $L: D(L)\subseteq X\to Y$ be a Fredholm mapping of index zero type, $\Gamma_0=(X_n,P_n)$ a projectionally complete scheme for X, and $\Omega\subset X$ an open bounded subset, and let $N:\overline{\Omega}\to Y$ be a mapping such that $I-(L+\lambda JP)^{-1}(N+\lambda JP)$ is an A-proper map with respect to Γ_0 for some $\lambda>0$. One can easily see that $0\in Lx-Nx$ iff $0\in(I-(L+\lambda JP)^{-1}(N+\lambda JP))x$. Assume that $0\notin(L-N)(\partial\Omega\cap D(L))$. Then $0\notin(I-(L+\lambda JP)^{-1}(N+\lambda JP))(\partial\Omega)$ for all $\lambda>0$, and we define a generalized degree

$$\deg_{\Gamma_0}(L-N,\Omega,0)=\cup_{0<\lambda}\deg(I-(L+\lambda JP)^{-1}(N+\lambda JP),\Omega,0), \tag{2.3}$$

where $\deg(I-(L+\lambda JP)^{-1}(N+\lambda JP),\Omega,0)$ is the generalized degree for A-proper maps if $I-(L+\lambda JP)^{-1}(N+\lambda JP)$ is A-proper with respect to Γ_0, otherwise $\deg(I-(L+\lambda JP)^{-1}(N+\lambda JP),\Omega,0)=\emptyset$.

Theorem 2.6. *The generalized degree defined by* (2.3) *has the following properties.*

(1) *If Ω_1 and Ω_2 are disjoint open subsets of Ω such that $0\notin(L-N)(D(L)\cap\overline{\Omega\setminus(\Omega_1\cup\Omega_2)})$, then*

$$\deg_{\Gamma_0}(L-N,\Omega,0)\subseteq\deg_{\Gamma_0}(L-N,\Omega_1)+\deg_{\Gamma_0}(L-N,\Omega_2,0).$$

(2) *If $H(t,x):[0,1]\times\overline{\Omega}\to Y$ satisfies $0\notin\cup_{t\in[0,1]}(L-H(t,\cdot))(D(L)\cap\partial\Omega)$, and $\{I-(L+\lambda JP)^{-1}(H(t,\cdot)+\lambda JP)\}_{t\in[0,1]}$ is a homotopy of A-proper maps with respect to Γ_0 for all $\lambda>0$, then $\deg_{\Gamma_0}(L-H(t,\cdot),\Omega,0)$ does not depend on $t\in[0,1]$.*

(3) *If $\deg_{\Gamma_0}(L-N,\Omega,0)\neq\{0\}$, then $0\in(L-N)(D(L)\cap\Omega)$.*

(4) *If Ω is a symmetric neighbourhood of 0, and $N:\overline{\Omega}\to Y$ is an odd mapping such that $I-(L+\lambda JP)^{-1}(N+\lambda JP)$ is A-proper with respect to Γ_0 for some $\lambda>0$, and $0\notin(L-N)(\partial\Omega\cap D(L))$, then $\deg_{\Gamma_0}(L-N,\Omega,0)$ does not contain even numbers.*

Proof. The proof is standard. We prove (2) and omit the others. Since $0\notin\cup_{t\in[0,1]}(L-H(t,\cdot))(D(L)\cap\partial\Omega)$, it follows that $0\notin\cup_{t\in[0,1]}(I-(L+\lambda JP)^{-1}(H(t,\cdot)+\lambda JP))(\partial\Omega)$ for all $\lambda>0$. By Theorem 2.1 of [**12**], we know that

$$\deg(I-(L+\lambda JP)^{-1}(H(t,\cdot)+\lambda JP),\Omega,0)$$

does not depend on $t\in[0,1]$ for each $\lambda>0$. Thus (2) follows from (2.3). □

Theorem 2.7. *Suppose that $(L+\lambda JP)^{-1}: Y\to X$ is a continuous compact mapping for each $\lambda>0$, and $0\in\Omega\subset X$ is an open bounded subset, $N:\overline{\Omega}\to Y$ is a continuous bounded mapping such that $Lx\neq Nx$, and $QNx\neq\eta JPx$ for all $x\in\partial\Omega\cap D(L)$, $\eta>0$, where P, Q are projections as in section 1. Then* $\deg(L-N,\Omega,0)=\{1\}$.

Proof. Let $\Gamma_0=(X_n,P_n)$ be a projectionally complete scheme for X. Since $(L+\lambda JP)^{-1}: Y\to X$ is continuous and compact for each $\lambda>0$, it follows that $\{I-(L+\lambda JP)^{-1}t(N+\lambda JP)\}_{t\in[0,1]}$ is a homotopy of A-proper maps with respect to Γ_0. We claim that $x\neq(L+\lambda JP)^{-1}t(N+\lambda JP)x$ for all $(t,x)\in[0,1]\times(\partial\Omega\cap D(L))$, $\lambda>0$. If this is not true, then there exist $\lambda_0>0$, $(t_0,x_0)\in[0,1)\times\partial\Omega$ such that $x_0=(L+\lambda_0JP)^{-1}t_0(Nx_0+\lambda JPx_0)$. Thus we have $x_0\in D(L)$, and

$$Lx_0+\lambda_0JPx_0=t_0(Nx_0+\lambda_0JPx_0).$$

Obviously, $t_0\neq1$, therefore $(1-t_0)\lambda_0JPx_0=t_0QNx_0$, which is a contradiction to one of our assumptions. Consequently, the A-proper degree $\deg(I-(L+\lambda JP)^{-1}(N+$

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

$\lambda JP), \Omega, 0) = \deg(I, \Omega, 0) = \{1\}$. By (2.3), we obtain

$$\deg_{\Gamma_0}(L - T, \Omega, 0) = \{1\}. \qquad \square$$

COROLLARY 2.8. *Suppose that H is a separable Hilbert space, and $(L + \lambda JP)^{-1} : H \to X$ is a continuous compact mapping for each $\lambda > 0$, and $0 \in \Omega \subset X$ is an open bounded subset, $N : \overline{\Omega} \to H$ is a continuous bounded mapping such that $Lx \neq Nx$ for all $x \in \partial\Omega \cap D(L)$, $QNx \neq 0$ for $x \in \partial\Omega \cap D(L) \cap \mathrm{Ker}(P)$, $(QNx, JPx) < 0$ for all $x \in \partial\Omega \cap D(L) \cap (\mathrm{Ker}(P))^c$, where P, Q are projections as in section 1. Then* $\deg(L - N, \Omega, 0) = \{1\}$.

Proof. From our assumptions, we have $QNx \neq \eta JPx$ for all $x \in \partial\Omega \cap D(L), \eta > 0$. Thus the conclusion follows from Theorem 2.7. $\square$

3. An Example. Consider the following wave equation

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) - h(u(t,x)) = f(t,x), & t \in (0, 2\pi), \quad x \in (0, \pi), \\ u(t,0) = u(t,\pi) = 0, & t \in (0, 2\pi), \\ u(0,x) = u(2\pi, x), & x \in (0,\pi), \end{cases} \tag{E 3.1}$$

where $h : R \to R$ is a continuous function satisfying

$$|h(u)| \leq \delta|u| + \gamma, \tag{3.1}$$

and $f(\cdot) \in L^2((0, 2\pi) \times (0, \pi))$, where $\delta > 0, \gamma > 0$ are constants.

We say $u \in L^2((0, 2\pi) \times (0, \pi))$ is a weak solution of (E 3.1) if

$$(u, v_{tt} - v_{xx}) - (h(u(t,x)), v) = (f(t,x), v)$$

for all $v \in C^2([0, 2\pi] \times [0, \pi])$ with $v(t, 0) = v(t, \pi) = 0$ for $t \in [0, 2\pi]$, and $v(2\pi, x) = v(0, x)$ for $x \in [0, \pi]$.

Let $L : D(L) \subset L^2((0, 2\pi) \times (0, \pi)) \to L^2((0, 2\pi) \times (0, \pi))$ be the wave operator $Lu = u_{tt} - u_{xx}$. Then it is well known that L is self-adjoint, densely defined, closed, and $\mathrm{Ker}(L)$ is infinite dimensional with $\mathrm{Ker}(L)^\perp = \mathrm{Im}(L)$. Thus L is a Fredholm mapping of zero index type. Let $P : L^2((0, 2\pi) \times (0, \pi)) \to \mathrm{Ker}(L)$ be the projection, then $(L + \lambda P)^{-1} : L^2((0, 2\pi) \times (0, \pi)) \to D(L)$ is compact for all $\lambda > 0$.

Let $N : L^2((0, 2\pi) \times (0, \pi)) \to L^2((0, 2\pi) \times (0, \pi))$ be defined by $Nu(t, x) = h(u(t, x)) + f(t, x)$ for $u(t, x) \in L^2((0, 2\pi) \times (0, \pi))$. By (3.1), N is a bounded continuous mapping. For each $\eta > 0$, consider the following equation

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + \eta u(t,x) - h(u(t,x)) = f(t,x), & t \in (0, 2\pi), \quad x \in (0, \pi), \\ u(t,0) = u(t,\pi) = 0, & t \in (0, 2\pi), \\ u(0,x) = u(2\pi, x), & x \in (0,\pi), \end{cases} \tag{E 3.2}$$

where h, f are as in (E 3.1). Let u_η be the weak solution of (E 3.2) if it exists, and we set $S = \{u_\eta : \eta > 0\}$. Now we have the following alternative result.

THEOREM 3.1. *S is unbounded in $L^2((0, 2\pi) \times (0, \pi))$ or* (E 3.1) *has a weak solution.*

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press

Proof. We may assume that S is bounded in $L^2((0, 2\pi) \times (0, \pi))$. So there exists $r_0 > 0$ such that

$$\|u_\eta\|_{L^2} < r_0, \text{ for all } u_\eta \in S. \tag{3.2}$$

Let $\Omega = \{u(t, x) \in L^2((0, 2\pi) \times (0, \pi)) : \|u\|_{L^2} < r_0\}$. By (3.2), we know $PNu \neq \eta Pu$ for all $u \in C^2([0, 2\pi] \times [0, \pi]) \cap \partial\Omega$, and $\eta > 0$. We may assume that $Lu \neq Nu$ for all $u \in C^2([0, 2\pi] \times [0, \pi]) \cap \partial\Omega$.

By Theorem 2.7, we have $\deg(L - N, \Omega, 0) = \{1\}$, thus (E 3.1) has a weak solution.

REFERENCES

1. J. Berkovits and V. Mustonen, An extension of Leray-Schauder degree and applications to nonlinear wave equations, *Diff. Int. Equat.* **3** (1990), 945–963.

2. H. Brézis, Periodic solutions of nonlinear vibrating strings and duality principles, *Bull. Amer. Math. Soc.* **8** (1983), 409–426.

3. F. E. Browder and W. V. Petryshyn, Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces, *J. Funct. Anal.* **3** (1969), 217–274.

4. M. Feckan and R. Kollar, Discontinuous wave equations and a topological degree for some classes of multi-valued mappings, *Appl. Math.* **44** (1999), 15–32.

5. P. M. Fitzpatrick, A generalized degree for uniform limits of A-proper mappings, *J. Math. Anal. Appl.* **35** (1971), 536–552.

6. R. E. Gaines and J. Mawhin, *Coincidence degree and nonlinear differential equations,* Lecture Notes in Mathematics No. 568 (Springer-Verlag, 1977).

7. J. Mawhin, Nonlinear perturbations of Fredholm mappings in normed spaces and applications to differential equations, Univ. de Brasilia, Trabalho de Matematica, No. 61 (1974).

8. J. Mawhin Topological degree methods in nonlinear boundary value problems (Amer. Math. Soc. Providence, RI, 1979).

9. W. V. Petryshyn, Fredholm theory for abstract and differential equations with noncompact nonlinear perturbations of Fredholm maps, *J. Math. Anal. Appl.* **72** (1979), 472–499.

10. W. V. Petryshyn, Using degree theory for densely defined A-proper maps in the solvability of semilinear equations with unbounded and noninvertible linear part, *Nonlinear Anal.* **4** (1980), 259–281.

11. W. V. Petryshyn and Z. S. Yu, Existence theorems for higher-order nonlinear periodic boundary value problems, *Nonlinear Anal.* **6** (1982), 943–969.

12. W. V. Petryshyn, Generalized topological degree and semilinear equations (Cambridge University Press, 1995).

https://doi.org/10.1017/S0017089505002879 Published online by Cambridge University Press