Note on Hypercomplex Numbers.

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The present note is an extension of a previous paper* on the same subject. In this paper a concise proof was given of a theorem by Scheffers to the effect that if a linear associative algebra contains the quaternion algebra as a subalgebra, both having the same modulus, then it can be expressed as the direct product of that quaternion algebra and another algebra. It was also shown that this theorem could be generalised to the extent of substituting a matric quadrate algebra for the quaternion algebra. In the present paper the theorem is extended to certain other types of algebras.

As the phrase *linear associative algebra* is rather cumbersome, I use throughout the word *algebra* in its place, and for the same reason I call a subalgebra which has the same modulus as the given algebra a *proper* subalgebra.

1. A number or element of an algebra is said to be rational with respect to a given field or domain if all its coefficients are rational in the same field and the algebra itself is said to be rational if the product of any two rational elements is also rational. Thus Hamilton's quaternions are rational in the field of rational numbers, and hence also in any subfield of the field of real numbers.

Let A be an algebra rational in a given field and having a rational proper subalgebra B which on extending the field is found to be equivalent to the matric quadrate algebra

$$e_{pq}\ (p,\ q=1,\ 2,\ldots n\,;\ e_{pq}e_{qr}=e_{pr},\ e_{pq}e_{rs}=0,\ q\neq r)\,;$$

then *the algebra A can be expressed as the direct product of B and some other rational subalgebra.*

By the extension of Scheffers' theorem mentioned in the introduction, A can be expressed in the extended field as the direct product of B and another algebra D. Let the bases of A and B be

$$\mathrm{A}=x_1,\ x_2,\ \ldots\ x_a$$
$$\mathrm{B}=y_1,\ y_2,\ \ldots\ y_b.$$

These elements can be chosen rational and the proof of the theorem consists in showing that a rational basis can also be chosen for D.

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* *Proceedings of the Royal Society of Edinburgh*, vol. 26, 1906.

Any element z of D can evidently be expressed in the form

$$z = \Sigma \xi_r x_r' + x'$$

where $x_1', \dots x_s', x'$ are linearly independent rational elements of A and $\xi_1, \dots \xi_s$ are irrational scalars which are linearly independent in the given field. If now y is any rational element of B

$$yz = zy$$

therefore $$0 = \Sigma \xi_r (y x_r' - x_r' y) + y x' - x' y.$$

Hence as the ξ's are independent we must have

$$y x_r' - x_r' y = 0 \quad (r = 1, 2, \dots s)$$
$$y x' - x' y = 0$$

for every rational element y of B and therefore $x_1', x_2', \dots x_s', x'$ are elements of D, *i.e.*, a rational basis can be chosen for D.

2. The second extension is as follows.

If an algebra A contains a rational proper subalgebra B in which (1) *every element has an inverse,* (2) *no element save the modulus is commutative with every other element, then A can be expressed as the direct product of B and a rational algebra C.*

Let $y_1, y_2 \dots y_b$ be a rational basis and

$$y = \Sigma \xi_r y_r$$

any rational element of B. If now we form

$$y' = y_r y - y y_r = \Sigma \xi_s (y_r y_s - y_s y_r) = \Sigma \xi_s y_s'$$

the coefficient ξ_r disappears possibly along with some others. Similarly if ξ_s is any coefficient which has not been eliminated by this process,

$$y'' = y_s' y' - y' y_s' = \Sigma \xi_t y_t''$$

does not contain the coefficient ξ_s and so on. This process may come to an end in two ways.

(1) If in $$y^{(r)} = \Sigma \xi_t y_t^{(r)}$$

$y_1^{(r)}, y_2^{(r)}, \dots$ are all commutative,

(2) If $$y^{(r)} = \xi_t y_t^{(r)}.$$

The first case reduces to the second if $\rho = 1$, so we may suppose $\rho > 1$. Now from the conditions imposed on B, there is an element x which is not commutative with $y_p^{(r)} y_q^{(r)-1}$, unless $y_p^{(r)} = y_q^{(r)}$. But by replacing $y^{(r-1)}$ by $x'y^{(r)}$ where x' is suitably chosen, we can always arrange that $y_p^{(r)} \neq y_q^{(r)}$, hence we may assume

$$y^{(r)}y_q^{(r)^{-1}}x - xy^{(r)}y_q^{(r)^{-1}} = \Sigma\xi_t(y_t^{(r)}y_q^{(r)^{-1}}x - xy_t^{(r)}y_q^{(r)^{-1}}) \neq 0.$$

The coefficient of ξ_q under the summation sign is, however, zero. Any other coefficient can be eliminated in the same way, and by this process we can reduce the number of terms under the summation sign step by step till an equation of the second type is reached. In both cases, therefore, we are led to an equation of the form

$$y_t = \xi_t y_t^{(r)}$$

or

$$\xi_t = y_t y_t^{(r)^{-1}}$$

where ξ_t is any preassigned coefficient of y. Hence remembering that y is an arbitrary element of B, we see that we can represent its rth coordinate in the form

$$\xi_r = f_r(y)$$

where the form of $f_r(\ )$ does not depend on y. Hence, as was proved in the paper referred to in the introduction, A can be expressed as the direct product of B and the algebra obtained from A by forming

$$f_r(x_s) \quad (r = 1, 2, \ldots b\ ;\ s = 1, 2, \ldots a)$$

$x_1, x_2, \ldots$ being a basis of A.