Glasgow Math. J. **46** (2004) 571–584. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504002009. Printed in the United Kingdom

HOMOLOGY OF POWERS OF REGULAR IDEALS

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(Received 30 October, 2003; accepted 27 April, 2004)

Abstract. For a commutative ring R with an ideal I, generated by a finite regular sequence, we construct differential graded algebras which provide R-free resolutions of I^s and of R/I^s for $s \geqslant 1$ and which generalise the Koszul resolution. We derive these from a certain multiplicative double complex $\mathbf{K}$. By means of a Cartan–Eilenberg spectral sequence we express $\mathrm{Tor}_*^R(R/I, R/I^s)$ and $\mathrm{Tor}_*^R(R/I, I^s)$ in terms of exact sequences and find that they are free as R/I-modules. Except for R/I, their product structure turns out to be trivial; instead, we consider an exterior product $\mathrm{Tor}_*^R(R/I, I^s) \otimes_R \mathrm{Tor}_*^R(R/I, I^t) \to \mathrm{Tor}_*^R(R/I, I^{s+t})$. This paper is based on ideas by Andrew Baker; it is written in view of applications to algebraic topology.

2000 *Mathematics Subject Classification.* 13D02, 13D07; 55U15.

0. Introduction. Let R be a commutative ring with unit and let $I \lhd R$ be an ideal generated by a finite regular sequence $r_1, \dots, r_n \in R$; i.e. r_1 is a non-zero divisor of R such that $R/(r_1) \neq 0$, r_2 a non-zero divisor of $R/(r_1)$ such that $R/(r_1, r_2) \neq 0$ and so on. The Koszul complex K, a differential graded algebra, provides a canonical R-free resolution of R/I. The aim of this paper is to construct explicit R-free resolutions of R/I^s and I^s, for $s > 1$, that generalise the Koszul resolution and enable us to compute $\mathrm{Tor}_*^R(R/I, R/I^s)$ and $\mathrm{Tor}_*^R(R/I, I^s)$.

We derive these from a certain second quadrant double complex $\mathbf{K}$ with R-free components that we call the *extended Koszul complex*. In formal analogy to the ordinary Koszul complex, $\mathbf{K}$ is constructed as a tensor product of elementary double complexes and carries a multiplicative structure. Its filtration by columns turns out to be a filtration by ideals $F^s(\mathbf{K}) \lhd \mathbf{K}$, so that the associated total complexes of $F^s(\mathbf{K})$ and $\mathbf{K}/s = \mathbf{K}/F^s(\mathbf{K})$ are differential graded algebras. Via suitable augmentations, they provide resolutions of I^s and R/I^s respectively.

The column-wise filtrations of $R/I \otimes_R F^s(\mathbf{K})$ and $R/I \otimes_R \mathbf{K}/s$ give rise to Cartan–Eilenberg type spectral sequences which converge to $\mathrm{Tor}_*^R(R/I, I^s)$ and $\mathrm{Tor}_*^R(R/I, R/I^s)$ respectively. We show that they collapse at E^2, express the Tor-groups by means of exact sequences and find that they are free over R/I.

Except for R/I, their multiplicative structure, induced by the R-algebra structure on R/I^s and on I^s, turns out to be trivial. However, the multiplications $I^s \otimes_R I^t \to I^{s+t}$ induce non-trivial exterior products

$$\mathrm{Tor}_*^R(R/I, I^s) \otimes_R \mathrm{Tor}_*^R(R/I, I^t) \longrightarrow \mathrm{Tor}_*^R(R/I, I^{s+t})$$

that allow us to view $\bigoplus_{s \geqslant 0} \mathrm{Tor}_*^R(R/I, I^s)$ as a bigraded algebra. It contains a copy of the polynomial ring $R/I[x_1, \dots, x_n]$ as a subalgebra, over which the whole algebra is generated by the R/I-basis elements of $\mathrm{Tor}_{k>0}^R(R/I, I)$.

This paper is based on ideas by Andrew Baker; he constructs in [**1**] resolutions of R/I^s and computes $\mathrm{Tor}_*^R(R/I, R/I^s)$. His approach to the resolutions, however, is conceptually different from the one presented here. He constructs them inductively, by pasting together copies of the Koszul complex.

Note that Tate constructs in [**11**] for *any* R-algebra of the form R/M, where R is a commutative Noetherian ring with unit and $M \lhd R$ an ideal, a differential graded algebra which provides a free resolution. Applying the construction to R/I^s, where I is as above, does not give the resolution considered here.

The present paper originated in the course of the work on my PhD thesis, in view of applications to algebraic topology. These will be discussed in [**13**], but at the suggestion of the referee, we briefly indicate their nature.

In algebraic topology, one considers ring spectra E, which are multiplicative objects in the stable homotopy category. They give rise to multiplicative homology and cohomology theories, $E^*(-)$ and $E^*(-)$ respectively, defined on topological spaces X. In particular, this means that the cohomology of a point, $E^* = E^*(*)$ is a graded commutative ring, and that $E^*(X)$ is a graded module over E^*, for any space X. (See [**3**] for a concise survey of cohomology theories.) Let $K^* = E^*/I$ be a quotient of E^* by an ideal I, generated by a regular sequence. If E is sufficiently well structured, it is possible to realize K^* topologically, by a module spectrum K over E [**7**, Section V]. In favourable cases, K is even a ring spectrum, but at any rate it defines homology and cohomology theories. There are natural transformations $E^*(X) \to K^*(X)$ of graded E^*-modules and similarly for homology. An important example are the Johnson–Wilson theories $E(n)$ and the Morava-K-theories $K(n)$ for a given prime p and $n \geqslant 0$. See [**3**]. Due to the simpler coefficients K^* of K, one expects $K^*(X)$ to be more accessible than $E^*(X)$. The natural question is then if there is a way to get back. One idea, first considered (in a special case) in [**2**], is to realize the I-adic tower

$$\begin{tikzcd} E^* \arrow[r] & \cdots \arrow[r] & E^*/I^3 \arrow[r] & E^*/I^2 \arrow[r] & E^*/I = K^* \\ & & I^2/I^3 \arrow[u] & I/I^2 \arrow[u] & \end{tikzcd} \tag{0.1}$$

topologically. More precisely, the aim is to construct a diagram

$$\begin{tikzcd} E \arrow[r] & \cdots \arrow[r] & E/I^3 \arrow[r] & E/I^2 \arrow[r] & E/I = K \\ & & I^2/I^3 \arrow[u] & I/I^2 \arrow[u] & \end{tikzcd} \tag{0.2}$$

of module spectra over E in such a way that

$$I^s/I^{s+1} \longrightarrow E/I^{s+1} \longrightarrow E/I^s$$

are cofibre sequences and that, paralleling algebra (compare Remark 3.3), the spectra I^s/I^{s+1} are coproducts of suspensions of K. For any space X, such a diagram gives rise to one of E^*-modules (an arrow with a circle denotes a homomorphism of degree -1).

$$\begin{tikzcd}
E^*(X) \arrow[r] & \cdots \arrow[r] & (E/I^3)^*(X) \arrow[r] & (E/I^2)^*(X) \arrow[r] \arrow[dl, "\circ" marking] & K^*(X) \arrow[dl, "\circ" marking] \\
 & & I^2/I^3 \otimes_{K^*} K^*(X) \arrow[u] & I/I^2 \otimes_{K^*} K^*(X) \arrow[u] &
\end{tikzcd}$$

This forms an unrolled exact couple (see [**4**]) and hence yields a spectral sequence

$$E_1^{s,t} = (I^s/I^{s+1} \otimes_{K_*} K^*(X))^t \implies E^{s+t}(X).$$

An understanding of the image of the tower (0.1) under the functor $\mathrm{Ext}_{E^*}^{*,*}(-, K^*)$, as provided by the present paper, makes the nature of a topological I-adic tower (0.2) transparent and leads to an easy construction.

It is a pleasure to express my gratitude to my supervisor Alain Jeanneret from the University of Berne and Andrew Baker from the University of Glasgow for many interesting discussions and for ongoing support. In particular, I would like to thank Andrew Baker for inviting me to spend a year at the University of Glasgow, as well as the Mathematics Institute in Berne for giving me the opportunity to do so. I would like to thank the people from the Mathematics Department in Glasgow, in general, for offering an inspiring and stimulating atmosphere during my stay, and the referee for useful suggestions.

1. Multiplicative double complexes. We define the notion of "double complex" which is most suitable for our purposes and describe a natural way of forming tensor products; this allows us to consider multiplicative double complexes. References for background material for this section are [**8**, VII.], [**9**, X.9] and [**12**].

Given an abelian category $\mathcal{A}$, we can form the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$, whose morphisms are the chain maps. As $\mathbf{Ch}(\mathcal{A})$ in turn is abelian, in a canonical way, we can iterate the construction.

DEFINITION 1.1. A *double complex* is an object C of $\mathbf{DC}(\mathcal{A}) = \mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$.

We display the components C_p of a double complex C as the columns of a lattice $C_{p,q}$. By definition, the differential $d_p\colon C_p \to C_{p-1}$ is a chain map, so that the squares in the lattice commute. Note that one often means by a double complex a lattice $C_{p,q}$ whose rows and columns are chain complexes, as here, but whose squares *anti*commute.

We denote the components $C_{p,q} \to C_{p-1,q}$ of the differential of C by $d^h_{p,q}$ (the *horizontal differentials*) and the components $C_{p,q} \to C_{p,q-1}$ of the differentials of the columns C_p of C by $d^v_{p,q}$ (the *vertical differentials*) when the indices are clear from the context, we omit them. We also use the convention $C^p_q = C_{-p,q}$.

By replacing the vertical differential $d^v_{p,q}$ by $(d^v_{p,q})' = (-1)^p d^v_{p,q}$, we can pass from a double complex C, as defined above, to one with anticommuting squares that we denote by C°.

The total complex $\mathrm{Tot}^\oplus(C^\circ)$ of C° has components

$$(\mathrm{Tot}^\oplus(C^\circ))_k = \bigoplus_{p+q=k} C_{p,q}$$

https://doi.org/10.1017/S0017089504002009 Published online by Cambridge University Press

and differential given by $d = d^h + (d^v)'$. Following MacLane, we call this complex the *condensation* of C and denote it by $C^\bullet$.

Assume now that $\langle \mathcal{A}, \otimes, e\rangle$ is an abelian, symmetric monoidal category (where e is the unit). We shall refer to the bifunctor $\otimes$ as the "tensor product".

Recall that $\mathbf{Ch}(\mathcal{A})$ inherits a symmetric monoidal structure from $\mathcal{A}$. Namely, for two chain complexes (C, d) and (D, d'), the tensor product $C \otimes D$ is defined by

$$(C \otimes D)_k = \bigoplus_{p+q=k} C_p \otimes D_q, \tag{1.2}$$

with differential given by

$$d_p \otimes 1 + (-1)^p\, 1 \otimes d'_q \colon C_p \otimes D_q \longrightarrow (C \otimes D)_{p+q-1}.$$

Embedding $\mathcal{A}$ in $\mathbf{Ch}(\mathcal{A})$ in the usual way, e is a unit in $\mathbf{Ch}(\mathcal{A})$ for $\otimes$. The symmetry isomorphism $\hat{\tau} : C \otimes D \longrightarrow D \otimes C$ is defined as

$$\hat{\tau}(c \otimes d) = (-1)^{kl}\tau(c \otimes d) \tag{1.3}$$

for $c \otimes d \in C_k \otimes D_l$, where $\tau : C_k \otimes D_l \longrightarrow D_l \otimes C_k$ is the given symmetry isomorphism in $\mathcal{A}$.

Iterating this procedure, we get a symmetric monoidal structure on $\mathbf{DC}(\mathcal{A})$. From the definition, the components (columns) of the tensor product $C \otimes D$ of two double complexes C and D consist of a direct sum of tensor products of columns of C and D. Note that $C \otimes D$ has the analogous property for the rows. We can express this in a more conceptual way by introducing the *transpose* TC of a double complex C, defined as $(TC)_{p,q} = C_{q,p}$, with differentials $(Td)^v_{p,q} = d^h_{q,p}$ and $(Td)^h_{p,q} = d^v_{q,p}$. The statement then amounts to the equation

$$TC \otimes TD = T(C \otimes D). \tag{1.4}$$

Note that the symmetry isomorphism $\hat{\hat{\tau}} : C \otimes D \to D \otimes C$ is given by

$$\hat{\hat{\tau}}(c \otimes d) = (-1)^{kp+lq}\tau(c \otimes d),$$

for $c \otimes d \in C_{k,l} \otimes D_{p,q}$.

Definition 1.5. A *multiplicative double complex* is a monoid in the monoidal category $\langle \mathbf{DC}(\mathcal{A}), \otimes, e\rangle$, i.e. a double complex C with a multiplication $\mu\colon C \otimes C \to C$ and a unit $\eta\colon e \to C$ such that the associativity and the two unit diagrams commute.

Unravelling the definitions, the product μ of a multiplicative double complex is defined by a collection of maps $C_{p,q} \otimes C_{k,l} \to C_{p+k,q+l}$ such that both the components of the horizontal and the vertical differential are derivations, in the sense that we have, for $c \in C_{p,q}$, $d \in C_{k,l}$,

$$d^h(\mu(c \otimes d)) = \mu(d^h(c) \otimes d) + (-1)^p\, \mu(c \otimes d^h(d)), \tag{1.6}$$

$$d^v(\mu(c \otimes d)) = \mu(d^v(c) \otimes d) + (-1)^q\, \mu(c \otimes d^v(d)). \tag{1.7}$$

Note that the condensation $C^\bullet$ of a multiplicative double complex is canonically a monoid in $\mathbf{Ch}(\mathcal{A})$, as a consequence of the natural isomorphism $(C \otimes D)^\bullet \cong C^\bullet \otimes D^\bullet$, which holds for any two double complexes C and D, and the fact that condensation is a functorial process.

Suppose that (C, μ, η) and (D, μ', η') are two multiplicative double complexes. As in any symmetric monoidal category, we can endow $C \otimes D$ with a multiplicative structure in a canonical way, by defining the product as the composition

$$(C \otimes D) \otimes (C \otimes D) \xrightarrow{C \otimes \hat{\tilde{\tau}} \otimes D} C \otimes C \otimes D \otimes D \xrightarrow{\mu \otimes \mu'} C \otimes D. \tag{1.8}$$

Explicitly, the product is given by the collection of maps

$$\begin{aligned}(C_{p,q} \otimes D_{k,l}) \otimes (C_{p',q'} \otimes D_{k',l'}) &\longrightarrow C_{p+p',q+q'} \otimes D_{k+k',l+l'} \\ (c \otimes d) \otimes (c' \otimes d') &\longmapsto (-1)^{kp'+lq'} \mu(c \otimes c') \otimes \mu'(d \otimes d').\end{aligned}$$

We have morphisms of monoids

$$C \cong C \otimes e \xrightarrow{C \otimes \eta'} C \otimes D, \quad D \cong e \otimes D \xrightarrow{\eta \otimes D} C \otimes D.$$

We shall also need the category $\partial\partial\mathcal{A}$ of ($\mathbb{Z}$–)bigraded objects in $\mathcal{A}$; by the construction of (1.2), $\partial\partial\mathcal{A}$ is monoidal. The forgetful functor $U\colon \mathbf{DC}(\mathcal{A}) \to \partial\partial\mathcal{A}$ is a strict morphism of monoidal categories and therefore restricts to a functor

$$U\colon \mathbf{Mon}_{\mathbf{DC}(\mathcal{A})} \longrightarrow \mathbf{Mon}_{\partial\partial\mathcal{A}}$$

between the categories of monoids. Whereas in $\mathbf{DC}(\mathcal{A})$ we were forced to introduce a sign when defining the symmetry isomorphisms in (1.3), we wouldn't need to do so in $\partial\partial\mathcal{A}$. However, we want the restriction of U to $\mathbf{Mon}_{\mathbf{DC}(\mathcal{A})}$ to be monoidal as well; therefore, we define symmetry isomorphisms in $\partial\partial\mathcal{A}$ as in (1.3). To stress that we use these symmetries for the definition of the tensor product of monoids in $\partial\partial\mathcal{A}$, as in (1.8), we denote it by $\tilde{\otimes}$ and call it the *twisted tensor product*.

In the next section, we apply these constructions to the symmetric monoidal category $\langle R\text{-}\mathbf{Mod}, \otimes_R, R\rangle$ of R-modules. The symmetry isomorphisms are given by the switch maps $\tau : M \otimes N \to N \otimes M$, $\tau(m \otimes n) = n \otimes m$. Monoids in R-$\mathbf{Mod}$ and $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ are R-algebras and differential graded R-algebras respectively. We shall refer to monoids in $\partial\partial(R\text{-}\mathbf{Mod})$ as *bigraded R-algebras.*

2. The extended Koszul complex and the resolutions. Without further notice, the ground ring is from now on understood to be R and so we omit the letter R in $\otimes_R$, Hom_R, Λ_R, Tor^R or Ext_R.

We briefly recall the definition of the Koszul complex K, in order to fix notations and to point out the formal similarity to the double complex $\mathbf{K}$ that we construct afterwards.

For a non-zero divisor $r \in R$, $K(r)$ is defined to be the differential graded algebra $(\Lambda(e), d)$ with $|e| = 1$ and $d(e) = r$. The projection $R \to R/(r)$ defines an augmentation $\varepsilon_r\colon K(r) \to R/(r)$, which exhibits $K(r)$ as an R-free resolution of $R/(r)$. For a regular sequence $r_1, \ldots, r_n$ generating an ideal I, the Koszul complex K is the differential graded algebra given by

$$K = K(r_1) \otimes \cdots \otimes K(r_n), \tag{2.1}$$

https://doi.org/10.1017/S0017089504002009 Published online by Cambridge University Press

according to a prescription similar to (1.8), but for chain complexes. As a graded algebra, K is the exterior algebra $\Lambda(e_1, \ldots, e_n)$. Together with the augmentation $\varepsilon = \otimes_{i=1}^{n} \varepsilon_{r_i}$, K is an R-free resolution of R/I of length n [**10**, Theorem 16.5].

Changing to two dimensions, we start by realizing $K(r)$ and $R[x]$, for a non-zero divisor $r \in R$ and a free variable x, as multiplicative double complexes, by assigning e and x bidegrees (0, 1) and (−1, 1) respectively. Hence $K(r)$ is concentrated on the y-axis and $R[x]$ on the secondary diagonal. Of course, all the differentials of $R[x]$ are trivial.

The multiplicative double complex $\mathbf{K}(r)$ corresponding to $K(r)$ is now given as follows. We take $K(r) \otimes R[x]$ and define horizontal differentials d^h by setting $d^h_{-k,k+1}(e \otimes x^k) = -1 \otimes x^{k+1}$. The reason for the sign will become clear later on. The other components of d^h are necessarily trivial. To see that d^h is compatible with the multiplicative structure canonically defined on $K(r) \otimes R[x]$, we have to check equation (1.6). We do the calculation for $c = e \otimes x^k$ and $d = e \otimes x^l$. The left-hand side of the equation is trivial, and the right-hand side is given by

$$\begin{aligned}&\mu(-(1 \otimes x^{k+1}) \otimes (e \otimes x^l)) + (-1)^{-k}\, \mu((e \otimes x^k) \otimes -(1 \otimes x^{l+1}))\\ &\quad = (-1)^{k+2}\, e \otimes x^{k+l+1} + (-1)^{-k+1}\, e \otimes x^{k+l+1} = 0.\end{aligned}$$

The vertical differentials in the kth column are induced by the ones of the shifted complex $\Sigma^k K(r)$. Here Σ denotes the suspension functor, defined on a chain complex C as $(\Sigma C)_k = C_{k-1}$, with differentials $(\Sigma d)_k = -d_k$. Explicitly, the non-trivial components are $d^v_{-k,k+1}(e \otimes x^k) = (-1)^k\, r \otimes x^k$.

Definition 2.2. The *extended Koszul complex* $\mathbf{K}$ associated to the regular sequence $r_1, \ldots, r_n$ is the multiplicative double complex

$$\mathbf{K} = \mathbf{K}(r_1) \otimes \cdots \otimes \mathbf{K}(r_n).$$

Let us describe the bigraded algebra $U(\mathbf{K})$ underlying $\mathbf{K}$. Underlying a building block $\mathbf{K}(r_i)$ is the bigraded algebra

$$U(\mathbf{K}(r_i; x_i)) = \Lambda(e_i) \mathbin{\tilde{\otimes}} R[x_i].$$

Note that the elements e_i and x_i *anti*commute (we have used this in the verification above). Moreover, we have the identifications (of bigraded algebras)

$$\begin{aligned}\Lambda(e_1) \mathbin{\tilde{\otimes}} \Lambda(e_2) &= \Lambda(e_1, e_2),\\ R[x_1] \mathbin{\tilde{\otimes}} R[x_2] &= R[x_1, x_2].\end{aligned}$$

Consequently, $U(\mathbf{K})$ is isomorphic to a twisted tensor product of an exterior and a polynomial algebra, concentrated on the y-axis and the secondary diagonal respectively, by means of a composition of symmetry isomorphisms. For $n = 2$, we have for instance

$$\begin{aligned}U(\mathbf{K}) &= U(\mathbf{K}(r_1)) \mathbin{\tilde{\otimes}} U(\mathbf{K}(r_2)) = \Lambda(e_1) \mathbin{\tilde{\otimes}} R[x_1] \mathbin{\tilde{\otimes}} \Lambda(e_2) \mathbin{\tilde{\otimes}} R[x_2]\\ &\cong \Lambda(e_1) \mathbin{\tilde{\otimes}} \Lambda(e_2) \mathbin{\tilde{\otimes}} R[x_1] \mathbin{\tilde{\otimes}} R[x_2] = \Lambda(e_1, e_2) \mathbin{\tilde{\otimes}} R[x_1, x_2],\end{aligned}$$

given on elements by

$$e_1^{j_1} \otimes x_1^{i_1} \otimes e_2^{j_2} \otimes x_2^{i_2} \longmapsto (-1)^{j_2 i_1} e_1^{j_1} \wedge e_2^{j_2} \otimes x_1^{i_1} x_2^{i_2},$$

for $j_1, j_2 \in \{0, 1\}$ and $i_1, i_2 \geqslant 0$. For arbitrary n, an isomorphism

$$U(\mathbf{K}) \cong \Lambda(e_1, \ldots, e_n) \mathbin{\tilde{\otimes}} R[x_1, \ldots, x_n] \tag{2.3}$$

is given by

$$e_1^{j_1} \otimes x_1^{i_1} \otimes \cdots \otimes e_n^{j_n} \otimes x_n^{i_n} \longmapsto (-1)^{(i,j)} e_1^{j_1} \wedge \cdots \wedge e_n^{j_n} \otimes x_1^{i_1} \cdots x_n^{i_n} \tag{2.4}$$

for $j_1, \ldots, j_n \in \{0, 1\}$, $i_1, \ldots, i_n \geqslant 0$, where (i, j) is defined as

$$(i, j) = j_2 i_1 + (j_3 i_2 + j_3 i_1) + \cdots + (j_n i_{n-1} + \cdots + j_n i_1).$$

Under this isomorphism, the diagonal line $p + q = k$ of $\mathbf{K}$ corresponds to the (right) $R[x_1, \ldots, x_n]$-submodule of $\Lambda(e_1, \ldots, e_n) \mathbin{\tilde{\otimes}} R[x_1, \ldots, x_n]$ generated by the homogeneous elements of degree k of $\Lambda(e_1, \ldots, e_n)$, for instance. We shall refer to elements of $\mathbf{K}$ by means of (2.3). The element $(e_1 \otimes 1) \otimes (1 \otimes x_2^4) \otimes (e_3 \otimes 1)$ for example is written as $-e_1 \wedge e_3 \otimes x_2^4$.

The column-wise filtration of $\mathbf{K}$ arises naturally as a filtration by ideals. Namely, define $F^s(\mathbf{K})$ for $s \geqslant 0$ to be the (left) ideal generated by J^s, where J is the maximal ideal $J = (x_1, \ldots, x_n) \lhd R[x_1, \ldots, x_n]$, so that

$$F^s(\mathbf{K}) = \mathbf{K} \cdot J^s.$$

We denote the quotients of $\mathbf{K}$ by these ideals by $\mathbf{K}/s$ and the components of the associated graded by $\mathbf{Q}^s$. We have

$$\mathbf{K}/s = \mathbf{K}/F^s(\mathbf{K}), \quad \mathbf{Q}^s = F^s(\mathbf{K})/F^{s+1}(\mathbf{K}).$$

Put more directly, $\mathbf{K}/s$ consists of the first s columns (with the differentials from $\mathbf{K}$) and $\mathbf{Q}^s$ of the sth column of $\mathbf{K}$; however, $\mathbf{Q}^s$ is still a double complex. Its condensation agrees with the sth column of $\mathbf{K}$ up to a shift, namely $(\mathbf{Q}^s)^\bullet = \Sigma^{-s}\mathbf{K}^s$.

Let us determine the homology of the columns and the rows of $\mathbf{K}$. The columns can be identified as direct sums of the Koszul complex $K(r_1, \ldots, r_n)$, indexed by the homogeneous monomials in $x_1, \ldots, x_n$ of degree s. Namely, we have

$$\mathbf{K}^s \cong \Sigma^s K \otimes J^s/J^{s+1}, \tag{2.5}$$

essentially by definition:

$$\begin{aligned}
\mathbf{K}^s &= \bigoplus_{i_1+\cdots+i_n=s} \mathbf{K}(r_1)^{i_1} \otimes \cdots \otimes \mathbf{K}(r_n)^{i_n} \\
&= \bigoplus_{i_1+\cdots+i_n=s} \Sigma^{i_1} K(r_1) \otimes R x_1^{i_1} \otimes \cdots \otimes \Sigma^{i_n} K(r_n) \otimes R x_n^{i_n} \\
&\cong \bigoplus_{i_1+\cdots+i_n=s} \Sigma^s K \otimes R x_1^{i_1} \otimes \cdots \otimes R x_n^{i_n} \\
&\cong \Sigma^s K \otimes J^s/J^{s+1}.
\end{aligned}$$

This implies that

$$H_p(\mathbf{K}^s) \cong \begin{cases} J^s/J^{s+1} & \text{if } p = s, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

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The rows of $\mathbf{K}(r_i)$ are all chain complexes of free R-modules; other than the zeroth one, which consists of R (concentrated in degree zero), they are exact. Therefore, using property (1.4), the Künneth theorem implies that

$$H_p(\mathbf{K}_t) = \begin{cases} R & \text{if } p = t = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Next, we define an augmentation on $\mathbf{K}^\bullet$. Note that $d_1 \colon \mathbf{K}_1^\bullet \to \mathbf{K}_0^\bullet$ is given by

$$d_1(e_j \otimes f) = r_j f - x_j f,$$

for a monomial $f \in R[x_1, \dots, x_n]$. Hence the evaluation map

$$\varepsilon \colon R[x_1, \dots, x_n] \longrightarrow R, \quad \varepsilon(x_i) = r_i$$

defines an augmentation. As ε is compatible with the filtrations given by powers of the ideals $J \lhd R[x_1, \dots, x_n]$ and $I \lhd R$ respectively, it induces augmentations

$$\varepsilon^s \colon (F^s(\mathbf{K}))_0^\bullet \longrightarrow I^s, \quad \varepsilon_s \colon (\mathbf{K}/s)_0^\bullet \longrightarrow R/I^s, \quad \varepsilon_{s+1}^s \colon (\mathbf{Q}^s)_0^\bullet \longrightarrow I^s/I^{s+1}$$

for the complexes $(F^s(\mathbf{K}))^\bullet$, $(\mathbf{Q}^s)^\bullet$, $(\mathbf{K}/s)^\bullet$ respectively.

Proposition 2.8 (Compare [**1**, Theorem 1.3].) *For $s \geqslant 1$, the differential graded algebras $(F^s(\mathbf{K}))^\bullet$, $(\mathbf{Q}^s)^\bullet = K \otimes J^s/J^{s+1}$ and $(\mathbf{K}/s)^\bullet$ provide R-free resolutions*

$$\begin{aligned} (F^s(\mathbf{K}))^\bullet &\xrightarrow{\varepsilon^s} I^s \longrightarrow 0, \\ (\mathbf{K}/s)^\bullet &\xrightarrow{\varepsilon_s} R/I^s \longrightarrow 0, \\ (\mathbf{Q}^s)^\bullet &\xrightarrow{\varepsilon_{s+1}^s} I^s/I^{s+1} \longrightarrow 0 \end{aligned}$$

of I^s, R/I^s and I^s/I^{s+1}, respectively.

Proof. Filtering the respective double complexes by rows gives rise to Cartan–Eilenberg spectral sequences [**5**, XV, §6]. They converge because the filtrations are bounded below and exhaustive, as the components of the condensation are defined as direct sums. See [**12**, 5.6]. The E^1-term, given by the homology of the columns, is concentrated on the diagonal $p+q=0$, as a consequence of (2.6).

Therefore, it only remains to check exactness of the complexes in degree zero. This is clear; the augmentations induce isomorphisms

$$\begin{aligned} H_0((F^s\mathbf{K})^\bullet) &= J^s/(x_1 - r_1, \dots, x_n - r_n) \xrightarrow{\cong} I^s, \\ H_0((\mathbf{K}/s)^\bullet) &= R[x_1, \dots, x_n]/(J^s, x_1 - r_1, \dots, x_n - r_n) \xrightarrow{\cong} R/I^s, \\ H_0((\mathbf{Q}^s)^\bullet) &= J^s/(J^{s+1}, x_1 - r_1, \dots, x_n - r_n) \xrightarrow{\cong} I^s/I^{s+1}. \end{aligned}$$ □

Remark 2.9. A similar argument shows that there are R-free resolutions of any subquotient of R of the form I^s/I^t, for $t > s \geqslant 0$, given by

$$(F^s(\mathbf{K})/F^t(\mathbf{K}))^\bullet \xrightarrow{\varepsilon_t^s} I^s/I^t \longrightarrow 0,$$

where ε_t^s is induced by ε.

REMARK 2.10. Note that the canonical inclusion $(\mathbf{Q}^s)^\bullet \to (\mathbf{K}/s)^\bullet$ covers (via the augmentations) the inclusion $I^s/I^{s+1} \to R/I^s$; similarly, the canonical projection $(F^s(\mathbf{K}))^\bullet \to (\mathbf{Q}^s)^\bullet$ covers the projection $I^s \to I^s/I^{s+1}$.

REMARK 2.11. We sketch how the complex $(\mathbf{K}/s)^\bullet$ can be constructed in a way that is explained in [**1**]. By definition, the underlying graded module of the condensation $(\mathbf{K}/s)^\bullet$ is the direct sum of the appropriately shifted columns of $\mathbf{K}/s$

$$\mathbf{K}^0 \oplus \Sigma^{-1}\mathbf{K}^1 \oplus \cdots \oplus \Sigma^{-(s-1)}\mathbf{K}^{s-1} = (\mathbf{Q}^0)^\bullet \oplus \cdots \oplus (\mathbf{Q}^{s-1})^\bullet.$$

To express the differentials, we interpret the components $d^h_{-k,*}$ of the horizontal differentials as chain maps

$$\partial^{(k)} : (\mathbf{Q}^k)^\bullet \to (\mathbf{Q}^{k+1})^\bullet$$

of degree -1. The differentials of $(\mathbf{K}/s)^\bullet$ are then given by

$$d(x_0, \ldots, x_{s-1}) = \big(d_0(x_0), d_1(x_1) + \partial^{(0)}(x_0), \ldots, d_{s-1}(x_{s-1}) + \partial^{(s-2)}(x_{s-2})\big),$$

where the d_k are the differentials of the complexes $(\mathbf{Q}^k)^\bullet$. The problem with this approach is that the expressions for the maps $\partial^{(k)}$ are rather unwieldy, which for instance makes the verfication that we actually do get a resolution already quite tedious.

REMARK 2.12. As remarked in the introduction, Tate describes a construction which yields some (non-explicit) multiplicative resolution $X_* \to R/I^s$ for Noetherian R [**11**, Theorem 1]. It will certainly be different from $(\mathbf{K}/s)^\bullet$, because it ends with $X_0 = R$, whereas $(\mathbf{K}/s)^\bullet_0 = R[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^s$. In contrast to $(\mathbf{K}/s)^\bullet$, the complex X_* will in most cases be unbounded, due to algebra generators in even degrees.

3. Computation of Tor-groups. Tensoring the Koszul resolution K with R/I kills all the differentials. Therefore Proposition 2.8 implies that

$$\mathrm{Tor}_*(R/I, I^s/I^{s+1}) = R/I \otimes K \otimes J^s/J^{s+1} \tag{3.1}$$

(recall that $J = (x_1, \ldots, x_n) \lhd R[x_1, \ldots, x_n]$), and we find the following well-known result.

PROPOSITION 3.2. *For $s \geqslant 0$, we have*

$$\mathrm{Tor}_*(R/I, I^s/I^{s+1}) = \Lambda_{R/I}(e_1, \ldots, e_n) \otimes J^s/J^{s+1}.$$

For $s = 0$, this is an identity of algebras.

REMARK 3.3. The statement for $s > 0$ follows in fact directly from the case $s = 0$, as a consequence of the splitting (of R-modules)

$$I^s/I^{s+1} \cong R/I \otimes J^s/J^{s+1} \cong \bigoplus_{x \in V_s} R/I\, x,$$

where V_s is the set of monomials of degree s in $x_1, \ldots, x_n$ [**10**, Theorem 16.2].

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It is convenient to introduce a reduced version of Tor for R/I^s. For this, let us first note that the projection $R \to R/I^s$ induces a split monomorphism

$$R/I \cong \mathrm{Tor}_*(R/I, R) \longrightarrow \mathrm{Tor}_*(R/I, R/I^s).$$

This can be seen as follows. Replace R/I and R/I^s by projective resolutions P and Q which are differential graded algebras, e.g. $P = (\mathbf{K}/1)^\bullet$ and $Q = (\mathbf{K}/s)^\bullet$. Then the map $\mathrm{Tor}_*(R/I, R) \to \mathrm{Tor}_*(R/I, R/I^s)$ is induced by $P \to P \otimes Q$ and split by $P \otimes Q \to P \otimes P \to P$, where the first map is the identity on P tensored with a lift of the projection $R/I^s \to R/I$. Defining

$$\widetilde{\mathrm{Tor}}_*(R/I, R/I^s) = \mathrm{coker}(R/I \longrightarrow \mathrm{Tor}_*(R/I, R/I^s)),$$

we therefore have

$$\mathrm{Tor}_*(R/I, R/I^s) \cong R/I \oplus \widetilde{\mathrm{Tor}}_*(R/I, R/I^s).$$

We need some notation. For $s \geqslant 0$, let ∂_s be the connecting homomorphism

$$\partial_s : \mathrm{Tor}_{*+1}(R/I, I^s/I^{s+1}) \longrightarrow \mathrm{Tor}_*(R/I, I^{s+1}/I^{s+2})$$

associated to the short exact sequence

$$0 \longrightarrow I^{s+1}/I^{s+2} \longrightarrow I^s/I^{s+2} \longrightarrow I^s/I^{s+1} \longrightarrow 0, \tag{3.4}$$

i_s the inclusion $I^s/I^{s+1} \to R/I^{s+1}$ and p_s the projection $I^s \to I^s/I^{s+1}$. We shall omit the index s if it is clear from the context. The projection $\mathrm{Tor}_*(R/I, R/I^s) \to \widetilde{\mathrm{Tor}}_*(R/I, R/I^s)$ is denoted by π.

THEOREM 3.5 ([**1**], Lemma 2.1, Proposition 2.2). *For $s > 1$, the sequence of graded R/I-modules*

$$\mathrm{Tor}_{*+1}(R/I, I^{s-2}/I^{s-1}) \xrightarrow{\partial} \mathrm{Tor}_*(R/I, I^{s-1}/I^s) \xrightarrow{\pi i_*} \widetilde{\mathrm{Tor}}_*(R/I, R/I^s) \longrightarrow 0$$

is exact; moreover, $\widetilde{\mathrm{Tor}}_*(R/I, R/I^s)$ is free over R/I. The product structure on $\widetilde{\mathrm{Tor}}_*(R/I, R/I^s)$, induced by the R-algebra structure on R/I^s, is trivial.

To simplify notation, we abbreviate $\mathrm{Tor}_*(R/I, -)$ to $H_*(-)$ in the following and refer to it as homology; similarly, $\widetilde{H}_*(-)$ stands for $\widetilde{\mathrm{Tor}}_*(R/I, -)$.

Proof. Making use of the free resolution $(\mathbf{K}/s)^\bullet$ of R/I^s constructed in the previous section, we can compute $H_*(R/I^s)$ as $H_*(R/I \otimes (\mathbf{K}/s)^\bullet)$. The filtration defined for $\mathbf{K}$ induces one on $\mathbf{K}/s$, of the form

$$0 = F^s \subseteq F^{s-1} \subseteq \cdots \subseteq F^0 = \mathbf{K}/s. \tag{3.6}$$

Setting $G^s = R/I \otimes (F^s)^\bullet$, this in turn gives rise to the filtration

$$0 = G^s \subseteq G^{s-1} \subseteq \cdots \subseteq G^0 = R/I \otimes (\mathbf{K}/s)^\bullet, \tag{3.7}$$

which is a filtration of the differential graded algebra $R/I \otimes (\mathbf{K}/s)^\bullet$ by ideals. It determines a Cartan–Eilenberg spectral sequence converging to $H_*(R/I^s)$, by the same argument as in the proof of Proposition 2.8. The E^1-term is given by the homology of the columns, which we have identified – up to a shift – as free resolutions of the quotients I^p/I^{p+1} and so $(E^1)^p_{*+p} = H_*(I^p/I^{p+1})$. The first differential is given by the horizontal differential and therefore (2.7) implies

$$(E^2)^p_* = \begin{cases} \operatorname{coker}((d^h)^{s-2}\colon (E^1)^{s-2}_* \longrightarrow (E^1)^{s-1}_*) & \text{if } p = s-1, \\ R/I & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For dimensional reasons, the spectral sequence collapses at E^2. Consequently, the composition

$$(E^2)^{s-1}_{*+s-1} = (E^\infty)^{s-1}_{*+s-1} = H_*(G^{s-1}) \hookrightarrow H_*(R/I^s) \xrightarrow{\pi} \widetilde{H}_*(R/I^s)$$

of the edge homomorphism with the projection π is an isomorphism, so that we have an exact sequence

$$(E^1)^{s-2}_{*+s-1} \longrightarrow (E^1)^{s-1}_{*+s-1} \longrightarrow \widetilde{H}_*(R/I^s) \longrightarrow 0.$$

Remark 2.10 shows that the map $(E^1)^{s-1}_{*+s-1} \to \widetilde{H}_*(R/I^s)$ can be identified with $\pi i_*\colon H_*(I^{s-1}/I^s) \to \widetilde{H}_*(R/I^s)$.

It remains to identify $R/I \otimes d^h\colon (E^1)^{s-2}_* \to (E^1)^{s-1}_*$ as a connecting homomorphism. The free resolutions of the terms in the short exact sequence

$$0 \longrightarrow I^{s-1}/I^s \longrightarrow I^{s-2}/I^s \longrightarrow I^{s-2}/I^{s-1} \longrightarrow 0,$$

described in Proposition 2.8 and Remark 2.9, fit into a short exact sequence of chain complexes

$$0 \longrightarrow (\mathbf{Q}^{s-1})^\bullet \longrightarrow (F^{s-2}(\mathbf{K})/F^s(\mathbf{K}))^\bullet \longrightarrow (\mathbf{Q}^{s-2})^\bullet \longrightarrow 0.$$

Going through the definition of the connecting homomorphism, we find that ∂_{s-2} is given by $R/I \otimes (d^h)^{s-2}$.

For the second statement, it suffices to observe that the image of ∂ is a free submodule of the free R/I-module $H_*(I^{s-1}/I^s)$. This is true because $\partial = R/I \otimes d^h$ maps basis elements of $R/I \otimes \mathbf{K}$ to sums of basis elements.

For the determination of the multiplicative structure of $H_*(R/I^s)$, recall from Section 1 that the product on $\mathbf{K}/s$ induces one on $(\mathbf{K}/s)^\bullet$ and hence on $H_*(R/I^s) = H_*(R/I \otimes (\mathbf{K}/s)^\bullet)$. The latter one indeed *is* the canonical internal product defined on $H_*(R/I^s)$. See [**9**, Corollary VIII.2.3]. As the product on $\mathbf{K}/s$ is compatible with the filtration (3.6), so are the induced ones on $R/I \otimes (\mathbf{K}/s)^\bullet$ and on $H_*(R/I^s)$. Now we have seen above that $\widetilde{H}_*(R/I^s)$ is concentrated in $H_*(G_{s-1})$, on which the product is trivial. □

We can express $H_*(I^s)$ in a similar manner. The proof is completely analogous.

THEOREM 3.8. *For $s \geqslant 0$, there is an exact sequence*

$$0 \longrightarrow \operatorname{Tor}_*(R/I, I^s) \xrightarrow{p_*} \operatorname{Tor}_*(R/I, I^s/I^{s+1}) \xrightarrow{\partial} \operatorname{Tor}_{*-1}(R/I, I^{s+1}/I^{s+2});$$

moreover, $\operatorname{Tor}_*(R/I, I^s)$ *is free over R/I. The product structure on* $\operatorname{Tor}_*(R/I, I^s)$*, induced by the R-algebra structure on I^s, is trivial.*

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REMARK 3.9. In particular, we have $R/I \otimes I^s \cong I^s/I^{s+1}$.

It is quite easy to make the connecting homomorphism

$$\partial\colon H_{*+1}(I^s/I^{s+1}) \to H_*(I^{s+1}/I^{s+2})$$

explicit. In the following, we slightly abuse notation and denote basis elements of

$$\bigoplus_{s\geqslant 0} H_*(I^s/I^{s+1}) = \bigoplus_{s\geqslant 0} \Sigma^{-s}(R/I \otimes \mathbf{K})^s$$

(see (3.1)) as expressions in the variables e_i and x_i, which are in fact generators of the components of $\mathbf{K}$. By means of illustration, consider first $\partial\colon H_{*+1}(R/I) \to H_*(I/I^2)$. The unit is mapped to zero and the elements e_i to x_i. For the elements of higher degree, we have

$$\begin{aligned}\partial(e_1 \wedge e_2) &= -e_2 \otimes x_1 + e_1 \otimes x_2,\\ \partial(e_1 \wedge e_2 \wedge e_3) &= e_2 \wedge e_3 \otimes x_1 - e_1 \wedge e_3 \otimes x_2 + e_1 \wedge e_2 \otimes x_3\end{aligned}$$

and so on. Considering $R/I \otimes \mathbf{K}$ under the isomorphism (2.3) as a right module over $R/I[x_1, \ldots, x_n]$, ∂ is a linear map. We have for instance

$$\partial(e_1 \wedge e_2 \otimes x_1) = -e_2 \otimes x_1^2 + e_1 \otimes x_1x_2.$$

In the general case, we have the following formula.

PROPOSITION 3.10. *The map* $\partial\colon H_{*+1}(I^s/I^{s+1}) \to H_*(I^{s+1}/I^{s+2})$ *is given by*

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_l} \otimes f) = \sum_{j=1}^{l} (-1)^{j+l}\, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_l} \otimes x_{i_j} f,$$

where $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\}$, f *is a monomial in* $x_1, \ldots, x_n$ *of degree* s *and the hat indicates that the entry underneath should be omitted.* □

REMARK 3.11. Identifying $H_*(I^s)$ with its image under p_* and recalling that $\ker \partial_s = \operatorname{im} \partial_{s-1}$ for $s \geqslant 1$, the proposition gives an explicit description of a basis of $H_*(I^s)$.

REMARK 3.12. Because all the Tor-groups we have computed are free R/I-modules, we have the Ext-groups for free, as a consequence of the following fact. If A is an R-module such that $\operatorname{Tor}_*^R(R/I, A)$ is free over R/I, there is a duality isomorphism

$$\operatorname{Ext}_R^*(A, R/I) \cong \operatorname{Hom}_{R/I}^*(\operatorname{Tor}_*^R(R/I, A), R/I).$$

It arises as an edge homomorphism of a Cartan–Eilenberg spectral sequence, which collapses under this condition. See [**5**, XVI, §6, Case 3].

4. An exterior multiplication. As mentioned in the introduction, we aim to study $\bigoplus_{s\geqslant 0} \operatorname{Tor}_*(R/I, I^s)$ as a bigraded R/I-algebra. We abbreviate $\operatorname{Tor}_*(R/I, -)$ by $H_*(-)$, as before.

https://doi.org/10.1017/S0017089504002009 Published online by Cambridge University Press

Ordinary ring multiplication induces pairings $I^s \otimes I^t \to I^{s+t}$. Taken all together, these give rise to a graded R-algebra structure on

$$B_I^*(R) = R \oplus I \oplus I^2 \oplus \cdots$$

called the blowup algebra of I in R in [**6**, 5.2]. On the other hand, we have the graded ring associated to the I-adic filtration

$$\mathrm{gr}_I^*(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

These product structures induce exterior multiplications on homology, giving both $H_*(B_I^*(R))$ and $H_*(\mathrm{gr}_I^*(R))$ the structure of bigraded R/I-algebras. We may compute these using the multiplicative double complexes deduced from $\mathbf{K}$. Proposition 3.2 immediately implies the following result.

PROPOSITION 4.1. *There is an isomorphism of bigraded algebras*

$$\mathrm{Tor}_*(R/I, \mathrm{gr}_I^*(R)) \cong \Lambda_{R/I}(e_1, \ldots, e_n) \mathbin{\widetilde{\otimes}} R/I[x_1, \ldots, x_n].$$

The projections $I^s \to I^s/I^{s+1}$ induce a map of graded R-algebras

$$p\colon B_I^*(R) \longrightarrow \mathrm{gr}_I^*(R).$$

On Tor this induces, by Theorem 3.8, a monomorphism of bigraded R/I-algebras

$$p_*\colon H_*(B_I^*(R)) \longrightarrow H_*(\mathrm{gr}_I^*(R)). \tag{4.2}$$

We identify $H_*(B_I^*(R))$ with its image under p_* in the following. By Remark 3.9,

$$\mathrm{Tor}_0(R/I, B_I^*(R)) \cong R/I[x_1, \ldots, x_n].$$

Hence we can consider $H_*(B_I^*(R))$ as a (bigraded) algebra over $R/I[x_1, \ldots, x_n]$. Note that $R/I[x_1, \ldots, x_n]$ is *not* contained in the centre of $H_*(B_I^*(R))$.

PROPOSITION 4.3. *Over* $R/I[x_1, \ldots, x_n]$, *the bigraded algebra* $\mathrm{Tor}_*(R/I, B_I^*(R))$ *is generated by the basis elements*

$$a_{(i_0,\ldots,i_k)} = \partial(e_{i_0} \wedge \cdots \wedge e_{i_k})$$

of $\mathrm{Tor}_k(R/I, I)$ *for* $0 < k < n$, *where* $\{i_0, \ldots, i_k\}$ *runs through the subsets of* $\{1, \ldots, n\}$ *of cardinality* $k+1$ *and we assume that* $1 \leqslant i_0 < \cdots < i_k \leqslant n$.

Proof. It is clear from the short exact sequence $0 \to I \to R \to R/I \to 0$ and Proposition 3.2 that the $a_{(i_0,\ldots,i_k)}$ defined in the statement generate $H_*(I)$ as an R/I-module. Now we claim that a set of generators of the sth column $H_*(I^s)$ of $H_*(B_I^*(R))$ is given by multiplying all monomials of degree s in $x_1, \ldots, x_n$ with all the $a_{(i_0,\ldots,i_k)}$. Namely, we know that, for $s > 0$,

$$\begin{aligned} H_*(I^s) &\cong \ker(\partial_s\colon H_*(I^s/I^{s+1}) \longrightarrow H_{*-1}(I^{s+1}/I^{s+2})) \\ &= \mathrm{im}(\partial_{s-1}\colon H_{*+1}(I^{s-1}/I^s) \longrightarrow H_*(I^s/I^{s+1})). \end{aligned}$$

The first equality is Theorem 3.8 and the second is equation (2.7) together with the recognition of $R/I \otimes d^h$ as the connecting homomorphism ∂ in the proof of Theorem 3.5. Also Proposition 3.10 implies that $\mathrm{im}\,\partial_{s-1} = J^{s-1}/J^s \cdot \mathrm{im}\,\partial_0$. $\square$

We finish by giving some examples for small n; we abbreviate $H_*(B_I^*(R))$ by A, R/I by E and $R/I[x_1, \ldots, x_n]$ by P.

- $n=1$: Clearly, we have $A \cong P = E[x]$.
- $n=2$: The basis element $a_{12} = -e_2x_1 + e_1x_2$ of $H_1(I)$ generates a free copy of P. More precisely, $A \cong \Lambda_E(a_{12}) \mathbin{\widetilde{\otimes}} P$, as bigraded algebras.
- $n=3$: Among the basis elements a_{12}, a_{13} and a_{23} we have the relation

$$x_1 \cdot a_{23} + x_2 \cdot a_{13} + x_3 \cdot a_{12} = 0$$

over P. As an example of a product, we have $a_{12} \cdot a_{23} = -x_2 \cdot a_{123}$.

- $n=4$: In addition to four relations of the type above, there is also

$$x_1 \cdot a_{234} + x_2 \cdot a_{134} + x_3 \cdot a_{124} + x_4 \cdot a_{123} = 0.$$

As a product, we have $a_{123} \cdot a_{234} = x_2x_3 \cdot a_{1234}$.

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