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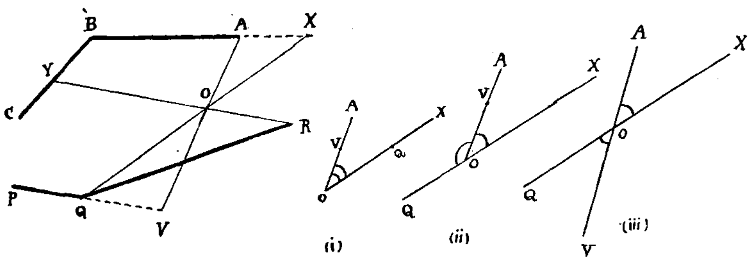
Two Extensions of Ceva's Theorem.

1. If $ABCDE$ ... is a polygon of an odd number of sides, O any point in its plane, and if AO, BO, CO, ... cut the sides opposite A, B, C, ... respectively, AB in X, BC in Y, CD in Z, ... then

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \ldots = +1.$$



Let PQ be the side opposite A. Then AB will be the side opposite Q.

Let AO cut PQ in V, QO cut AB in X, RO cut BC in Y. Then according as V and A, X and Q are on the same or different sides of O, $A\widehat{O}X$ will be identical with, supplementary, or vertically opposite to $V\widehat{O}Q$. In any case, $\sin AOX = \sin VOQ$.

$$\text{Now} \quad \frac{AX}{XB} = \frac{\triangle AOX}{\triangle XOB} = \frac{OA \,.\sin AOX}{OB \,.\sin XOB},$$

$$\frac{BY}{YC} = \frac{OB \,.\sin BOY}{OC \sin YOC},$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{PV}{VQ} = \frac{OP \,.\sin POV}{OQ \,.\sin VOQ},$$

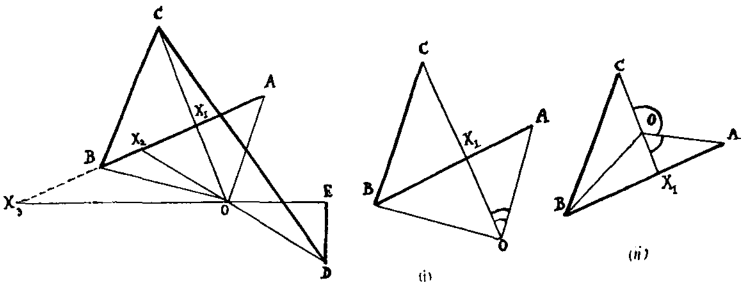
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots, \text{ etc.}$$

In the continued product OB and sin AOX in the numerator cancel OB and sin VOQ in the denominator, and so on, so that

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdots = +1.$$

2 If $ABCDE \ldots HK$ be a polygon of n sides, O any point in its plane, and if CO, DO, EO, ... KO cut AB in the $n-2$ points X_1, X_2, X_3, ... X_{n-2}, and DO, EO, ... AO cut BC in the $n-2$ points Y_1, Y_2, ... Y_{n-2}, ..., and lastly if BO, CO, ... HO cut KA in V_1, V_2, ... V_{n-2}, then

$$\left(\frac{AX_1}{X_1B} \cdot \frac{AX_2}{X_2B} \cdot \frac{AX_3}{X_3B} \cdots \frac{AX_{n-2}}{X_{n-2}B}\right)\left(\frac{BY_1}{Y_1C} \cdot \frac{BY_2}{Y_2C} \cdots \frac{BY_{n-2}}{Y_{n-2}C}\right)$$
$$\left(\ldots\right)\left(\frac{KV_1}{V_1A} \cdots \frac{KV_{n-2}}{V_{n-2}A}\right) = +1.$$



For $A\widehat{O}C$ is identical with or supplementary to $A\widehat{O}X_1$.

$$\therefore \quad \sin AOX_1 = \sin AOC.$$

$$\text{Now} \quad \frac{AX_1}{X_1B} = \frac{OA \, . \sin AOX_1}{OB \, . \sin X_1OB} = \frac{OA \, . \sin AOC}{OB \, . \sin COB}.$$

So for all the other ratios.

$$\therefore \text{ the continued product} = \left[\frac{OA}{OB} \cdot \frac{OB}{OC} \cdots \frac{OK}{OA}\right]^{n-2} \times$$

$$\left[\left(\frac{\sin AOC}{\sin COB} \cdot \frac{\sin AOD}{\sin DOB} \cdots \frac{\sin AOK}{\sin KOB}\right)\left(\frac{\sin BOD}{\sin DOC} \cdot \frac{\sin BOE}{\sin EOC} \cdots\right)\left(\ldots\right) \times \right.$$
$$\left.\left(\frac{\sin KOB}{\sin BOA} \cdots \frac{\sin KOH}{\sin HOA}\right)\right] = +1.$$

When $n = 3$, each of the above theorems reduces to Ceva's Theorem.

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