Bull. Aust. Math. Soc. **110** (2024), 158–160
doi:10.1017/S0004972723001120

ANY DUAL OPERATOR SPACE IS WEAKLY LOCALLY REFLEXIVE

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(Received 10 October 2023; accepted 16 October 2023; first published online 12 December 2023)

Abstract

We introduce the notion of weakly local reflexivity in operator space theory and prove that any dual operator space is weakly locally reflexive.

2020 *Mathematics subject classification*: primary 46L07.

Keywords and phrases: dual operator space, weakly local reflexivity.

1. Introduction

The theory of operator spaces is a natural noncommutative quantisation of Banach space theory. Many problems in operator spaces are naturally motivated from both Banach space theory and operator algebra theory. Some properties such as local reflexivity, exactness, nuclearity and injectivity have been intensively studied (see [5, 6, 10]). In particular, for any operator space V,

$$V \text{ is nuclear} \Rightarrow V \text{ is exact} \Rightarrow V \text{ is locally reflexive.}$$

The first implication was proved in [10] and the second in [6]. In [6], Effros *et al.* showed that an operator space V is nuclear if and only if V is locally reflexive and V^{**} is injective. As pointed out in [6], local reflexivity is an essential condition in this result since Kirchberg [8] had constructed a separable nonnuclear operator space V for which $V^{**} = \prod_{n=1}^{+\infty} M_n$. Turning to C^*-algebra theory, using Conne's deep work in [3], Choi and Effros proved the following result in [1, 2]:

$$\text{A } C^*\text{-algerba } \mathcal{A} \text{ is nuclear} \Leftrightarrow \text{its second dual } \mathcal{A}^{**} \text{ is injective.}$$

In [4], Dong and Ruan showed that an operator space V is exact if and only if V is locally reflexive and V^{**} is weak* exact.

In [5], Effros *et al.* used the technique of mapping spaces to prove the most surprising result in the theory of operator spaces: the dual $\mathcal{A}^*$ of any C^*-algebra $\mathcal{A}$

Project partially supported by the National Natural Science Foundation of China (No. 11871423) and Zhejiang Provincial Natural Science Foundation of China (No. LQ21A010015).

is locally reflexive. In light of the fact that C^*-algebras need not be locally reflexive, it was thought that the same would be true for their dual operator spaces. It therefore came as quite a surprise to find that all such dual spaces, as well as all von Neumann algebraic preduals, are locally reflexive. In this short paper, we introduce the notion of weakly local reflexivity in operator space theory. We prove that any dual operator space is weakly locally reflexive.

2. Weakly local reflexivity

We first recall the definition of local reflexivity in operator space theory (see [7]).

DEFINITION 2.1. Suppose that V is an operator space. We say that V is locally reflexive if for any finite dimensional operator space L, every complete contraction $\varphi : L \to V^{**}$ is the point-weak* limit of a net of complete contractions $\varphi_\alpha : L \to V$.

DEFINITION 2.2. We say that a dual operator space V^* is weakly locally reflexive if for any finite dimensional operator space L and every complete contraction $\varphi : L \to V^{***}$, there exists a net of complete contractions $\varphi_\alpha : L \to V^*$ such that

$$\langle \varphi_\alpha(x), f\rangle \longrightarrow \langle \varphi(x), f\rangle \quad \text{for all } x \in L, f \in V.$$

It is well known that $\mathcal{B}(\mathcal{H})$ is not locally reflexive for any infinite dimensional Hilbert space $\mathcal{H}$. However, the following result implies that $\mathcal{B}(\mathcal{H})$ is weakly locally reflexive.

THEOREM 2.3. *Any dual operator space V^* is weakly locally reflexive.*

PROOF. For any finite dimensional subspaces $E \subseteq V^{***}$ and $F \subseteq V \subseteq V^{**}$, it follows from [7, Lemma 14.3.4] that for each $n \in \mathbb{N}$, we can find a mapping $\psi^{(n)} : E \to V^*$ such that $\|(\psi^{(n)})_n\| < 1 + 1/n$ and

$$\langle \psi^{(n)}(x), f\rangle = \langle x, f\rangle \quad \text{for all } x \in E, f \in F.$$

Thus, $\{\psi^{(n)}\}$ is a sequence in the closed ball of radius 2 of $B(E, V^*) = (E \overset{\gamma}{\otimes} V)^*$. From Alaoglu's theorem and [9, Lemma 7.2], we may choose a limit point $\psi : E \to V^*$ of the sequence $\{\psi^{(n)}\}$ in the point-weak* topology. If $r \leq n$, then

$$\|(\psi^{(n)})_r\| \leq \|(\psi^{(n)})_n\| \leq 1 + \frac{1}{n}$$

and thus $\|\psi_r\| \leq 1$. It follows that $\|\psi\|_{cb} \leq 1$. Furthermore,

$$\langle \psi(x), f\rangle = \langle x, f\rangle \quad \text{for all } x \in E, f \in F.$$

Now for any finite dimensional operator space L and every complete contraction $\varphi : L \to V^{***}$, we fix $E = \varphi(L) \subseteq V^{***}$. For any finite dimensional subspaces $F \subseteq V$, it follows from the above proof that there exist complete contractions $\psi_F : E \to V^*$ such that

$$\langle \psi_F(y), f\rangle = \langle y, f\rangle \quad \text{for all } y \in E = \varphi(L), f \in F.$$

Thus, the net $\psi_F \circ \varphi : L \to V^*$ satisfies

$$\langle \psi_F \circ \varphi(x), f \rangle \longrightarrow \langle \varphi(x), f \rangle \quad \text{for all } x \in L, f \in V,$$

with $\|\psi_F \circ \varphi\|_{cb} \leq 1$. This implies that the dual operator space V^* is weakly locally reflexive. □

References

[1] M.-D. Choi and E. G. Effros, 'Separable nuclear C^*-algebras and injectivity', *Duke Math. J.* **43** (1976), 309–322.

[2] M.-D. Choi and E. G. Effros, 'Nuclear C^*-algebras and injectivity: The general case', *Indiana Univ. Math. J.* **26** (1977), 443–446.

[3] A. Connes, 'Classification of injective factors', *Ann. of Math. (2)* **104** (1976), 585–609.

[4] Z. Dong and Z.-J. Ruan, 'Weak* exactness for dual operator spaces', *J. Funct. Anal.* **253** (2007), 373–397.

[5] E. G. Effros, M. Junge and Z.-J. Ruan, 'Integral mappings and the principle of local reflexivity for non-commutative L^1-spaces', *Ann. of Math. (2)* **151** (2000), 59–92.

[6] E. G. Effros, N. Ozawa and Z.-J. Ruan, 'On injectivity and nuclearity for operator spaces', *Duke Math. J.* **110** (2001), 489–521.

[7] E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Mathematical Society Monographs, New Series, 23 (The Clarendon Press, Oxford–New York, 2000).

[8] E. Kirchberg, 'On subalgebras of the CAR-algebra', *J. Funct. Anal.* **129** (1995), 35–63.

[9] V. Paulsen, *Completely Bounded Maps and Operator Algebras* (Cambridge University Press, Cambridge, 2002).

[10] G. Pisier, 'Exact operator spaces', *Astérisque* **232** (1995), 159–186.

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