NONOSCILLATION CRITERIA FOR ELLIPTIC EQUATIONS

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Sufficient conditions will be derived for the linear elliptic partial differential equation

$$(1) \qquad Lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + 2 \sum_{i=1}^{n} b_i D_i u + cu = 0$$

to be nonoscillatory in an unbounded domain R in n-dimensional Euclidean space E^n. The boundary ∂R of R is supposed to have a piecewise continuous unit normal vector at each point. There is no essential loss of generality in assuming that R contains the origin. Otherwise no special assumptions are needed regarding the shape of R : it is not necessary for R to be quasiconical (as in [2]), quasicylindrical, or quasibounded [1].

Our results are generalizations of the one-dimensional nonoscillation theorems of Hille [3], Moore [5], Potter [6], and others. An example of a nonoscillation criterion for (1) in the selfadjoint case ($b_i \equiv 0$, $i = 1, 2, \ldots, n$) was given recently by Headley and the author [2]. Nonoscillation criteria are obtained here for the general linear elliptic equation (1) as a consequence of the author's comparison theorem [7] and the one-dimensional theorems cited above. In the special case that (1) is the Schrödinger equation $\nabla^2 u + cu = 0$ and R coincides with E^n, Theorem 2 below reduces to a result of Glazman [1]. Specialization of our results to the case $n = 1$ immediately yields new nonoscillation criteria for general second order ordinary linear differential equations.

Points in E^n are denoted by $x = (x^1, x^2, \ldots, x^n)$ and differentiation with respect to x^i is denoted by D_i, $i = 1, 2, \ldots, n$. It is assumed

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that the functions a_{ij}, b_i, and c involved in (1) are real-valued and continuous on $R \cup \partial R$, that the b_i are differentiable in R, and that the matrix (a_{ij}) is symmetric and positive definite in R. A "solution" of (1) is supposed to be continuous in $R \cup \partial R$ and have uniformly continuous first partial derivatives in R, and all derivatives involved in (1) are supposed to exist, be continuous, and satisfy (1) at every point in R.

The following notations will be used:

$$R_r = R \cap \{x \in E^n : |x| > r\} ; \quad S_r = \{x \in R \cup \partial R : |x| = r\} .$$

A bounded domain $N \subset R$ is said to be a <u>nodal domain</u> of a nontrivial solution u of (1) if and only if $u = 0$ on ∂N. The differential equation (1) is said to be <u>nonoscillatory</u> in R if and only if there exists a number $s > 0$ such that no nontrivial solution has a nodal domain contained in R_s [1, p. 158].

Let g be the function defined by

$$\text{(2)} \qquad g(r) = \max_{x \in S_r} [c(x) - \operatorname{div} b(x)], \qquad 0 < r < \infty ,$$

where $b(x) = (b_1(x), b_2(x), \ldots, b_n(x))$, $x \in R$, and let C be the function in R defined by the equation $C(x) = g(|x|)$. Let $\lambda(x)$ denote the smallest eigenvalue of the matrix $(a_{ij}(x))$, $x \in R$. Let f be an arbitrary positive-valued function of class $C^1(0, \infty)$ such that

$$f(r) \leq \min_{x \in S_r} \lambda(x), \qquad 0 < r < \infty ,$$

and define the function A in R by the equation $A(x) = f(|x|)$. Then

$$\text{(3)} \qquad \sum_{i,j=1}^{n} a_{ij} z^i z^j \geq \lambda(x) |z|^2 \geq A(x) |z|^2$$

for all $x \in R$ and all $z \in E^n$. The following theorem is obtained by comparison of (1) with the separable equation

$$\sum_{i=1}^{n} D_i(AD_i v) + Cv = 0 . \tag{4}$$

THEOREM 1. <u>Equation</u> (1) <u>is nonoscillatory in</u> R <u>if the ordinary differential equation</u>

$$(r^{n-1} f(r) \zeta')' + r^{n-1} g(r) \zeta = 0 \tag{5}$$

<u>is nonoscillatory at</u> $r = \infty$, <u>i.e. if there exists a number</u> a <u>such that every nontrivial solution of</u> (5) <u>has at most one zero in</u> (a, ∞).

<u>Proof</u>. Suppose to the contrary that (1) is oscillatory in R. Then there exists a nontrivial solution u_r of (1) with a nodal domain N_r contained in R_r for all $r > 0$. The variation between (1) and (4) is [7]

$$V[u] = \int_{N_r} \left[\sum_{i,j} a_{ij} D_i u D_j u - A|\nabla u|^2 + (C - c + \operatorname{div} b)u^2 \right] dx,$$

which is positive by (2) and (3). Since (1) is majorized by (4), it follows from the author's comparison theorem [7] that every solution of (4) has a zero at some point of $\bar{N}_r$, and hence at some point of R_r.

However, a routine separation of variables of (4) in hyperspherical coordinates $r, \theta_1, \theta_2, \ldots, \theta_{n-1}$ [4, p. 58] shows that (4) has radial solutions $v(x) = \zeta(r)$ $(r = |x|)$, where ζ satisfies (5). Since (5) is nonoscillatory, there exists a solution $v(x) = \zeta(r)$ of (4) and a number r_0 such that $v(x)$ is free of zeros in R_r for all $r > r_0$. The contradiction establishes Theorem 1.

As a consequence of Theorem 1, any one of the known sets of sufficient conditions for (5) to be nonoscillatory generates a nonoscillation criterion for (1), for example, Moore's conditions

$$\int_1^\infty \frac{dr}{r^{n-1} f(r)} < \infty \quad \text{and} \quad \limsup_{r \to \infty} \left| \int_1^r x^{n-1} g(x) dx \right| < \infty,$$

or Potter's conditions

$$\int_1^\infty \frac{dr}{r^{n-1} f(r)} = +\infty \text{ and } L > 2$$

where $L = \lim_{r \to \infty} r^{n-1} f(r) \left\{ r^{1-n} [f(r) g(r)]^{-\frac{1}{2}} \right\}'$ (whenever the limit exists) [5; 6].

In the case $n = 1$, the differential equation (1) has the form

$$(1') \qquad [a(x) u']' + 2b(x) u' + c(x) u = 0, \qquad 0 \leq x < \infty.$$

The definitions of f and g reduce to

$$f(x) = \lambda(x) = a(x), \quad g(x) = c(x) - b'(x),$$

and substitution into any nonoscillation criterion for (5) [e.g. Moore's criterion above] immediately yields a nonoscillation criterion for (1').

The nonoscillation theorems below are obtained from Theorem 1 in the case that the differential operator L is <u>uniformly elliptic</u> in R_s for some $s > 0$, i.e. there exists a positive number λ_0 (the ellipticity constant) such that $\lambda(x) \geq \lambda_0$ for all $x \in R_s$.

THEOREM 2. *Equation (1) is nonoscillatory in R if L is uniformly elliptic in R_s for some $s > 0$ and*

$$(6) \qquad \limsup_{r \to \infty} r^2 g(r) < (n-2)^2 \lambda_0 / 4,$$

where λ_0 is the ellipticity constant.

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In the special case that (a_{ij}) is the unit matrix and $b_i = 0$, $i = 1, 2, \ldots, n$, equation (1) reduces to the Schrödinger equation

$$\nabla^2 u + c(x) u = 0, \qquad x \in R.$$

In this case, $f(r) = \lambda(x) = 1$, $0 < r = |x| < \infty$, and if R coincides with E^n,

$$g(r) = \max_{|x| = r} c(x).$$

Theorem 2 then becomes Glazman's theorem [1, p. 158]: *The Schrödinger equation is nonoscillatory in* E^n *if*

$$\limsup_{r \to \infty} r^2 g(r) < \frac{(n-2)^2}{4}.$$

THEOREM 3. *Equation* (1) *is nonoscillatory in* R *if* L *is uniformly elliptic in* R_s *for some* $s > 0$ *and*

(7) $$\limsup_{r \to \infty} r \int_r^\infty h^+(t)\,dt < \lambda_0 / 4,$$

where $h^+(t) = \max[h(t), 0]$ *and*

$$h(t) = g(t) - \frac{1}{4}(n-1)(n-3)\lambda_0 t^{-2}, \qquad 0 < t < \infty.$$

To prove Theorem 2, it is sufficient to prove that (5) is nonoscillatory at $r = \infty$ in the case $f(r) = \lambda_0$. The hypothesis (6) implies that there exist constants r_0 and γ such that $r^2 g(r) < \gamma < (n-2)^2 \lambda_0 / 4$ for all $r > r_0$. Thus the Euler equation $(\lambda_0 r^{n-1} \zeta')' + \gamma r^{n-3} \zeta = 0$ is nonoscillatory, and also (5) is nonoscillatory by Sturm's comparison theorem.

To prove Theorem 3 we observe that $\zeta(r)$ satisfies (5) if and only if $\phi(r) = r^{(n-1)/2}\zeta(r)$ satisfies the differential equation

$$(8) \qquad \lambda_0 \phi'' + h(r)\phi = 0 .$$

On account of the hypothesis (7), the equation

$$\lambda_0 y'' + h^+(r)y = 0 .$$

is nonoscillatory by Hille's theorem [3]. Since $h^+(r) \geq h(r)$, (8) also is nonoscillatory by Sturm's comparison theorem.

REFERENCES

1. I.M. Glazman, Direct methods of qualitative spectral analysis of singular differential operators. (Israel Program for Scientific Translations, Daniel Davey and Co., New York, 1965.)

2. V.B. Headley and C.A. Swanson, Oscillation criteria for elliptic equations. Pacific J. Math. 27 (1968) 501-506.

3. E. Hille, Non-oscillation theorems. Trans. Amer. Math. Soc. 64 (1948) 234-252.

4. S.G. Mikhlin, The problem of the minimum of a quadratic functional. (Holden-Day, San Francisco, 1965.)

5. R.A. Moore, The behavior of solutions of a linear differential equation of second order. Pacific J. Math. 5 (1955) 125-145.

6. R.L. Potter, On self-adjoint differential equations of second order. Pacific J. Math. 3 (1953) 467-491.

7. C.A. Swanson, A comparison theorem for elliptic differential equations. Proc. Amer. Math. Soc. 17 (1966) 611-616.

8. C.A. Swanson, Comparison theorems for elliptic equations on unbounded domains. Trans. Amer. Math. Soc. 126 (1967) 278-285.

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