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INERTIA INVARIANTS OF A SET OF PARTICLES

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The formal object of this note is the calculation of the principal moments of inertia of a set of particles at their mass centre, in terms of their mutual distances; the calculation brings in some identities which although simple may be in part novel.

If I_{ij} is the inertia tensor at a point O, the quantities (with S denoting a sum over axes 1, 2, 3)

$$I = \mathrm{S}I_{11},\ I' = \mathrm{S}(I_{11}I_{22}-I_{12}^2),\ I'' = \det(I_{ij}) \quad \text{.................(1)}$$

are invariant for rotation of axes; and the principal moments at O are the roots in λ of

$$\lambda^3 - I\lambda^2 + I'\lambda - I'' = 0. \quad \text{................................(2)}$$

The problem, then, is to calculate these *inertia invariants* (1) when O is the mass centre.

The masses and positions of particles will be denoted by a, b, c, ... and A, B, C, ... respectively, and Σ will imply a sum over all the particles. Thus the inertia tensor at O is

$$I_{ij} = \Sigma\, a\, OA^2 \delta_{ij} - \Sigma\, a\, OA_i OA_j \quad \text{..........................(3)}$$

where OA_i is a component of the vector $\boldsymbol{OA}$.

The invariants (1) are dimensionally the squares of distances, areas and volumes respectively; so it is natural to relate them to squares of the extensions of lines, triangles and tetrahedra determined by sets of points or particles. The three *geometrical invariants*, (4) below, are of these types. The first forms in (4) refer to an arbitrary origin O; as we shall see later, these may be transformed, when O is the mass centre, into the second forms which involve configurations AB, ABC, $ABCD$ of the particles alone:

$$\left.\begin{aligned} J &= \Sigma\, a\, OA^2 && (\equiv)\ m^{-1}\,\Sigma'\, ab\, AB^2 \\ J' &= \Sigma'\, ab\, OAB^2 && (\equiv)\ m^{-1}\,\Sigma'\, abc\, ABC^2 \\ J'' &= \Sigma'\, abc\, OABC^2 && (\equiv)\ m^{-1}\,\Sigma'\, abcd\, ABCD^2 \end{aligned}\right\}. \quad \text{...............(4)}$$

Here $m = \Sigma a$ is the total mass, Σ' denotes a sum over all distinct choices (without reference to order) of sets of two, three or four particles in the respective summands; and

$$ABC = AB \times AC, \quad ABCD = AB\,.\,(AC \times AD). \quad \text{..............(5)}$$

Deferring consideration of the second forms in (4) and allowing O to be arbitrary, we find that the inertia invariants (1) at O are in terms of the first forms in (4)

$$I = 2J, \quad I' = J^2+J', \quad I'' = JJ'-J''. \qquad \text{.....................(6)}$$

The first relation is trivial. To prove the second, it is observed that a separation of the two sums in (3) puts $I_{11}I_{22}-I_{12}^2$ into the form

$$\Sigma a\,OA^2\,\Sigma b\,OB_3^2+\begin{vmatrix} \Sigma a\,OA_1^2 & \Sigma b\,OB_1OB_2 \\ \Sigma a\,OA_1OA_2 & \Sigma b\,OB_2^2 \end{vmatrix}. \qquad \text{...............(7)}$$

A pair A,B of distinct particles contributes to this determinant terms which add up to $ab\{(\boldsymbol{OA}\times\boldsymbol{OB})_3\}^2$. Summation of (7) over the axes now gives the second relation (6). The third relation follows by a slight extension.

When O is the mass centre G, the corresponding inertia invariants satisfy (6), with now the two identifiable forms (4) of J, J' and J''. The identity of the two forms follows from the condition that, when O is at G,

$$\Sigma a\,\boldsymbol{OA} = 0. \qquad \text{....................................(8)}$$

The identity (4) for J is well known, the second form being Jacobi's function. In the second form of J', the sum Σ' may be replaced by one-sixth of a complete summation over all A, B and C. This gives (using $\boldsymbol{AB} = \boldsymbol{OB}-\boldsymbol{OA}$ etc. in $\boldsymbol{ABC}$)

$$J' = (6m)^{-1}\Sigma\Sigma\Sigma abc(\boldsymbol{OB}\times\boldsymbol{OC}+\boldsymbol{OC}\times\boldsymbol{OA}+\boldsymbol{OA}\times\boldsymbol{OB})^2. \qquad \text{.........(9)}$$

Cross terms like $(\boldsymbol{OB}\times\boldsymbol{OC})\,.\,(\boldsymbol{OC}\times\boldsymbol{OA})$ vanish on summation by (8), and so we arrive at the first form of J' in (4). The two forms of J'' may be identified similarly.

By (4) and (6) the inertia invariants at G are expressed in terms of the squared extensions of the distances, triangles and tetrahedra determined by the particles. These quantities can all be expressed in squares of mutual distances; for example

$$ABC^2 = -\tfrac{1}{4}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \gamma & \beta \\ 1 & \gamma & 0 & \alpha \\ 1 & \beta & \alpha & 0 \end{vmatrix}, \qquad \text{...........................(10)}$$

where α, β, γ = BC^2, CA^2, AB^2. There is a similar form for $ABCD^2$. This completes the formal solution of the problem stated.

The following points may be noticed about the identities (4). (i) The identities hold also for " components ", that is, we may replace the squares of OA, OAB, $OABC$ by the squares of OA_i, $(\boldsymbol{OA}\times\boldsymbol{OB})_i$ and $OA_i(\boldsymbol{OB}\times\boldsymbol{OC})_i$ if we make similar replacements in the second forms. (ii) The identities may be put in algebraic or determinant forms. (iii) A geometrical form of the J' and J'' identities for n equal masses is that the sum of the squares of the triangles or tetrahedra on n points is n times the sum of the squares of the triangles or tetrahedra the points subtend at their mean centre. This could be generalised to higher dimensions.

An alternative approach. The initial problem was first solved by using the inertia tensor at G in the form involving solely relative positions, namely

$$I_{ij} = m^{-1}\Sigma' ab\, AB^2\delta_{ij} - m^{-1}\Sigma' ab\, AB_i AB_j. \qquad \text{(11)}$$

If this is used in (1) with a method of expansion as in (7), we find that the inertia invariants are given by (6), with J as in (4) (second form) and with

$$J' = m^{-2}\Sigma'' abcd(\mathbf{AB}\times\mathbf{CD})^2, \qquad \text{(12)}$$

$$J'' = m^{-3}\Sigma''' abcdef[(\mathbf{AB}\times\mathbf{CD})\,.\,\mathbf{EF}]^2. \qquad \text{(13)}$$

Here Σ'' denotes a sum over distinct pairs of distinct pairs A,B and C,D, and Σ''' over distinct triples of distinct pairs. These forms (12) and (13) arise naturally in a vibration problem; their direct conversion into the second forms of (4) is of some interest.

To convert (12), we replace the sum Σ'' by one-eighth of a complete sum over all A, B, C, D, and then use the identity (compare (5))

$$(\mathbf{AB}\times\mathbf{CD})^2 + (\mathbf{AC}\times\mathbf{BD})^2 + (\mathbf{AD}\times\mathbf{BC})^2 = ABC^2 + ABD^2 + ACD^2 + BCD^2, \qquad \text{(14)}$$

which gives J' as in (4). To convert (13), the sum Σ''' is first replaced by a complete sum. The identity (14) is true also for squares of components (replacing ABC^2 by $\{(\mathbf{ABC})_i\}^2$ etc.), and so it can be used to replace sums of six-point vector expressions in (13) by the five-point type $(\mathbf{ABC}\,.\,\mathbf{EF})^2$. These may be reduced to four-point or tetrahedral type $ABCD$ by the identity

$$(\mathbf{AB}\,.\,\mathbf{CDE})^2 + (\mathbf{AC}\,.\,\mathbf{BDE})^2 + (\mathbf{AD}\,.\,\mathbf{BCE})^2 + (\mathbf{AE}\,.\,\mathbf{BCD})^2 = ABCD^2 + ABCE^2 + ABDE^2 + ACDE^2 + 2BCDE^2. \qquad \text{(15)}$$

In this way the expression J'' of (13) is reduced to the second form in (4).

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