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INVERSION OF OPERATOR PENCILS ON HILBERT SPACE

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Abstract

We consider a linear operator pencil with complex parameter mapping one Hilbert space onto another. It is known that the resolvent is analytic in an open annular region of the complex plane centred at the origin if and only if the coefficients of the Laurent series satisfy a doubly-infinite set of left and right fundamental equations and are suitably bounded. If the resolvent has an isolated singularity at the origin we propose a recursive orthogonal decomposition of the domain and range spaces that enables us to construct the key nonorthogonal projections that separate the singular and regular components of the resolvent and subsequently allows us to find a formula for the basic solution to the fundamental equations. We show that each Laurent series coefficient in the singular part of the resolvent can be approximated by a weakly convergent sequence of finite-dimensional matrix operators and we show how our analysis can be extended to find a global expression for the resolvent of a linear pencil in the case where the resolvent has only a finite number of isolated singularities.

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1. Introduction

We wish to calculate the generalized resolvent for a linear pencil that is singular in the unperturbed state but is nonsingular for small perturbations. Our main aim is to reconcile two different methods that have been used to define the resolvent operator. The first is an indirect method based on a system of fundamental equations. This method is valid for linear pencils that map one Banach space onto another. The second uses a sequence of unitary transformations to progressively reduce the original resolvent problem to an equivalent problem on a smaller space. This method is valid for linear pencils that map one Hilbert space onto another.

1.1. Applications. We outline one application and cite two others.

1.1.1. Input retrieval in linear control systems. For a general discussion about semigroups, we refer to the classic texts by Kato [12] and Yosida [21]. The following description was originally published in [10] and can also be found in [3, pages 261–262]. Let H be a Banach space and let $A \in \mathcal{B}(H)$ be a bounded linear map on H. Suppose that there exists some $\omega > 0$ and further suppose that for each ϵ with $0 < \epsilon < \omega$ we can find $M_\epsilon > 0$ such that

$$\|(sI - A)^{-1}\| \leq \frac{M_\epsilon}{|s|}$$

for all $s \in \mathbb{C}$ with $|\arg s| < \pi/2 + \omega - \epsilon$. Then A generates a bounded holomorphic semigroup e^{At} in the region $|\arg t| < \omega$ and the resolvent of A is given by the formula

$$(sI - A)^{-1} = \int_0^\infty e^{-st} \cdot e^{At} \cdot dt$$

for $s \in \mathbb{C}$ with real part $\Re(s) > 0$. Thus, the resolvent of A can be interpreted as the Laplace transform of the semigroup generated by A. The theory of one-parameter semigroups is described clearly and concisely in Kato [12, pages 479–495]. The integral in the above expression is a Bochner integral. For more information about the Bochner integral, consult Yosida [21, pages 132–135]. If $r_\sigma > 0$ is the spectral radius of A, then Yosida [21, Theorem 3, page 211] showed that

$$(sI - A)^{-1} = \frac{1}{s}\left[I + \frac{A}{s} + \left(\frac{A}{s}\right)^2 + \cdots\right]$$

for all $s \in \mathbb{C}$ with $|s| > r_\sigma$. Now suppose that G and K are Banach spaces and that $B \in \mathcal{B}(G, H)$ and $C \in \mathcal{B}(H, K)$ are bounded linear transformations. Let $u : [0, \infty) \mapsto G$ be an analytic function defined by

$$u(t) = u_0 + u_1 t + \frac{u_2 t^2}{2!} + \cdots$$

for all $t \in [0, \infty)$, where $\{u_j\} \subset G$ and $\|u_j\| \leq a^{j+1}$ for some $a \in \mathbb{R}$ with $a > 0$. The Laplace transform of u will be

$$U(s) = \frac{1}{s}\left[u_0 + \frac{u_1}{s} + \frac{u_2}{s^2} + \cdots\right]$$

for $|s| > a$. We will consider an infinite-dimensional linear control system

$$x' = Ax + Bu,$$
$$y = Cx,$$

where $u = u(t)$ is the input, $x = x(t)$ is the state and $y = y(t)$ is the output. We assume that the system is initially at rest with $x(0) = 0$. If the input to the system is assumed

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to be analytic (as described above), it follows from Kato [12, Theorem 1.27, pages 493–494] that the output from the system is determined by the formula

$$y(t) = \int_0^t Ce^{A(t-\tau)}B \cdot u(\tau)\, d\tau$$

or equivalently by the corresponding Laplace transform formula

$$Y(s) = C(sI - A)^{-1}B \cdot U(s).$$

The latter formula will be well defined in the region $|s| > \max[r_\sigma, a]$ by the series expansion

$$Y(s) = \frac{1}{s}\Big[CB + \frac{CAB}{s} + \frac{CA^2B}{s^2} + \cdots\Big] \cdot U(s).$$

Thus, the problem of input retrieval can be formulated as a power series inversion problem with

$$U(s) = s \cdot \Big[CB + \frac{CAB}{s} + \frac{CA^2B}{s^2} + \cdots\Big]^{-1} \cdot Y(s).$$

If we write $z = 1/s$ and define $A_0 = CB$ and $A_1 = CAB$, then we can certainly find the desired inverse operator if we can find an expression for $R(z) = (A_0 + A_1z)^{-1}$ in some region $0 < |z| < r$. We are particularly interested in the case where $A_0 = CB$ is singular.

1.1.2. Singularly perturbed Markov processes. The resolvent operator is used to calculate mean first passage times in the theory of singularly perturbed Markov processes. We refer to [3, pages 262–267] for two particular examples. For an extended discussion of perturbed Markov processes, see [3, Ch. 6, pages 151–208 and Ch. 7, pages 209–244]. See also [5].

1.1.3. The generalized Sylvester equation. Let F, G, H and K be Banach spaces, let $A_0, A_1 \in \mathcal{B}(F, G)$ and $B_0, B_1 \in \mathcal{B}(H, K)$ and define linear pencils $A(z) = A_0 + A_1z$ and $B(z) = B_0 + B_1z$ for all $z \in \mathbb{C}$. Let $\rho(A_0, A_1) = \{z \in \mathbb{C} \mid R(z) = A(z)^{-1} \in \mathcal{B}(G, F)\}$ and $\rho(B_0, B_1) = \{z \in \mathbb{C} \mid S(z) = B(z)^{-1} \in \mathcal{B}(K, H)\}$ denote the resolvent sets for $A(z)$ and $B(z)$ respectively and let $\sigma(A_0, A_1) = \mathbb{C} \setminus \rho(A_0, A_1)$ and $\sigma(B_0, B_1) = \mathbb{C} \setminus \rho(B_0, B_1)$ denote the corresponding spectral sets. If $\sigma(A_0, A_1) \cap \sigma(B_0, B_1) = \emptyset$, then for all $C \in \mathcal{B}(F, K)$ the generalized Sylvester equation

$$A_1XB_0 - A_0XB_1 = C$$

has a unique solution $X \in \mathcal{B}(G, H)$. More precisely, if Γ is a Cauchy contour with $\sigma(A_0, A_1)$ included in the interior domain of Γ and $\sigma(B_0, B_1)$ included in the exterior domain of Γ, then

$$X = \frac{1}{2\pi i}\int_\Gamma R(z)CS(z)\, dz.$$

For a full treatment, readers are referred to Gohberg *et al.* [8, pages 54–56].

1.2. Organization of the paper. The paper is organized as follows. In Section 2 we outline the necessary prerequisite material and discuss some of the basic ideas. We review the relevant literature in Section 3. There is a vast literature on this topic and hence we have restricted our review to those papers that are directly relevant to our discussion. For each of these papers we have outlined the relevant contributions. In Section 4 we present the new results. We show how the new results are related to the fundamental equations in Section 5. We illustrate our results in Section 6 with a particular example based on a speculative model of decay and growth. We discuss matters relating to numerical calculations in Section 7 and conclude with a brief summary in Section 8.

2. Preliminaries

We wish to reconcile two seemingly unrelated methods that have been used to define the resolvent operator $R(z) = A(z)^{-1} = (A_0 + A_1 z)^{-1}$ when the unperturbed operator $A(0) = A_0$ is singular. The first method is quite general but is essentially indirect and uses a doubly-infinite system of fundamental equations to define the resolvent. The results are valid in Banach space but the method is nonconstructive and there is currently no general procedure to solve the fundamental equations. The second method is less general but more direct and uses a sequence of unitary transformations to progressively reduce the original problem to an equivalent problem on a smaller space. If the process terminates after a finite number of steps, then the inversion can be completed using a Neumann expansion. The method is restricted to Hilbert space but is constructive and is amenable to numerical calculation provided the procedure terminates after a finite number of steps.

The main difficulty with the fundamental equations in their most general form is that the left and right systems are each doubly-infinite. If the systems have a unique solution, it is known that there exist corresponding key projection operators that separate each system into two singly-infinite subsystems. Each subsystem can then be solved recursively. Thus, the problem of calculating the resolvent reduces to one of finding the key projection operators.

In this paper we restrict our attention to Hilbert spaces. Our aim is to use the progressive reduction procedure and the associated orthogonal projections to find a formula for the key nonorthogonal projections that separate the fundamental equations into singly-infinite systems. We are particularly interested in the case where the reduction continues *ad infinitum*.

2.1. Prerequisite theory. The proposed reduction procedure relies on one problem-specific result and two general results.

Let H and K be Hilbert spaces with $A_0, A_1 \in \mathcal{B}(H, K)$ and $A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$. Suppose that there is some $r > 0$ such that $R(z) = A(z)^{-1}$ is analytic for all $z \in \mathcal{U}_{0,r} = \{z \in \mathbb{C} \mid 0 < |z| < r\}$. Let $H_1 = A_0^{-1}(\{\mathbf{0}\})$ be the null space of A_0 and define $K_1 = A_1(H_1) \subseteq K$ to be the image of H_1 under A_1. In order to show that $H \cong H_1 \times H_1^{\perp}$ and $K \cong K_1 \times K_1^{\perp}$, we must prove that H_1 and K_1 are closed subspaces. Since H_1 is the null space of A_0,

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it must be closed. To show that K_1 is closed, the following lemma is crucial. The proof can be found in [11] or in [3, page 271].

Lemma 2.1. *Suppose that* $H_1 = A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$*. Define* $K_1 = A_1(H_1) \subseteq K$*. If* $R(z_0) = A(z_0)^{-1} \in \mathcal{B}(K, H)$ *is well defined for some* $z_0 \in \mathbb{C}$ *with* $z_0 \neq 0$*, then* A_1 *is bounded below on* H_1 *and* K_1 *is a closed subspace.* □

The Banach inverse theorem [15, pages 149–150] justifies the existence of a bounded linear inverse operator. The theorem is used repeatedly during the reduction procedure.

Theorem 2.2 (Banach inverse theorem). *Let* X *and* Y *be Banach spaces and suppose that* $A \in \mathcal{B}(X, Y)$ *is a bounded linear operator. If* $A(X) = Y$ *and if* $A^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}$*, then there exists* $\epsilon > 0$ *such that* $\|A\boldsymbol{x}\| > \epsilon\|\boldsymbol{x}\|$ *for all* $\boldsymbol{x} \in X$ *and the inverse operator* $A^{-1} \in \mathcal{B}(Y, X)$ *is well defined.* □

Zorn's lemma [21, pages 2–3] guarantees the existence of a maximal element in a partially ordered set. The lemma is used to justify the existence of a maximal orthogonal projection. In this regard we need firstly to define a partial ordering on the set of all projections. If X is a Banach space and $E_1, E_2 \in \mathcal{B}(X)$ are projections, then we say that $E_1 \preceq E_2$ if $E_1E_2 = E_2E_1 = E_1$. See [6, pages 481–482] for more discussion.

Lemma 2.3 (Zorn's lemma). *Let* P *be a nonempty partially ordered set with the property that every linearly ordered subset of* P *has an upper bound in* P*. Then* P *contains at least one maximal element.*

2.2. The fundamental equations. We follow the presentation in [1]. Let H, K be complex Banach spaces and let $A_0, A_1 \in \mathcal{B}(H, K)$ be bounded linear operators, where A_0 is singular. Note that A_0 is nonsingular if and only if $A_0(H) = K$ and $A_0^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}$. Define a linear operator pencil $A : \mathbb{C} \to \mathcal{B}(H, K)$ by the formula $A(z) = A_0 + A_1 z$ and suppose that the resolvent $R : \mathcal{U}_{s,r} \to \mathcal{B}(K, H)$ defined by $R(z) = A(z)^{-1}$ is analytic for $z \in \mathcal{U}_{s,r} = \{z \in \mathbb{C} \mid s < |z| < r\}$, where $s, r \in \mathbb{R}$ and satisfy $0 \le s < r \le \infty$. Hence, the resolvent can be represented on the annular region $\mathcal{U}_{s,r}$ by a Laurent series $R(z) = \sum_{j\in\mathbb{Z}} R_j z^j$, where $R_j \in \mathcal{B}(K, H)$ for all $j \in \mathbb{Z}$. Furthermore, for each δ, ϵ with $s < s + \epsilon < r - \delta < r$, there are constants c_δ, d_ϵ such that

$$\|R_j\| \le c_\delta/(r-\delta)^j \quad \text{and} \quad \|R_{-j}\| \le d_\epsilon(s+\epsilon)^j \quad \text{for } j \in \mathbb{N}. \tag{2.1}$$

By equating coefficients for each power of z in the identities $R(z)A(z) = I \in \mathcal{B}(H)$ and $A(z)R(z) = I \in \mathcal{B}(K)$, we can see that $\{R_j\}_{j\in\mathbb{Z}}$ must satisfy the left fundamental equations

$$R_{j-1}A_1 + R_jA_0 = \begin{cases} I & \text{if } j = 0, \\ 0 & \text{if } j \neq 0 \end{cases} \tag{2.2}$$

and the right fundamental equations

$$A_1R_{j-1} + A_0R_j = \begin{cases} I & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases} \tag{2.3}$$

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Conversely, if $\{R_j\}_{j\in\mathbb{Z}}$ satisfies the bounds given in (2.1) and solves the fundamental equations (2.2) and (2.3), then $R(z) = \sum_{j\in\mathbb{Z}} R_j z^j$ is well defined for $z \in \mathcal{U}_{s,r}$ and satisfies the identities $R(z)A(z) = I$ and $A(z)R(z) = I$.

2.3. The reduced resolvent. We follow the presentation in [11]. Let H, K be complex Hilbert spaces and let $A_0, A_1 \in \mathcal{B}(H, K)$ with $A_0(H) = K$ but $A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$. Define $A : \mathbb{C} \to \mathcal{B}(H, K)$ by the formula $A(z) = A_0 + A_1 z$ and suppose that there is some $r > 0$ such that $R(z) = A(z)^{-1} \in \mathcal{B}(K, H)$ is analytic for $z \in \mathcal{U}_{0,r}$. Thus, $R(z_0) = A(z_0)^{-1} \in \mathcal{B}(K, H)$ is well defined for some $z_0 \neq 0$. Since $H_1 = A_0^{-1}(\{\mathbf{0}\}) \subseteq H$ is closed, it follows that H is isomorphic to $H_1 \times H_1^{\perp}$, where $H_1^{\perp}$ denotes the orthogonal complement of $H_1 \subseteq H$. We write $H \cong H_1 \times H_1^{\perp}$. We assume that $A_0 \neq 0$, which means that $H_1 \neq H$ and hence that $H_1^{\perp} \neq \{\mathbf{0}\}$. Let $P_1 \in \mathcal{B}(H)$ denote the natural orthogonal projection onto the subspace $H_1 \subseteq H$ and define $U_1 \in \mathcal{B}(H, H_1 \times H_1^{\perp})$ by the formula

$$U_1 = \begin{bmatrix} P_1 \\ I - P_1 \end{bmatrix} \iff U_1 \boldsymbol{x} = \begin{bmatrix} P_1 \boldsymbol{x} \\ (I - P_1)\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_1^{\perp} \end{bmatrix}$$

for each $\boldsymbol{x} \in H$. Now $P_1^* = P_1$ and $U_1^* = [P_1, I - P_1] \in \mathcal{B}(H_1 \times H_1^{\perp}, H)$. The mapping U_1 defines a unitary equivalence between H and $H_1 \times H_1^{\perp}$.

By Lemma 2.1, we know that $K_1 = A_1(H_1) \subseteq K$ is also closed and so $K \cong K_1 \times K_1^{\perp}$. We may assume that $K_1^{\perp} \neq \{\mathbf{0}\}$, as the following argument shows. Choose $\boldsymbol{x}_1^{\perp} \in H_1^{\perp}$ with $\boldsymbol{x}_1^{\perp} \neq \mathbf{0}$ and suppose that $\boldsymbol{y}_1 = A(z_0)\boldsymbol{x}_1^{\perp} \in K_1$. We know that $\boldsymbol{y}_1 \neq \mathbf{0}$ because $A(z_0)$ is bounded below. Since $A_1(H_1) = K_1$, we can find $\boldsymbol{x}_1 \in H_1$ such that $A_1(\boldsymbol{x}_1) = \boldsymbol{y}_1/z_0$. Thus, $A(z_0)\boldsymbol{x}_1 = \boldsymbol{y}_1$. We must have $\boldsymbol{x}_1 \neq \mathbf{0}$ because $\boldsymbol{y}_1 \neq \mathbf{0}$. However, this implies that $A(z_0)\boldsymbol{x}_1 = A(z_0)\boldsymbol{x}_1^{\perp}$ with $\boldsymbol{x}_1 \neq \boldsymbol{x}_1^{\perp}$. Since $A(z_0)$ is 1–1, this is a contradiction. Therefore, $A(z_0)\boldsymbol{x}_1^{\perp} = \boldsymbol{y}_1 + \boldsymbol{y}_1^{\perp}$ for some $\boldsymbol{y}_1^{\perp} \in K_1^{\perp}$ with $\boldsymbol{y}_1^{\perp} \neq \mathbf{0}$. Thus, $K_1^{\perp} \neq \{\mathbf{0}\}$.

Let $Q_1 \in \mathcal{B}(K)$ denote the natural orthogonal projection onto the subspace $K_1 \subseteq K$ and define $V_1 \in \mathcal{B}(K, K_1 \times K_1^{\perp})$ by the formula

$$V_1 = \begin{bmatrix} Q_1 \\ I - Q_1 \end{bmatrix} \iff V_1 \boldsymbol{y} = \begin{bmatrix} Q_1 \boldsymbol{y} \\ (I - Q_1)\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_1^{\perp} \end{bmatrix}$$

for each $\boldsymbol{y} \in K$. Thus, $Q_1^* = Q_1$ and $V_1^* = [Q_1, I - Q_1] \in \mathcal{B}(K_1 \times K_1^{\perp}, K)$. The mapping V_1 defines a unitary equivalence between K and $K_1 \times K_1^{\perp}$. Now we use the unitary transformations U_1, V_1 to write the operator $A_i \in \mathcal{B}(H, K)$ in the operator matrix form defined by $A_i{}^{(1)} = V_1 A_i U_1^* \in \mathcal{B}(H_1 \times H_1^{\perp}, K_1 \times K_1^{\perp})$ for each $i = 0, 1$. Thus,

$$A_0{}^{(1)} = \begin{bmatrix} Q_1 A_0 P_1 & Q_1 A_0 (I - P_1) \\ (I - Q_1) A_0 P_1 & (I - Q_1) A_0 (I - P_1) \end{bmatrix} = \begin{bmatrix} 0 & A_{0,(1,2)}{}^{(1)} \\ 0 & A_{0,(2,2)}{}^{(1)} \end{bmatrix},$$

where we have used the fact that $A_0 P_1(H) = A_0(H_1) = 0$ and

$$A_1{}^{(1)} = \begin{bmatrix} Q_1 A_1 P_1 & Q_1 A_1 (I - P_1) \\ (I - Q_1) A_1 P_1 & (I - Q_1) A_1 (I - P_1) \end{bmatrix} = \begin{bmatrix} A_{1,(1,1)} & A_{1,(1,2)}{}^{(1)} \\ 0 & A_{1,(2,2)}{}^{(1)} \end{bmatrix},$$

where we note that $(I - Q_1) A_1 P_1(H) = (I - Q_1)(K_1) = 0$. Now $A^{(1)}(z) = V_1 A(z) U_1^* \in \mathcal{B}(H_1 \times H_1^{\perp}, K_1 \times K_1^{\perp})$ can be written in the equivalent operator matrix form as

$$A^{(1)}(z) = \begin{bmatrix} A_{1,(1,1)} z & A_{(1,2)}{}^{(1)}(z) \\ 0 & A_{(2,2)}{}^{(1)}(z) \end{bmatrix},$$

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where we note the simplified form ${A_{(1,1)}}^{(1)}(z) = A_{1,(1,1)}z$ for the block on the leading diagonal in the $(1, 1)$ position. Since A_1 is bounded below on H_1, we deduce that $[A_{1,(1,1)}]^{-1} \in \mathcal{B}(K_1, H_1)$ is well defined. From $A_0(H) = K$, it follows that ${A_{0,(1,2)}}^{(1)}(H_1^\perp) = K_1$ and ${A_{0,(2,2)}}^{(1)}(H_1^\perp) = K_1^\perp$.

If $[{A_{0,(2,2)}}^{(1)}]^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}$, then ${A_{0,(2,2)}}^{(1)}$ is a 1–1 mapping of $H_1^\perp$ onto $K_1^\perp$ and so, by Theorem 2.2, we deduce that ${A_{0,(2,2)}}^{(1)}$ is bounded below. Hence, $[{A_{0,(2,2)}}^{(1)}]^{-1} \in \mathcal{B}(K_1^\perp, H_1^\perp)$ is well defined. Since $R(z)$ is analytic for $z \in \mathcal{U}_{0,r}$, it follows that $[{A_{(2,2)}}^{(1)}(z)]^{-1}$ is also analytic on $\mathcal{U}_{0,r}$. Thus, we can use the Neumann expansion to represent ${A_{(2,2)}}^{(1)}(z)^{-1} \in \mathcal{B}(K, H)$ on the region $|z| < r$. If $[{A_{0,(2,2)}}^{(1)}]^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$, then the reduced problem to calculate the resolvent ${R_{(2,2)}}^{(1)}(z) = [{A_{(2,2)}}^{(1)}(z)]^{-1}$ is precisely the same problem as the original resolvent problem but on a smaller space. Thus, we could repeat the process on the reduced resolvent.

3. Related work

In studying the invertibility of time-invariant linear control systems, Sain and Massey [16] used Laplace transforms to show that the desired system inversion could be formulated as a matrix power series inversion problem in the form

$$X(s)[C(sI - A)^{-1}B + D] = \frac{1}{s^p}I. \tag{3.1}$$

By assuming a Laurent series expansion in the form

$$X(s) = \frac{1}{s^p}[X_0 + X_1 s + X_2 s^2 + \cdots]$$

and equating coefficients for the various powers of s in (3.1), they obtained a system of matrix fundamental equations. Their analysis of the system included a rank test on an augmented system matrix that provided necessary and sufficient conditions for existence of a unique solution but did not include a suggested calculation procedure.

More generally, studies relating to the spectral theory of bounded linear operators [12, 21] had posed similar problems relating to representation of the resolvent operator by a Laurent series. For the most part these problems were adequately solved using a standard Neumann expansion

$$R(\lambda) = (\lambda I - A)^{-1} = \frac{1}{\lambda}\Big[I + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\Big]$$

for $\lambda \in \mathcal{U}_{r,\infty} = \{\lambda \in \mathbb{C} \mid r < |\lambda| < \infty\}$, where r is the spectral radius of A.

3.1. Inversion of matrix pencils. There is a substantial literature on the inversion of matrix pencils. We will not attempt a comprehensive review of the more recent work but rather refer readers to [4, 7, 20] and references listed therein. For our purposes, a more fruitful approach is to review the papers [9, 13, 14, 16, 17] that are directly related to our work. Throughout this section we suppose that $A_0, A_1 \in \mathbb{C}^{n\times n}$, where A_0 is singular. We write $A(z) = A_0 + A_1 z \in \mathbb{C}^{n\times n}$ and $R(z) = A(z)^{-1} \in \mathbb{C}^{n\times n}$ for all $z \in \mathbb{C}$ where the inverse exists.

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3.1.1. Generalized Jordan chains. Langenhop [13, 14] used generalized Jordan chains of subspaces to find necessary and sufficient conditions for the existence of an analytic resolvent $R(z)$ when $z \in \mathcal{U}_{0,r}$ for some $r > 0$. In [13], Langenhop defined $N_{0,-1} = \{\mathbf{0}\} \subseteq \mathbb{C}^n$ and $R_{0,-1} = \mathbb{C}^n$ and then recursively defined subspaces

$$R_{1,k} = \{\boldsymbol{y} = A_1\boldsymbol{x} \mid \boldsymbol{x} \in N_{0,k}\} = A_1(N_{0,k}) \subseteq \mathbb{C}^n,$$
$$N_{0,k+1} = \{\boldsymbol{x} \in \mathbb{C}^n \mid A_0\boldsymbol{x} \in R_{1,k}\} = A_0^{-1}(R_{1,k}) \subseteq \mathbb{C}^n$$

and

$$N_{1,k} = \{\boldsymbol{x} \in \mathbb{C}^n \mid A_1\boldsymbol{x} \in R_{0,k}\} = A_1^{-1}(R_{0,k}) \subseteq \mathbb{C}^n,$$
$$R_{0,k+1} = \{\boldsymbol{y} = A_0\boldsymbol{x} \mid \boldsymbol{x} \in N_{1,k}\} = A_0(N_{1,k}) \subseteq \mathbb{C}^n$$

for each $k \in \mathbb{N} - 2$. Langenhop showed that $\{\mathbf{0}\} = N_{0,0} \subseteq N_{0,k-1} \subseteq N_{0,k}$ and $\mathbb{C}^n \supseteq N_{1,k-1} \supseteq N_{1,k} \supseteq N_1 = A_1^{-1}(\{\mathbf{0}\}) = \{\boldsymbol{x} \in \mathbb{C}^n \mid A_1\boldsymbol{x} = \mathbf{0}\}$ for each $k \in \mathbb{N}$. In particular, he proved that if $R(z)$ is well defined for some $z \neq 0$, then

$$N_1 \cap N_{0,k-1} = \{\mathbf{0}\} \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

Conversely, if (3.2) is true, he showed it was then possible to construct matrices $R_{-1}, R_0 \in \mathbb{C}^{n\times n}$ such that for some $r > 0$ the resolvent is given by

$$R(z) = \sum_{j=-m}^{\infty} R_j z^j \tag{3.3}$$

for all $z \in \mathcal{U}_{0,r}$, where $m \leq n$ and where $R_{-k} = (-1)^{k-1}(R_{-1}A_0)^{k-1}R_{-1}$ for $1 \leq k \leq m$ and $R_\ell = (-1)^\ell(R_0A_1)^\ell R_0$ for $\ell \in \mathbb{N} - 1$. Thus, Langenhop showed that (3.2) is necessary and sufficient for the existence of $R(z)$ in some region $\mathcal{U}_{0,r}$.

Although the fundamental equations are not highlighted in [13], Langenhop made the connection more explicit in [14] by showing that $R(z)$ is analytic on some deleted neighbourhood of the origin $z \in \mathcal{U}_{0,r}$ if and only if there exist $R_{-1}, R_0 \in \mathbb{C}^{n\times n}$ such that

$$R_0A_0 + R_{-1}A_1 = I \quad \text{and} \quad A_0R_0 + A_1R_{-1} = I. \tag{3.4}$$

Thus, (3.2) is equivalent to (3.4) for $A_0, A_1 \in \mathbb{C}^{n\times n}$. Nevertheless, we note that (3.4) may have more than one solution $\{R_{-1}, R_0\}$ and the series (3.3) generated by a particular solution may not converge on $\mathcal{U}_{0,r}$. Consider the following example taken from [1, Example 3].

Example 3.1. Define $A_0, A_1 \in \mathbb{C}^{3\times 3}$ by setting

$$A_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The resolvent $R(z)$ is singular at $z_1 = 0$, $z_2 = -1$ and $z_3 = -3$ and hence can be represented in the form

$$R(z) = \frac{B}{z} + \frac{C}{z+1} + \frac{D}{z+3},$$

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where $B, C, D \in \mathbb{C}^{3\times 3}$. Thus, we expect to find three different Laurent series centred at $z = 0$ for the three annular regions $\mathcal{U}_{0,1}$, $\mathcal{U}_{1,3}$ and $\mathcal{U}_{3,\infty}$. The Laurent series on each annular region is generated by a different pair $\{R_{-1}, R_0\}$. Thus, for instance, there is a solution

$$R_{-1} = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \quad \text{and} \quad R_0 = \begin{bmatrix} 0 & \frac{1}{9} & \frac{1}{9} \\ 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{18} & \frac{1}{18} \end{bmatrix}.$$

The associated Laurent series $R(z) = \sum_{k\in\mathbb{N}}(-1)^{k-1}(R_{-1}A_0)^{k-1}R_{-1}/z^k + \sum_{j\in\mathbb{N}}(-1)^j (R_0A_1)^j R_0 z^j$ converges for $z \in \mathcal{U}_{1,3}$. See [1, Example 3] for further details. □

3.1.2. The fundamental matrix equations. Howlett [9] used the fundamental equations described in [16] to solve the problem of input retrieval in finite-dimensional linear systems. Two methods were proposed. In each case an expression was found for the inverse transfer function. The first method used a general procedure for the inversion of matrix power series to express the solution as a Laurent series. The second method used elementary row and column operations on a modified Rosenbrock system matrix to find a closed form for the solution. For a linear pencil, Howlett showed that $R(z) \in \mathbb{C}^{n\times n}$ has a simple pole at $z = 0$ and is well defined on some $\mathcal{U}_{0,r}$ if and only if there exist nonsingular matrices M and N such that

$$A_0 \cong MA_0N = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_1 \cong MA_1N = \begin{bmatrix} A_{1,11} & A_{1,12} \\ A_{1,21} & I_q \end{bmatrix},$$

where $p + q = n$. For a linear pencil with a higher-order pole, a similar decomposition was applied to an augmented coefficient matrix to find the Laurent series. A rank test proposed by Sain and Massey [16] was used to determine whether the inversion was possible and to find the order of the pole. Howlett also showed that the coefficients of the resolvent have a characteristic geometric form and satisfy a finite recurrence relationship. The method can be implemented numerically using Gaussian elimination to calculate the coefficients of the resolvent.

3.1.3. The reduced resolvent. Suppose that $R(z) \in \mathbb{C}^{n\times n}$ is analytic on some region $\mathcal{U}_{0,r}$. Schweitzer and Stewart [17] used the resolvent equation

$$R(w) - R(z) = (z - w)R(w)A_1R(z)$$

to provide an alternative derivation of (3.3) which avoids overt use of the technical machinery in [13]. They also described a two-stage computational algorithm based on a decomposition proposed by Van Dooren [19]. The first stage used a progressive sequence of unitary transformations to find an equivalent problem with a simplified structure where the transformed matrix $A_0{}^{(1)} = VA_0U$ is strictly block upper triangular and has a zero block diagonal, and where the transformed matrix $A_1{}^{(1)} = VA_1U$ is block upper triangular and has nonsingular blocks on the leading diagonal except possibly in the final position. The second stage used multiplication on the left and right by nonsingular elementary matrices to further simplify the final block structure.

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3.2. Inversion of operator pencils. The ideas underlying the inversion of matrix pencils have since been extended to operator pencils on infinite-dimensional spaces, but the original arguments involving the ranks of various matrices or the values of certain determinants can no longer be used. In addition, it is necessary to prove that the subspaces used to decompose the operators are all closed. There are three relevant techniques that have been suggested. The mainstream research on spectral theory for operator pencils on Banach space used complex function theory with the principal results established by Stummel [18]. Subsequently, Howlett *et al.* [11] showed that the progressive decomposition proposed by Schweitzer and Stewart [17] could also be used for operators on Hilbert space provided the process terminated after a finite number of steps. More recently, Albrecht *et al.* [1] used the fundamental equations to obtain a more general but also more explicit form of the original matrix result obtained by Langenhop in [14].

3.2.1. Spectral theory for operator pencils. We outline relevant results from the paper by Stummel [18]. Let H and K be complex Banach spaces and suppose that $A_0, A_1 \in \mathcal{B}(H, K)$. Define $A(z) = A_0 + A_1 z \in \mathcal{B}(K, H)$ for all $z \in \mathbb{C}$. Stummel considered the spectral set $\sigma = \{z \in \mathbb{C} \mid R(z) = A(z)^{-1} \notin \mathcal{B}(H, K)\}$ for $A(z)$ and used line integrals to define complementary projections that isolate separate components of the spectrum. In the case where $R(z) \in \mathcal{B}(K, H)$ is analytic on an annular region $\mathcal{U}_{s,r}$, the spectral set has two disjoint components—an *interior* compact subset $\sigma_1 \subseteq \{z \in \mathbb{C} \mid |z| \le s\}$ and an *exterior* closed subset $\sigma_2 \subseteq \{z \in \mathbb{C} \mid |z| \ge r\}$. Following Stummel, we define $R_{-1} \in \mathcal{B}(K, H)$ by the contour integral formula

$$R_{-1} = \frac{1}{2\pi i} \int_\Gamma R(\zeta)\, d\zeta,$$

where $\Gamma = \{\zeta = \rho e^{i\theta} \mid \theta \in [0, 2\pi)\}$ and $s < \rho < r$. Stummel defined corresponding projections $P = R_{-1} A_1 \in \mathcal{B}(H)$ and $Q = A_1 R_{-1} \in \mathcal{B}(K)$ to establish the formal separation of σ_1 and σ_2. The method proposed by Stummel is described clearly by Gohberg *et al.* [8, pages 49–54]. The earlier underlying concept of spectral separation for a bounded linear operator is elegantly presented in the classic book by Kato [12, pages 178–179].

3.2.2. The fundamental equations for operator pencils. Let H and K be complex Banach spaces and suppose that $A_0, A_1 \in \mathcal{B}(H, K)$. Define $A(z) = A_0 + A_1 z \in \mathcal{B}(K, H)$ for all $z \in \mathbb{C}$ and let $R(z) = A(z)^{-1}$ denote the resolvent. Albrecht *et al.* [1] showed that $R(z)$ is analytic on an annular region $\mathcal{U}_{s,r}$ if and only if there exists an appropriate solution to the fundamental equations. More specifically, they established the following results. We refer readers to the original article [1] for proofs of these results. See also [2, 10] for some preliminary work.

THEOREM 3.2. *The coefficients $\{R_j\}_{j \in \mathbb{Z}} \in \mathcal{B}(K, H)$ satisfy (2.1), (2.2) and (2.3) if and only if the following are all satisfied: (i) $P = R_{-1} A_1 \in \mathcal{B}(H)$ and $I - P = R_0 A_0 \in \mathcal{B}(H)$ are projections on H; and $Q = A_1 R_{-1} \in \mathcal{B}(K)$ and $I - Q = A_0 R_0 \in$*

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$\mathcal{B}(K)$ are corresponding projections on K; (ii) $A_i = QA_iP + (I-Q)A_i(I-P)$ for $i = 0, 1$; (iii) $R_{-k} = PR_{-k}Q$ for $k \in \mathbb{N}$ and $R_\ell = (I-P)R_\ell(I-Q)$ for $\ell \in \mathbb{N}-1$; (iv) $R_{-k} = (-1)^{k-1}(R_{-1}A_0)^{k-1}R_{-1}$ and $R_\ell = (-1)^\ell (R_0A_1)^\ell R_0$ for $k, \ell \in \mathbb{N}$; and (v) *$\lim_{k\to\infty} \|(R_{-1}A_0)^k\|^{1/k} \le s$ and $\lim_{\ell\to\infty} \|(R_0A_1)^\ell\|^{1/\ell} \le 1/r$.* □

COROLLARY 3.3. *Let $s, r \in \mathbb{R}$ with $0 \le s < r \le \infty$. The resolvent $R : \mathcal{U}_{s,r} \to \mathcal{B}(K, H)$ is analytic if and only if there exist $R_{-1}, R_0 \in \mathcal{B}(K, H)$ such that (i) $R_{-1}A_1 + R_0A_0 = I$ and $A_1R_{-1} + A_0R_0 = I$; (ii) $R_{-1}A_iR_0 = 0$ and $R_0A_iR_{-1} = 0$ for each $i = 0, 1$; and (iii) $\lim_{k\to\infty} \|(R_{-1}A_0)^k\|^{1/k} \le s$ and $\lim_{\ell\to\infty} \|(R_0A_1)^\ell\|^{1/\ell} \le 1/r$. If these conditions are satisfied, then R_{-1}, R_0 are uniquely defined and the resolvent can be written in the form*

$$R(z) = R_\sigma(z) + R_\rho(z)$$

for all $z \in \mathcal{U}_{s,r}$, where $R_\sigma(z) = PR(z)Q$ given by

$$R_\sigma(z) = (Iz + R_{-1}A_0)^{-1}R_{-1} = \sum_{k=1}^{\infty} (-1)^{k-1}(R_{-1}A_0)^{k-1}R_{-1}/z^k \tag{3.5}$$

for $|z| > s$ is the singular part and $R_\rho(z) = (I-P)R(z)(I-Q)$ given by

$$R_\rho(z) = (I + R_0A_1z)^{-1}R_0 = \sum_{\ell=0}^{\infty} (-1)^\ell (R_0A_1)^\ell R_0 z^\ell \tag{3.6}$$

for $|z| < r$ is the regular part. The operators $P = R_{-1}A_1 \in \mathcal{B}(H)$ and $Q = A_1R_{-1} \in \mathcal{B}(K)$ are the corresponding key projections that separate the singular and regular parts of the resolvent. If we define linear operators $\mathcal{R}_\lambda = \lambda^{-1}R(-\lambda^{-1})A_0 \in \mathcal{B}(H)$ and $\mathcal{S}_\lambda = \lambda^{-1}A_0R(-\lambda^{-1}) \in \mathcal{B}(K)$, then $\mathcal{R}_\lambda, \mathcal{S}_\lambda$ satisfy the resolvent equations $\mathcal{R}_\lambda - \mathcal{R}_\mu = (\mu-\lambda)\mathcal{R}_\lambda\mathcal{R}_\mu$ and $\mathcal{S}_\lambda - \mathcal{S}_\mu = (\mu-\lambda)\mathcal{S}_\lambda\mathcal{S}_\mu$ for $\lambda, \mu \in \mathcal{U}_{r^{-1},s^{-1}}$. □

Corollary 3.3 is a more general form of the result established by Langenhop in [14]. At the same time it is more precise because conditions (ii) and (iii) ensure that the solution is unique. This leads us to the following definition of a basic solution to the fundamental equations.

DEFINITION 3.4. If $\{R_{-1}, R_0\} \subseteq \mathcal{B}(K, H)$ satisfy (i)–(iii) in Corollary 3.3, then we say that $\{R_{-1}, R_0\}$ is the basic solution to (2.2) and (2.3) on $\mathcal{U}_{s,r}$. □

3.2.3. Global structure of the resolvent. Suppose that $R(z)$ has isolated singularities at $z = z_s$ for each $s = 1, 2, \ldots, m$ for some $m \in \mathbb{N}$ but is analytic elsewhere. Write $A(z) = A_{s,0} + A_{s,1}(z - z_s)$, where $A_{s,0} = A_0 + A_1z_s$ and $A_{s,1} = A_1$. Let $\{R_{s,-1}, R_{s,0}\}$ denote the basic solution to the fundamental equations on $z_s + \mathcal{U}_{0,r_s}$, where $r_s = \min_{t\neq s} |z_t - z_s|$. Let $P_s = R_{s,-1}A_1$ and $Q_s = A_1R_{s,-1}$ be the corresponding projections and write $R(z) = R_{s,\sigma}(z) + R_{s,\rho}(z)$, where

$$R_{s,\sigma}(z) = P_sR(z)Q_s \quad (z \neq z_s)$$

is singular at $z = z_s$ and where

$$R_{s,\rho}(z) = (I - P_s)R(z)(I - Q_s) \quad (z \neq z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_m)$$

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is regular at $z = z_s$. Albrecht *et al.* [1] used the fundamental equations to show that $P_s P_t = 0$ and $Q_s Q_t = 0$ for $s \neq t$ and hence defined corresponding complementary projections $P_\infty = \prod_{s=1}^m (I - P_s) = I - \sum_{s=1}^m P_s \in \mathcal{B}(H)$ and $Q_\infty = \prod_{s=1}^m (I - Q_s) = I - \sum_{s=1}^m Q_s \in \mathcal{B}(K)$. By writing

$$H = M_1 \oplus \cdots \oplus M_m \oplus M_\infty \quad \text{and} \quad K = N_1 \oplus \cdots \oplus N_m \oplus N_\infty,$$

where $M_s = P_s(H)$, $N_s = Q_s(K)$, $M_\infty = P_\infty(H) = (I - \sum_{s=1}^m P_s)(H)$ and $N_\infty = Q_\infty(K) = (I - \sum_{s=1}^m Q_s)(K)$, they established the full spectral decomposition

$$R(z) = \sum_{s=1}^m R_{s,\sigma}(z) + R_\infty(z), \tag{3.7}$$

where each term $R_{s,\sigma}(z) = P_s R(z) Q_s \in \mathcal{B}(N_s, M_s)$ is analytic for $z \neq z_s$ and the remainder $R_\infty(z) = P_\infty R(z) Q_\infty \in \mathcal{B}(N_\infty, M_\infty)$ is entire.

3.2.4. Separation of the fundamental equations. The key projections $P = R_{-1}A_1 \in \mathcal{B}(H)$ and $Q = A_1 R_{-1} \in \mathcal{B}(K)$ can be used to rewrite each doubly-infinite system of fundamental equations as two separate singly-infinite systems—one for the singular part of the resolvent and the other for the regular part. Define corresponding partitions of $H \cong P(H) \times (I - P)(H) = P(H) \times P^c(H) = M \times M^c$ and $K \cong Q(K) \times (I - Q)(K) = Q(K) \times Q^c(K) = N \times N^c$. Theorem 3.2 shows that

$$A_i \cong \begin{bmatrix} Q \\ Q^c \end{bmatrix} A_i [P \quad P^c] = \begin{bmatrix} QA_iP & QA_iP^c \\ Q^cA_iP & Q^cA_iP^c \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_i & 0 \\ 0 & \mathfrak{A}_i^c \end{bmatrix}$$

with

$$R_{-k} \cong \begin{bmatrix} P \\ P^c \end{bmatrix} R_{-k} [Q \quad Q^c] = \begin{bmatrix} PR_{-k}Q & PR_{-k}Q^c \\ P^cR_{-k}Q & P^cR_{-k}Q^c \end{bmatrix} = \begin{bmatrix} \mathfrak{R}_{-k} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$R_\ell \cong \begin{bmatrix} P \\ P^c \end{bmatrix} R_\ell [Q \quad Q^c] = \begin{bmatrix} PR_\ell Q & PR_\ell Q^c \\ P^cR_\ell Q & P^cR_\ell Q^c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{R}_\ell^c \end{bmatrix},$$

where $\mathfrak{A}_i \in \mathcal{B}(M, N)$, $\mathfrak{A}_i^c \in \mathcal{B}(M^c, N^c)$ for each $i = 0, 1$, $\mathfrak{R}_{-k} \in \mathcal{B}(N, M)$ for $k \in \mathbb{N}$ and $\mathfrak{R}_\ell^c \in \mathcal{B}(N^c, M^c)$ for $\ell \in \mathbb{N} - 1$. If we restrict our attention to M and note that $P\boldsymbol{m} = \boldsymbol{m}$ for all $\boldsymbol{m} \in M$, then the left fundamental equations for the singular part of the resolvent can be rewritten as

$$\begin{aligned} \mathfrak{R}_{-1}\mathfrak{A}_1 &= \mathfrak{I}, \\ \mathfrak{R}_{-k-1}\mathfrak{A}_1 + \mathfrak{R}_{-k}\mathfrak{A}_0 &= 0 \quad \text{for } k \in \mathbb{N}, \end{aligned} \tag{3.8}$$

where $\mathfrak{I} \in \mathcal{B}(M)$ denotes the identity operator on M. If we restrict our attention to M^c and note that $P^c\boldsymbol{m}^c = \boldsymbol{m}^c$ for all $\boldsymbol{m}^c \in M^c$, then the left fundamental equations for the regular part of the resolvent become

$$\begin{aligned} \mathfrak{R}_0^c\mathfrak{A}_0^c &= \mathfrak{I}^c, \\ \mathfrak{R}_{\ell-1}^c\mathfrak{A}_1^c + \mathfrak{R}_\ell^c\mathfrak{A}_0^c &= 0 \quad \text{for } \ell \in \mathbb{N}, \end{aligned} \tag{3.9}$$

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where $\mathfrak{I}^c \in \mathcal{B}(M^c)$ denotes the identity operator on M^c. The systems (3.8) and (3.9) are completely separate. If we restrict our attention to N and note that $Q\boldsymbol{n} = \boldsymbol{n}$ for all $\boldsymbol{n} \in N$, then the right fundamental equations for the singular part of the resolvent are

$$\begin{aligned} \mathfrak{A}_1\mathfrak{R}_{-1} &= \mathfrak{I}, \\ \mathfrak{A}_1\mathfrak{R}_{-k-1} + \mathfrak{A}_0\mathfrak{R}_{-k} &= 0 \quad \text{for } k \in \mathbb{N}, \end{aligned} \tag{3.10}$$

where $\mathfrak{I} \in \mathcal{B}(N)$ denotes the identity operator on N and, if we restrict our attention to N^c and note that $Q^c\boldsymbol{n}^c = \boldsymbol{n}^c$ for all $\boldsymbol{n}^c \in N^c$, then the right fundamental equations for the regular part of the resolvent are reduced to

$$\begin{aligned} \mathfrak{A}_0^c\mathfrak{R}_0^c &= \mathfrak{I}^c, \\ \mathfrak{A}_1^c\mathfrak{R}_{\ell-1}^c + \mathfrak{A}_0^c\mathfrak{R}_{\ell}^c &= 0 \quad \text{for } \ell \in \mathbb{N}, \end{aligned} \tag{3.11}$$

where $\mathfrak{I}^c \in \mathcal{B}(N^c)$ denotes the identity operator on N^c. The systems (3.10) and (3.11) are completely separate. In infinite-dimensional space the analysis depends on *both* the left and right sets of fundamental equations. It is not sufficient to use only one of the two sets. Indeed, we need both $\mathfrak{R}_{-1}\mathfrak{A}_1 = \mathfrak{I}$ and $\mathfrak{A}_1\mathfrak{R}_{-1} = \mathfrak{I}$ in order to deduce that $\mathfrak{R}_{-1} = \mathfrak{A}_1^{-1} \in \mathcal{B}(N, M)$ is the uniquely defined inverse of $\mathfrak{A}_1$. Now we can see that the systems (3.8) and (3.10) have the unique solution

$$\mathfrak{R}_{-k} = (-1)^{k-1}(\mathfrak{A}_1^{-1}\mathfrak{A}_0)^{k-1}\mathfrak{A}_1^{-1}$$

for $k \in \mathbb{N}$. A similar argument using $\mathfrak{R}_0^c\mathfrak{A}_0^c = \mathfrak{I}^c$ and $\mathfrak{A}_0^c\mathfrak{R}_0^c = \mathfrak{I}^c$ shows that $\mathfrak{R}_0^c = [\mathfrak{A}_0^c]^{-1} \in \mathcal{B}(N^c, M^c)$ is the uniquely defined inverse of $\mathfrak{A}_0^c$. Thus, the systems (3.9) and (3.11) have the unique solution

$$\mathfrak{R}_\ell^c = (-1)^\ell([\mathfrak{A}_0^c]^{-1}\mathfrak{A}_1^c)^\ell[\mathfrak{A}_0^c]^{-1}$$

for $\ell \in \mathbb{N} - 1$. Note that the existence of $\mathfrak{A}_1^{-1}$ means that M is isomorphic to N and the existence of $[\mathfrak{A}_0^c]^{-1}$ means that M^c is isomorphic to N^c. Thus, $H = M \times M^c$ and $K = N \times N^c$ are isomorphic. Nevertheless, there are situations where we may wish to regard these isomorphic spaces as different. See [1, 11] and [3, pages 282–285] for some specific instances.

3.2.5. The reduced resolvent for an operator pencil. Howlett *et al.* [11] used the unitary operators $U_1^* = [P_1, I - P_1]$ and $V_1^* = [Q_1, I - Q_1]$ described in Section 2.3 to show that

$$A^{(1)}(z) = V_1A(z)U_1^* = \begin{bmatrix} A_{1,(1,1)}z & A_{(1,2)}{}^{(1)}(z) \\ 0 & A_{(2,2)}{}^{(1)}(z) \end{bmatrix} \tag{3.12}$$

and hence deduce that if $R(z) \in \mathcal{B}(K, H)$ is well defined for $z \in \mathcal{U}_{0,r}$ for some $r > 0$, then

$$\begin{aligned} R(z) = P_1[A_{1,(1,1)}]^{-1}Q_1/z& \\ &- P_1[A_{1,(1,1)}]^{-1}A_{(1,2)}{}^{(1)}(z)R_{(2,2)}{}^{(1)}(z)(I - Q_1)/z \\ &+ (I - P_1)R_{(2,2)}{}^{(1)}(z)(I - Q_1) \end{aligned}$$

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for all $z \in \mathcal{U}_{0,r}$, where $R_{(2,2)}{}^{(1)}(z) = [A_{(2,2)}{}^{(1)}(z)]^{-1}$. For each $z \neq 0$, it is clear from (3.12) that $R(z)$ exists if and only if $[A_{1,(1,1)}]^{-1}$ and $R_{(2,2)}{}^{(1)}(z)$ both exist.

If $[A_{0,(2,2)}{}^{(1)}]^{-1}$ exists, then the Neumann expansion can be used to find a Maclaurin series for $R_{(2,2)}{}^{(1)}(z)$ when $z \in \mathcal{U}_{0,r}$ and hence find a corresponding Laurent series for $R(z)$. In this case $R(z)$ has a pole of order 1 at $z = 0$. If $[A_{0,(2,2)}{}^{(1)}]^{-1}$ does not exist, then the reduction procedure is applied to the pencil $A_{(2,2)}{}^{(1)}(z)$. If the process terminates after n steps, then $R(z)$ has a pole of order n at $z = 0$. In this paper we are interested in what happens when the reduction continues *ad infinitum*.

The assumption that $R(z)$ is analytic for $z \in \mathcal{U}_{0,r}$ is nontrivial, as the following example shows.

Example 3.5. Let $H = K = \ell^2$ and let $A_0 \in \mathcal{B}(H, K)$, $A_1 \in \mathcal{B}(H, K)$ and $A(z) = A_0 + A_1 z \in \mathcal{B}(H, K)$ be defined by the infinite matrices

$$A_0 = \begin{bmatrix} 0 & \epsilon & 0 & 0 & \cdots \\ 0 & 0 & \epsilon & 0 & \cdots \\ 0 & 0 & 0 & \epsilon & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \epsilon J \quad \text{and} \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = I,$$

where $\epsilon \in (0, 1)$ is some known real number. Now for $z \neq 0$ a formal reduction $[A(z) \mid I\,] \to [I \mid R(z)\,]$ using elementary row operations gives

$$R(z) = \begin{bmatrix} 1/z & -\epsilon/z^2 & \epsilon^2/z^3 & -\epsilon^3/z^4 & \cdots \\ 0 & 1/z & -\epsilon/z^2 & \epsilon^2/z^3 & \cdots \\ 0 & 0 & 1/z & -\epsilon/z^2 & \cdots \\ 0 & 0 & 0 & 1/z & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \sum_{j \in \mathbb{N}} (-1)^{n-1} (\epsilon J)^{n-1} / z^n.$$

It is relatively straightforward to argue that $R(z)A(z)\boldsymbol{e}_j = A(z)R(z)\boldsymbol{e}_j = \boldsymbol{e}_j$ for all $z \neq 0$ and each $j \in \mathbb{N}$, where $\{\boldsymbol{e}_j\}_{j \in \mathbb{N}} \in \ell^2$ is the usual orthonormal basis for ℓ^2. Although this might suggest at first glance that $R(z) = A(z)^{-1}$ is well defined for $z \neq 0$, this is not so. Indeed, we have $J^k \boldsymbol{e}_{k+1} = \boldsymbol{e}_1$ for each $k \in \mathbb{N}$. Thus, $\|J^k\| \geq 1$ for all $k \in \mathbb{N}$ and so the series expression above for $R(z)$ does not converge in the operator norm for $|z| < \epsilon$. □

See [1, Section 3.2] for an example where $R(z)$ has an isolated essential singularity at $z = 0$. An infinite-dimensional problem where $R(z)$ has a first-order pole can be found in [3, pages 264–266].

4. The new results

We stated earlier that our aim was to reconcile two different methods that have been used to define the generalized resolvent of a linear operator pencil that is singular at the origin. Let H and K be Hilbert spaces and suppose that $A_0, A_1 \in \mathcal{B}(H, K)$, where $A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$. Let $A(z) = A_0 + A_1 z \in \mathcal{B}(H, K)$ be a linear pencil and let

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$R(z) = A(z)^{-1} \in \mathcal{B}(K, H)$ denote the resolvent. The resolvent $R(z)$ is analytic on the region $\mathcal{U}_{s,r}$ where $0 \le s < r \le \infty$ if and only if there exists a corresponding solution to the fundamental equations. We will show that the method of progressive reduction using a sequence of unitary transformations will find the resolvent if and only if it is analytic on a deleted neighbourhood of the origin $\mathcal{U}_{0,r}$ for some $0 < r \le \infty$. Thus, we can only truly reconcile the two methods on a deleted neighbourhood of the origin.

However, it may be possible to reconcile the two methods indirectly. It is often the case that the resolvent has only a finite number of isolated singularities. In this case we have already noted (3.7) that for each $s = 1, \ldots, m$ the resolvent can be expressed in the form

$$R(z) = R_{s,\sigma}(z) + \sum_{t \neq s} R_{t,\sigma}(z) + R_{\infty}(z),$$

where $R(z)$ is analytic for all $z \in z_s + \mathcal{U}_{0,r_s}$ and

$$R_{s,\sigma}(z) = \sum_{k \in \mathbb{N}} R_{s,-k}/(z - z_s)^k$$

is the singular part of the expansion at $z = z_s$. Consequently, we will be able to use the progressive reduction to calculate $R_{s,\sigma}(z)$ for $z \in z_s + \mathcal{U}_{0,r_s}$. Since $R_{s,\sigma}(z)$ is analytic for all $z \neq z_s$, it follows that this expansion will actually converge for $z \in z_s + \mathcal{U}_{0,\infty}$. The expansion for $R_{s,\sigma}(z)$ can now be converted into a Maclaurin series in nonnegative powers of z for $z \in \mathcal{U}_{0,|z_s|}$ or into a Laurent series in negative powers of z for $z \in \mathcal{U}_{|z_s|,\infty}$. Thus, we can ultimately obtain a Laurent series representation for $R(z)$ on each annular region $\mathcal{U}_{|z_s|,|z_t|}$ where $R(z)$ is analytic.

4.1. The first new result. To explain the first new result, we need to explain the notation. Suppose that $A_0(H) = K$ and $A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$. Suppose too that $R(z)$ is analytic on $\mathcal{U}_{0,r}$ for some $r > 0$. In Section 2.3, we showed that in this case we can write $H \cong H_1 \times H_1^{\perp}$ and $K \cong K_1 \times K_1^{\perp}$ and $A(z) \in \mathcal{B}(H, K)$ can be represented in the equivalent form $A^{(1)}(z) \in \mathcal{B}(H_1 \times H_1^{\perp}, K_1 \times K_1^{\perp})$, where

$$A^{(1)}(z) = \begin{bmatrix} A_{1,(1,1)}z & A_{(1,2)}{}^{(1)}(z) \\ 0 & A_{(2,2)}{}^{(1)}(z) \end{bmatrix}$$

for all $z \in \mathcal{U}_{0,r}$. Clearly, $R^{(1)}(z) = [A^{(1)}(z)]^{-1}$ exists on $\mathcal{U}_{0,r}$ if and only if $[A_{1,(1,1)}]^{-1}$ exists and $R_{(2,2)}{}^{(1)}(z) = [A_{(2,2)}{}^{(1)}(z)]^{-1}$ exists on $\mathcal{U}_{0,r}$.

If $[A_{0,(2,2)}{}^{(1)}]^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}$, then we can calculate the inverse operator $A_{(2,2)}{}^{(1)}(z)^{-1} = (A_{0,(2,2)}{}^{(1)} + A_{1,(2,2)}{}^{(1)}z)^{-1}$ as a Maclaurin series by using a Neumann expansion and the process terminates. Thus, we suppose that $[A_{0,(2,2)}{}^{(1)}]^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$. Since we already observed in Section 2.3 that $A_0(H) = K$ implies that $A_{0,(2,2)}{}^{(1)}(H_1^{\perp}) = K_1^{\perp}$, the problem to calculate the reduced resolvent $R_{(2,2)}{}^{(1)}(z) \in \mathcal{B}(K_1^{\perp}, H_1^{\perp})$ has precisely the same form as the original problem to calculate the resolvent $R^{(1)}(z) \in \mathcal{B}(K, H)$. Hence, the process can be repeated on the reduced problem.

Since the reduced problem has the same form as the original, we can also see that the inversion can be completed using a Neumann expansion if the process terminates

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after a finite number of steps. Thus, we restrict our attention to the case where the process continues *ad infinitum*.

After $m \in \mathbb{N}$ applications we have the orthogonal decompositions $H = H_1 \times \cdots \times H_m \times H_m^\perp$, where $H_n = [A_{0,(n+1,n+1)}{}^{(n)}]^{-1}(\{\mathbf{0}\}) \subseteq H_{n-1}^\perp$ with $H_{n-1}^\perp = H_n \times H_n^\perp$ for all $n \leq m$ and $K = K_1 \times \cdots \times K_m \times K_m^\perp$, where $K_n = A_{1,(n+1,n+1)}{}^{(n)}(H_n) \subseteq K_{n-1}$ with $K_{n-1} = K_n \times K_n^\perp$ for all $n \leq m$. Now $A(z) \in \mathcal{B}(H, K)$ can be represented in the equivalent form $A^{(m)}(z) \in \mathcal{B}(H_1 \times \cdots \times H_m \times H_m^\perp, K_1 \times \cdots \times K_m \times K_m^\perp)$, where

$$A^{(m)}(z) \cong \begin{bmatrix} A_{1,(1,1)}z & A_{(1,2)}(z) & \cdots & A_{(1,m)}(z) & A_{(1,m+1)}{}^{(m)}(z) \\ 0 & A_{1,(2,2)}z & \cdots & A_{(2,m)}(z) & A_{(2,m+1)}{}^{(m)}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{1,(m,m)}z & A_{(m,m+1)}{}^{(m)}(z) \\ 0 & 0 & \cdots & 0 & A_{(m+1,m+1)}{}^{(m)}(z) \end{bmatrix},$$

where $[A_{1,(n,n)}]^{-1}$ is well defined for $n \leq m$ and $A_{0,(m+1,m+1)}{}^{(m)}(H_m^\perp) = K_m^\perp$. We also have $A_{0,(m+1,m+1)}{}^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$.

We assume that the reduction continues *ad infinitum* and define increasing sequences of orthogonal projections $\{S_m\}_{m\in\mathbb{N}} \in \mathcal{B}(H)$ and $\{T_m\}_{m\in\mathbb{N}} \in \mathcal{B}(K)$ by setting $S_m = P_1 + \cdots + P_m \in \mathcal{B}(H)$ and $T_m = Q_1 + \cdots + Q_m \in \mathcal{B}(K)$, where $P_n \in \mathcal{B}(H)$ and $Q_n \in \mathcal{B}(K)$ are the natural orthogonal projections onto the subspaces $H_n \subseteq H$ and $K_n \subseteq K$ respectively for all $n \in \mathbb{N}$. We write $H = S_n(H) \times (I - S_n)(H) = M_n \times M_n^\perp$ and $K = T_n(K) \times (I - T_n)(K) = N_n \times N_n^\perp$ for each $n \in \mathbb{N}$. Note that $M_n \cong H_1 \times \cdots \times H_n$ and $N_n \cong K_1 \times \cdots \times K_n$. Thus, the operator $A_i \in \mathcal{B}(H, K)$ can be represented in the equivalent form $\mathcal{A}_i{}^{(n)} \in \mathcal{B}(M_n \times M_n^\perp, N_n \times N_n^\perp)$, where

$$\mathcal{A}_i{}^{(n)} = \begin{bmatrix} T_n A_i S_n & T_n A_i (I - S_n) \\ (I - T_n) A_i S_n & (I - T_n) A_i (I - S_n) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{i,(1,1)}{}^{(n)} & \mathcal{A}_{i,(1,2)}{}^{(n)} \\ 0 & \mathcal{A}_{i,(2,2)}{}^{(n)} \end{bmatrix}$$

for each $i = 0, 1$ and each $n \in \mathbb{N}$. Thus, $\mathcal{A}^{(n)}(z) \in \mathcal{B}(M_n \times M_n^\perp, N_n \times N_n^\perp)$ is given by

$$\mathcal{A}^{(n)}(z) = \mathcal{A}_0{}^{(n)} + \mathcal{A}_1{}^{(n)} z = \begin{bmatrix} \mathcal{A}_{(1,1)}{}^{(n)}(z) & \mathcal{A}_{(1,2)}{}^{(n)}(z) \\ 0 & \mathcal{A}_{(2,2)}{}^{(n)}(z) \end{bmatrix}$$

for each $n \in \mathbb{N}$. Moreover, the blocks $\mathcal{A}_{(1,1)}^{(n)}(z)$ and $\mathcal{A}_{(2,2)}^{(n)}(z)$ are always invertible for $z \in \mathcal{U}_{0,r}$. Our first new result can now be stated as follows.

Theorem 4.1. *Suppose that $A_0(H) = K$ and $A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$ and suppose further that $[A_{0,(m+1,m+1)}{}^{(m)}]^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$ for all $m \in \mathbb{N}$. Suppose too that $R(z)$ is analytic on $\mathcal{U}_{0,r}$ for some $r > 0$. For the sequences of orthogonal projections $\{S_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(H)$ and $\{T_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(K)$, there exist maximal orthogonal projections $S \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$ with $S_n \leq S$ and $T_n \leq T$ for all $n \in \mathbb{N}$. If $S \neq I$ and $T \neq I$, then we have corresponding orthogonal decompositions $H \cong S(H) \times (I - S)(H) = M \times M^\perp$ and $K \cong T(K) \times (I - T)(K) = N \times N^\perp$. If we define $\mathcal{A}_{1,(1,1)} = T A_1 S \in \mathcal{B}(M, N)$ and $\mathcal{A}_{0,(2,2)} = (I - T) A_0 (I - S) \in \mathcal{B}(M^\perp, N^\perp)$, then the inverse mappings $[\mathcal{A}_{1,(1,1)}]^{-1} \in \mathcal{B}(N, M)$ and $[\mathcal{A}_{0,(2,2)}]^{-1} \in \mathcal{B}(N^\perp, M^\perp)$ are each well defined.* □

PROOF. Since $S_n \le S_m \le I \in \mathcal{B}(H)$ and $T_n \le T_m \le I \in \mathcal{B}(K)$ for all $m, n \in \mathbb{N}$ with $m > n$, it follows from Zorn's lemma that we can find maximal orthogonal projections $S \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$ for the respective partially ordered sequences $\{S_n\}$ and $\{T_n\}$. There are two cases to consider. If the sequences $\{S_n(H)\} \subseteq H$ and $\{T_n(K)\} \subseteq K$ do not exhaust the respective spaces, then $S \neq I$ and $T \neq I$ and there exist corresponding nontrivial partitions $H \cong S(H) \times (I - S)(H) = M \times M^\perp$ and $K \cong T(K) \times (I - T)(K) = N \times N^\perp$ for the spaces H and K. If the sequences $\{S_n(H)\}$ and $\{T_n(K)\}$ do exhaust the respective spaces, then $S = I$ and $T = I$ and there is no corresponding partition.

Case 1: If $S \neq I$ and $T \neq I$, define $\mathcal{A}_i \in \mathcal{B}(M \times M^\perp, N \times N^\perp)$ by

$$\mathcal{A}_i = \begin{bmatrix} T \\ I - T \end{bmatrix} A_i \begin{bmatrix} S & I - S \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{i,(1,1)} & \mathcal{A}_{i,(1,2)} \\ 0 & \mathcal{A}_{i,(2,2)} \end{bmatrix}$$

for each $i = 0, 1$ and $\mathcal{A}(z) = \mathcal{A}_0 + \mathcal{A}_1 z \in \mathcal{B}(M \times M^\perp, N \times N^\perp)$ by

$$\mathcal{A}(z) = \begin{bmatrix} T \\ I - T \end{bmatrix} A(z) \begin{bmatrix} S & I - S \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{(1,1)}(z) & \mathcal{A}_{(1,2)}(z) \\ 0 & \mathcal{A}_{(2,2)}(z) \end{bmatrix},$$

where the individual blocks are defined by $\mathcal{A}_{(i,j)}(z) = \mathcal{A}_{0,(i,j)} + \mathcal{A}_{1,(i,j)} z$ for all $z \in \mathbb{C}$. We have assumed that $R(z) \in \mathcal{B}(K, H)$ is analytic for $z \in \mathcal{U}_{0,r}$ and so

$$\mathcal{R}(z) = \begin{bmatrix} S \\ I - S \end{bmatrix} R(z) \begin{bmatrix} T & I - T \end{bmatrix} = \begin{bmatrix} \mathcal{R}_{(1,1)}(z) & \mathcal{R}_{(1,2)}(z) \\ \mathcal{R}_{(2,1)}(z) & \mathcal{R}_{(2,2)}(z) \end{bmatrix}$$

is also analytic for $z \in \mathcal{U}_{0,r}$. The equations $\mathcal{R}(z)\mathcal{A}(z) = I$ and $\mathcal{A}(z)\mathcal{R}(z) = I$ for $z \in \mathcal{U}_{0,r}$ have a unique solution given by

$$\mathcal{R}_{(2,2)}(z) = [\mathcal{A}_{(2,2)}(z)]^{-1}, \quad \mathcal{R}_{(2,1)}(z) = 0, \quad \mathcal{R}_{(1,1)}(z) = [\mathcal{A}_{(1,1)}(z)]^{-1}$$

and

$$\mathcal{R}_{(1,2)}(z) = (-1)[\mathcal{A}_{(1,1)}(z)]^{-1} \mathcal{A}_{(1,2)}(z) [\mathcal{A}_{(2,2)}(z)]^{-1}$$

for all $z \in \mathcal{U}_{0,r}$. In particular, we note that $[\mathcal{A}_{(1,1)}(z)]^{-1} \in \mathcal{B}(N, M)$ is well defined. We have $M = S(H) \cong H_1 \times H_2 \times \cdots$ and $N = T(K) \cong K_1 \times K_2 \times \cdots$ and so the operator $\mathcal{A}_{(1,1)}(z) \in \mathcal{B}(M, N)$ can be represented in the form

$$\mathcal{A}_{(1,1)}(z) \cong \begin{bmatrix} A_{1,(1,1)} z & A_{(1,2)}(z) & A_{(1,3)}(z) & \cdots \\ 0 & A_{1,(2,2)} z & A_{(2,3)}(z) & \cdots \\ 0 & 0 & A_{1,(3,3)} z & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $\mathcal{R}_{(1,1)}(z) = [\mathcal{A}_{(1,1)}(z)]^{-1} \in \mathcal{B}(N, M)$ is well defined for $z \in \mathcal{U}_{0,r}$, it has an analogous representation

$$\mathcal{R}_{(1,1)}(z) \cong \begin{bmatrix} R_{(1,1)}(z) & R_{(1,2)}(z) & R_{(1,3)}(z) & \cdots \\ 0 & R_{(2,2)}(z) & R_{(2,3)}(z) & \cdots \\ 0 & 0 & R_{(3,3)}(z) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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on $\mathcal{U}_{0,r}$. If we equate corresponding blocks in the equations

$$\mathcal{R}_{(1,1)}(z)\mathcal{A}_{(1,1)}(z) = I \quad \text{and} \quad \mathcal{A}_{(1,1)}(z)\mathcal{R}_{(1,1)}(z) = I,$$

we obtain a doubly-infinite but naturally ordered set of equations. In the (j, j) position,

$$R_{(j,j)}(z)A_{1,(j,j)}z = I \quad \text{and} \quad A_{1,(j,j)}zR_{(j,j)}(z) = I$$

for each $j \in \mathbb{N}$. It follows that $[A_{1,(j,j)}]^{-1} \in \mathcal{B}(K_j, H_j)$ exists and that

$$R_{(j,j)}(z) = \frac{[A_{1,(j,j)}]^{-1}}{z} \tag{4.1}$$

for each $j \in \mathbb{N}$. In the $(j, j + 1)$ position a similar argument shows that

$$R_{(j,j+1)}(z) = (-1)\frac{[A_{1,(j,j)}]^{-1}A_{(j,j+1)}(z)[A_{1,(j+1,j+1)}]^{-1}}{z^2} \tag{4.2}$$

for each $j \in \mathbb{N}$. We can continue this process to progressively solve for all blocks in the operator matrix $\mathcal{R}_{(1,1)}(z)$ for all $z \in \mathcal{U}_{0,r}$. It is important to observe that each block $R_{(j,j+n)}(z)$ contains only terms in $1/z^k$ where $k \in \{1, 2, \ldots, n\}$. Because $\mathcal{R}_{(1,1)}(z)$ is analytic for $z \in \mathcal{U}_{0,r}$, it is now clear that there must be a convergent Laurent expansion in the form

$$\mathcal{R}_{(1,1)}(z) = \frac{\mathcal{R}_{-1,(1,1)}}{z} + \frac{\mathcal{R}_{-2,(1,1)}}{z^2} + \cdots , \tag{4.3}$$

where $\mathcal{R}_{-k,(1,1)} \in \mathcal{B}(N, M)$ for all $k \in \mathbb{N}$ and $\|\mathcal{R}_{-k,(1,1)}\|^{1/k} \to 0$ as $k \to \infty$. Hence, the Laurent series actually converges for all $z \in \mathcal{U}_{0,\infty}$. By using some tedious algebra to compare the solution (4.3) with our solution computed above in (4.1) and (4.2), it can be shown that

$$\mathcal{R}_{-k,(1,1)} = (-1)^{k-1}[\mathcal{A}_{1,(1,1)}]^{-1}(\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^{k-1}$$

for all $k \in \mathbb{N}$. This means that $\|[\mathcal{A}_{1,(1,1)}]^{-1}(\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^{k-1}\|^{1/k} \to 0$ as $k \to \infty$. Thus, we can now justify the formal Neumann expansion

$$(\mathcal{A}_{0,(1,1)} + \mathcal{A}_{1,(1,1)}z)^{-1} = \sum_{k=1}^{\infty}(-1)^{k-1}[\mathcal{A}_{1,(1,1)}]^{-1}(\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^{k-1}/z^k.$$

The maximality of S means that $[\mathcal{A}_{0,(2,2)}]^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}$. Since $A_0(H) = K$, it follows that $\mathcal{A}_{0,(2,2)}(M^\perp) = N^\perp$ and so $\mathcal{A}_{0,(2,2)}$ is a 1–1 mapping of $M^\perp$ onto $N^\perp$. Thus, by the Banach inverse theorem, $[\mathcal{A}_{0,(2,2)}]^{-1} \in \mathcal{B}(N^\perp, M^\perp)$ is also a well-defined bounded linear mapping.

Case 2: If $S = I$ and $T = I$, then we have $H \cong H_1 \times H_2 \times \cdots$ and $K \cong K_1 \times K_2 \times \cdots$ and similar arguments can be used to show that

$$\mathcal{A}(z) \cong \begin{bmatrix} A_{1,(1,1)}z & A_{(1,2)}(z) & A_{(1,3)}(z) & \cdots \\ 0 & A_{1,(2,2)}z & A_{(2,3)}(z) & \cdots \\ 0 & 0 & A_{1,(3,3)}z & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathcal{B}(H, K)$$

is invertible for $z \in \mathcal{U}_{0,r}$ and that $[\mathcal{A}_{1,(1,1)}]^{-1} \in \mathcal{B}(K, H)$ is well defined. □

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4.2. The second new result. Suppose that the conditions of Theorem 4.1 are satisfied. Since

$$\mathcal{A}(z) = \begin{bmatrix} T \\ I-T \end{bmatrix} A(z) \begin{bmatrix} S & I-S \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{(1,1)}(z) & \mathcal{A}_{(1,2)}(z) \\ 0 & \mathcal{A}_{(2,2)}(z) \end{bmatrix}$$

and since $\mathcal{A}(z)^{-1}$ exists if and only if $R(z) = A(z)^{-1}$ exists, it follows that the inverse mapping $\mathcal{R}(z) = \mathcal{A}(z)^{-1} \in \mathcal{B}(N \times N^{\perp}, M \times M^{\perp})$ is well defined with

$$\mathcal{R}(z) = \mathcal{A}(z)^{-1} = \begin{bmatrix} [\mathcal{A}_{(1,1)}(z)]^{-1} & -[\mathcal{A}_{(1,1)}(z)]^{-1}\mathcal{A}_{(1,2)}(z)[\mathcal{A}_{(2,2)}(z)]^{-1} \\ 0 & [\mathcal{A}_{(2,2)}(z)]^{-1} \end{bmatrix} \tag{4.4}$$

for all $z \in \mathcal{U}_{0,r}$. We have $\mathcal{A}_{(j,j)}(z) = \mathcal{A}_{0,(j,j)} + \mathcal{A}_{1,(j,j)}z$ for each $j = 1, 2$. From Theorem 4.1, we know that $[\mathcal{A}_{1,(1,1)}]^{-1} \in \mathcal{B}(N, M)$ and $[\mathcal{A}_{0,(2,2)}]^{-1} \in \mathcal{B}(N^{\perp}, M^{\perp})$ are both well defined. Hence, we can use the relevant Neumann expansions to show that

$$[\mathcal{A}_{(1,1)}(z)]^{-1} = [\mathcal{A}_{1,(1,1)}]^{-1} \sum_{k=1}^{\infty} (-1)^{k-1} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^{k-1}/z^k \tag{4.5}$$

for $|z| > s_1 = \lim_{k\to\infty} \|(\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^k\|^{1/k}$ and

$$[\mathcal{A}_{(2,2)}(z)]^{-1} = [\mathcal{A}_{0,(2,2)}]^{-1} \sum_{j=0}^{\infty} (-1)^j (\mathcal{A}_{1,(2,2)}[\mathcal{A}_{0,(2,2)}]^{-1})^j z^j \tag{4.6}$$

for $|z| < r_1 = 1/\lim_{j\to\infty} \|(\mathcal{A}_{1,(2,2)}[\mathcal{A}_{0,(2,2)}]^{-1})^j\|^{1/j}$. We initially assumed that $A(z)^{-1} \in \mathcal{B}(K, H)$ is well defined for $z \in \mathcal{U}_{0,r}$ and so there must be a Laurent series expansion for the singular part of the resolvent that converges for $z \in \mathcal{U}_{0,\infty}$. The Laurent series (4.5) converges for $z \in \mathcal{U}_{s_1,\infty}$ for some finite s_1 and so both expansions are valid for $z \in \mathcal{U}_{s_1,\infty}$. Since the Laurent series representation is unique, it follows that the two series must be identical. Hence, $s_1 = 0$. A similar argument applied to the regular part means that we must also have $r_1 = r$.

The expansions (4.5) and (4.6) can be used to extract the crucial coefficients $R_{-1}, R_0 \in \mathcal{B}(K, H)$ from (4.4). We can now state our second new result.

Theorem 4.2. *Suppose that $A_0(H) = K$ and $A_0^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$ and suppose further that $[A_{0,(m+1,m+1)}{}^{(m)}]^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$ for all $m \in \mathbb{N}$. Suppose too that $R(z)$ is analytic on $\mathcal{U}_{0,r}$ for some $r > 0$. If we substitute the series expansions (4.5) and (4.6) into (4.4) and extract the coefficients of $1/z$ and the constant coefficients, then*

$$R_{-1} \cong \mathcal{R}_{-1} = \begin{bmatrix} R_{-1,(1,1)} & R_{-1,(1,2)} \\ 0 & 0 \end{bmatrix},$$

where $R_{-1,(1,1)} = [\mathcal{A}_{1,(1,1)}]^{-1}$ and

$$\begin{aligned} R_{-1,(1,2)} &= [\mathcal{A}_{1,(1,1)}]^{-1} \Big\{ \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^{j+1} \mathcal{A}_{1,(1,2)} ([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^j \\ &\quad - \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^j \mathcal{A}_{0,(1,2)} ([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^j \Big\} [\mathcal{A}_{0,(2,2)}]^{-1} \end{aligned}$$

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and

$$R_0 \cong \mathcal{R}_0 = \begin{bmatrix} 0 & R_{0,(1,2)} \\ 0 & R_{0,(2,2)} \end{bmatrix},$$

where

$$\begin{aligned} &R_{0,(1,2)} \\ &\quad = [\mathcal{A}_{1,(1,1)}]^{-1}\bigg\{ \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^j \mathcal{A}_{0,(1,2)}([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^{j+1} \\ &\qquad - \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^j \mathcal{A}_{1,(1,2)}([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^{j}\bigg\}[\mathcal{A}_{0,(2,2)}]^{-1} \end{aligned}$$

and $R_{0,(2,2)} = [\mathcal{A}_{0,(2,2)}]^{-1}$. *The key projections* $P = R_{-1}A_1 \in \mathcal{B}(H)$ *and* $Q = A_1R_{-1} \in \mathcal{B}(K)$ *are given by*

$$P \cong \begin{bmatrix} I & P_{(1,2)} \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} P_{(1,2)} &= R_{-1,(1,1)}\mathcal{A}_{1,(1,2)} + R_{-1,(1,2)}\mathcal{A}_{1,(2,2)} \\ &= [\mathcal{A}_{1,(1,1)}]^{-1}\bigg\{ \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^j \mathcal{A}_{1,(1,2)}([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^{j} \\ &\qquad - \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^j \mathcal{A}_{0,(1,2)}([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^{j+1}\bigg\} \end{aligned}$$

and

$$Q \cong \begin{bmatrix} I & Q_{(1,2)} \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} Q_{(1,2)} &= \mathcal{A}_{1,(1,1)}R_{-1,(1,2)} \\ &= \bigg\{ \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^{j+1} \mathcal{A}_{1,(1,2)}([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^{j} \\ &\qquad - \sum_{j=0}^{\infty} (\mathcal{A}_{0,(1,1)}[\mathcal{A}_{1,(1,1)}]^{-1})^j \mathcal{A}_{0,(1,2)}([\mathcal{A}_{0,(2,2)}]^{-1}\mathcal{A}_{1,(2,2)})^{j}\bigg\}[\mathcal{A}_{0,(2,2)}]^{-1}. \qquad \square \end{aligned}$$

PROOF. It is easy to check that $P^2 = P$ and $Q^2 = Q$ and, although the details are complicated, only elementary algebra is required to check that $\{R_{-1}, R_0\}$ is a basic solution to the fundamental equations (2.2) and (2.3). We remind the reader that the requirements for a basic solution are given in Definition 3.4. Note that $P(H) = M$ and $Q(K) = N$ but in general $(I - P)(H) = M^c \neq M^\perp$ and $(I - Q)(K) = N^c \neq N^\perp$. $\square$

4.3. The sequence of unitary transformations. We described the first stage of the reduction process in Section 2.3 and observed that the original inversion problem could be replaced by an equivalent inversion problem on a smaller space. We concluded that if the unperturbed operator remains singular in the reduced pencil, then the process could be repeated. Although this high-level argument is correct, we believe that when the reduction continues *ad infinitum* the detailed inductive argument is also instructive.

The inductive hypothesis. Suppose that after stage $m \in \mathbb{N}$ we have the orthogonal decompositions $H \cong H_1 \times \cdots \times H_m \times H_m^\perp = F_m$ and $K \cong K_1 \times \cdots \times K_m \times K_m^\perp = G_m$ and that for each $j \leq m$ the associated natural orthogonal projections are denoted by $P_j \in \mathcal{B}(H)$ mapping H onto H_j and $Q_j \in \mathcal{B}(K)$ mapping K onto K_j. Let $U_m \in \mathcal{B}(H, F_m)$ and $V_m \in \mathcal{B}(K, G_m)$ be unitary transformations defined by

$$U_m = \begin{bmatrix} P_1 \\ \vdots \\ P_m \\ I - S_m \end{bmatrix} \quad \text{and} \quad V_m = \begin{bmatrix} Q_1 \\ \vdots \\ Q_m \\ I - T_m \end{bmatrix},$$

where we have written $S_m = P_1 + \cdots + P_m$ and $T_m = Q_1 + \cdots + Q_m$. We suppose that $A_0, A_1 \in \mathcal{B}(H, K)$ are represented in the equivalent form $A_0^{(m)}, A_1^{(m)} \in \mathcal{B}(F_m, G_m)$ defined by $A_0^{(m)} = V_m A_0 U_m^*$ and $A_1^{(m)} = V_m A_1 U_m^*$ and given by

$$A_0^{(m)} = \begin{bmatrix} 0 & A_{0,(1,2)} & A_{0,(1,3)} & \cdots & A_{0,(1,m)} & A_{0,(1,m+1)}{}^{(m)} \\ 0 & 0 & A_{0,(2,3)} & \cdots & A_{0,(2,m)} & A_{0,(2,m+1)}{}^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{0,(m-1,m)} & A_{0,(m-1,m+1)}{}^{(m)} \\ 0 & 0 & 0 & \cdots & 0 & A_{0,(m,m+1)}{}^{(m)} \\ 0 & 0 & 0 & \cdots & 0 & A_{0,(m+1,m+1)}{}^{(m)} \end{bmatrix}$$

and

$$A_1^{(m)} = \begin{bmatrix} A_{1,(1,1)} & A_{1,(1,2)} & \cdots & A_{1,(1,m)} & A_{1,(1,m+1)}{}^{(m)} \\ 0 & A_{1,(2,2)} & \cdots & A_{1,(2,m)} & A_{1,(2,m+1)}{}^{(m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{1,(m,m)} & A_{1,(m,m+1)}{}^{(m)} \\ 0 & 0 & \cdots & 0 & A_{1,(m+1,m+1)}{}^{(m)} \end{bmatrix},$$

where we have written $A_{i,(r,s)}{}^{(m)} = A_{i,(r,s)}$ for blocks that remain unchanged by subsequent transformations. Thus, $A^{(m)}(z) = V_m A(z) U_m^* \in \mathcal{B}(F_m, G_m)$ is given by

$$A^{(m)}(z) = \begin{bmatrix} A_{1,(1,1)}z & A_{(1,2)}(z) & \cdots & A_{(1,m)}(z) & A^{(m)}_{(1,m+1)}(z) \\ 0 & A_{1,(2,2)}z & \cdots & A_{(2,m)}(z) & A^{(m)}_{(2,m+1)}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{1,(m,m)}z & A^{(m)}_{(m,m+1)}(z) \\ 0 & 0 & \cdots & 0 & A^{(m)}_{(m+1,m+1)}(z) \end{bmatrix}.$$

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We suppose that $[A_{1,(j,j)}]^{-1} \in \mathcal{B}(K_j, H_j)$ is well defined for each $j \leq m$ and that ${A_{0,(m+1,m+1)}}^{(m)}(H_m^\perp) = K_m^\perp$ and $[{A_{0,(m+1,m+1)}}^{(m)}]^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$.

The induction. It follows that $H_{m+1} = [{A_{0,(m+1,m+1)}}^{(m)}]^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$ is a nontrivial closed subspace of $H_m^\perp$. Thus, we may write $H_m^\perp \cong H_{m+1} \times H_{m+1}^\perp$. Now from our original hypothesis that $R(z) = A(z)^{-1}$ is analytic for $z \in \mathcal{U}_{0,r}$ we know that $[{A_{(m+1,m+1)}}^{(m)}(z)]^{-1}$ must be well defined for $z \in \mathcal{U}_{0,r}$ and so by Lemma 2.1 we deduce that ${A_{1,(m+1,m+1)}}^{(m)}$ is bounded below on H_{m+1}. Hence, we also know that $K_{m+1} = {A_{1,(m+1,m+1)}}^{(m)}(H_{m+1})$ is closed and so we can write $K_m^\perp \cong K_{m+1} \times K_{m+1}^\perp$.

Let $P_{m+1} \in \mathcal{B}(H)$ denote the natural orthogonal projection onto the subspace $H_{m+1} \subseteq H$ and let $Q_{m+1} \in \mathcal{B}(K)$ denote the natural orthogonal projection onto the subspace $K_{m+1} \subseteq K$. Note that $P_jP_{m+1} = P_{m+1}P_j = 0$ and $Q_jQ_{m+1} = Q_{m+1}Q_j = 0$ for all $j \leq m$. If we write $F_{m+1} = H_1 \times \cdots \times H_{m+1} \times H_{m+1}^\perp$ and $G_{m+1} = K_1 \times \cdots \times K_{m+1} \times K_{m+1}^\perp$, then we have unitary operators $U_{m+1} \in \mathcal{B}(H, F_{m+1})$ and $V_m \in \mathcal{B}(K, G_{m+1})$ defined by

$$U_{m+1} = \begin{bmatrix} P_1 \\ \vdots \\ P_{m+1} \\ I - S_{m+1} \end{bmatrix} \quad \text{and} \quad V_{m+1} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{m+1} \\ I - T_{m+1} \end{bmatrix},$$

where we have written $S_{m+1} = P_1 + \cdots + P_{m+1}$ and $T_{m+1} = Q_1 + \cdots + Q_{m+1}$. The operators ${A_{i,(j,m+1)}}^{(m)} \in \mathcal{B}(H_j, K_m^\perp)$ can now be represented in the form

$$\begin{bmatrix} A_{i,(j,m+1)} & {A_{i,(j,m+2)}}^{(m+1)} \end{bmatrix} \in \mathcal{B}(H_j, K_{m+1} \times K_{m+1}^\perp)$$

for each $i = 0, 1$ and $j = 1, \ldots, m$. Since H_{m+1} is the null space of the operator ${A_{0,(m+1,m+1)}}^{(m)} \in \mathcal{B}(H_m^\perp, K_m^\perp)$, we can represent the operator in the equivalent form

$$\begin{bmatrix} 0 & {A_{0,(m+1,m+2)}}^{(m+1)} \\ 0 & {A_{0,(m+2,m+2)}}^{(m+1)} \end{bmatrix} \in \mathcal{B}(H_{m+1} \times H_{m+1}^\perp, K_{m+1} \times K_{m+1}^\perp).$$

Because $K_{m+1} = {A_{1,(m+1,m+1)}}^{(m)}(H_{m+1})$, the operator ${A_{1,(m+1,m+1)}}^{(m)} \in \mathcal{B}(H_m^\perp, K_m^\perp)$ can be represented in the equivalent form

$$\begin{bmatrix} A_{1,(m+1,m+1)} & {A_{1,(m+1,m+2)}}^{(m+1)} \\ 0 & {A_{1,(m+2,m+2)}}^{(m+1)} \end{bmatrix} \in \mathcal{B}(H_{m+1} \times H_{m+1}^\perp, K_{m+1} \times K_{m+1}^\perp).$$

Therefore, ${A_{(m+1,m+1)}}^{(m)}(z) \in \mathcal{B}(H_m^\perp, K_m^\perp)$ can be represented as

$$\begin{bmatrix} A_{1,(m+1,m+1)}z & {A_{(m+1,m+2)}}^{(m+1)}(z) \\ 0 & {A_{(m+2,m+2)}}^{(m+1)}(z) \end{bmatrix} \in \mathcal{B}(H_{m+1} \times H_{m+1}^\perp, K_{m+1} \times K_{m+1}^\perp).$$

The operator $[{A_{(m+1,m+1)}}^{(m)}(z)]^{-1} \in \mathcal{B}(K_m^\perp, H_m^\perp)$ is analytic for $z \in \mathcal{U}_{0,r}$. Hence, we deduce that $[A_{1,(m+1,m+1)}]^{-1} \in \mathcal{B}(H_{m+1}, K_{m+1})$ is well defined and that $[{A_{(m+2,m+2)}}^{(m+1)}(z)]^{-1} \in \mathcal{B}(H_{m+1}^\perp, K_{m+1}^\perp)$ is analytic for $z \in \mathcal{U}_{0,r}$. Thus, we have established that $A_0^{(m+1)}, A_1^{(m+1)} \in \mathcal{B}(F_{m+1}, G_{m+1})$ have the desired structural properties.

The conclusion. Since these properties are true for $n = 1$, they must be true for all $n \in \mathbb{N}$.

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4.4. Generalization of the first new result. In general, if $A_0(H) \neq K$ the recursive elimination is applied in exactly the same way until the null space of the unperturbed operator has been reduced to the zero element. This may or may not happen after a finite number of steps. If the elimination continues *ad infinitum*, then an application of Zorn's lemma similar to the one described in Theorem 4.1 will be required. Either way the problem of inverting the original pencil $A(z) = A_0 + A_1 z \in \mathcal{B}(H, K)$ where A_0 is neither onto nor 1–1 will be reduced to the inversion of a similar pencil $\mathcal{A}_{(2,2)}(z) = \mathcal{A}_{0,(2,2)} + \mathcal{A}_{1,(2,2)} z \in \mathcal{B}(M^\perp, N^\perp)$ on smaller spaces $M^\perp \subseteq H$ and $N^\perp \subseteq K$ where the unperturbed operator $\mathcal{A}_{0,(2,2)}$ is 1–1 but is not onto. Now in this case the adjoint operator $\mathcal{A}^*_{0,(2,2)} \in \mathcal{B}(N^\perp, M^\perp)$ is onto with $\mathcal{A}^*_{0,(2,2)}(N^\perp) = M^\perp$ but is not 1–1. Since $\mathcal{R}_{(2,2)}(z) = \mathcal{A}_{(2,2)}(z)^{-1} = [\mathcal{A}^*_{(2,2)}(z)^{-1}]^*$, we apply the inversion process to the adjoint operator $\mathcal{A}^*_{(2,2)}(z)$ to find $\mathcal{A}^*_{(2,2)}(z)^{-1}$ and then take the adjoint.

5. Solving the fundamental equations

In Section 3.2.4, we showed that if the key projections $P \in \mathcal{B}(H)$ and $Q \in \mathcal{B}(K)$ are known, then we can write $H \cong P(H) \times (I - P)(H) = M \times M^c$ and $K \cong Q(K) \times (I - Q)(K) = N \times N^c$. We also defined operators $\mathfrak{A}_i = QA_iP \in \mathcal{B}(M, N)$ and $\mathfrak{A}_i^c = Q^c A_i P^c = (I - Q)A_i(I - P) \in \mathcal{B}(M^c, N^c)$ for each $i = 0, 1$ in order to show that the solution to the fundamental equations can be written in the form

$$\mathfrak{R}_{-k} = (-1)^{k-1}(\mathfrak{A}_1^{-1}\mathfrak{A}_0)^{k-1}\mathfrak{A}_1^{-1},$$

where $\mathfrak{R}_{-k} = PR_{-k}Q \cong R_{-k}$ for $k \in \mathbb{N}$ and

$$\mathfrak{R}_j^c = (-1)^j([\mathfrak{A}_0^c]^{-1}\mathfrak{A}_1^c)^j(\mathfrak{A}_0^c)^{-1},$$

where $\mathfrak{R}_j^c = P^c R_j Q^c \cong R_j$ for $j \in \mathbb{N} - 1$. In Section 4.2, we used a sequence of unitary transformations to find general formulæ (4.4), (4.5) and (4.6) for the resolvent and hence found corresponding series expansions for the key projections $P \in \mathcal{B}(H)$ and $Q \in \mathcal{B}(K)$. We will now revisit these expansions to discuss numerical calculation of the basic solution. We assume throughout this section that $R(z) \in \mathcal{B}(K, H)$ is analytic for $z \in \mathcal{U}_{0,r}$ for some $r > 0$ and hence that $\lim_{k\to\infty} \|(\mathfrak{A}_1^{-1}\mathfrak{A}_0)^k\|^{1/k} = 0$ and $\lim_{j\to\infty} \|([\mathfrak{A}_0^c]^{-1}\mathfrak{A}_1^c)^j\|^{1/j} = 1/r$.

5.1. The singular part of the resolvent. The singular part of the resolvent is formally represented as a series of negative powers. We define

$$\mathfrak{R}_\sigma(z) = (\mathfrak{A}_0 + \mathfrak{A}_1 z)^{-1} = \sum_{k\in\mathbb{N}} \mathfrak{R}_{-k} z^{-k}$$

for all $z \neq 0$, where $\mathfrak{R}_{-k} = (-1)^{k-1}[\mathfrak{A}_1^{-1}\mathfrak{A}_0]^{k-1}\mathfrak{A}_1^{-1}$ for all $k \in \mathbb{N}$.

In the case where the decomposition continues *ad infinitum*, we know that $M \cong H_1 \times H_2 \times \cdots$ and $N \cong K_1 \times K_2 \times \cdots$ and so we can represent these operators as

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infinitely extending operator matrices

$$\mathfrak{A}_0 \cong \mathcal{A}_{0,(1,1)} \cong \begin{bmatrix} 0 & A_{0,(1,2)} & A_{0,(1,3)} & A_{0,(1,4)} & \cdots \\ 0 & 0 & A_{0,(2,3)} & A_{0,(2,4)} & \cdots \\ 0 & 0 & 0 & A_{0,(3,4)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathfrak{A}_1 \cong \mathcal{A}_{1,(1,1)} \cong \begin{bmatrix} A_{1,(1,1)} & A_{1,(1,2)} & A_{1,(1,3)} & A_{1,(1,4)} & \cdots \\ 0 & A_{1,(2,2)} & A_{1,(2,3)} & A_{1,(2,4)} & \cdots \\ 0 & 0 & A_{1,(3,3)} & A_{1,(3,4)} & \cdots \\ 0 & 0 & 0 & A_{1,(4,4)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The inverse operator $\mathfrak{A}_1^{-1} \in \mathcal{B}(N, M)$ is well defined and we already showed in the proof of Theorem 4.1 that the operator $[A_{1,(j,j)}]^{-1} \in \mathcal{B}(K_j, H_j)$ is well defined for each $j \in \mathbb{N}$. Because $\mathcal{A}_{1,(1,1)}$ is block upper triangular, the inverse operator can also be represented as a block upper triangular operator matrix in the form

$$\mathfrak{A}_1^{-1} = [\mathcal{A}_{1,(1,1)}]^{-1} \cong \begin{bmatrix} R_{1,(1,1)} & R_{1,(1,2)} & R_{1,(1,3)} & R_{1,(1,4)} & \cdots \\ 0 & R_{1,(2,2)} & R_{1,(2,3)} & R_{1,(2,4)} & \cdots \\ 0 & 0 & R_{1,(3,3)} & R_{1,(3,4)} & \cdots \\ 0 & 0 & 0 & R_{1,(4,4)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $R_{1,(i,j)} \in \mathcal{B}(K_j, H_i)$ for each $i, j \in \mathbb{N}$ and $R_{1,(j,j)} \cong [A_{1,(j,j)}]^{-1}$ for each $j \in \mathbb{N}$. It follows that

$$\mathfrak{A}_1^{-1}\mathfrak{A}_0\mathfrak{A}_1^{-1} \cong \begin{bmatrix} 0 & S_{0,(1,2)} & S_{0,(1,3)} & S_{0,(1,4)} & \cdots \\ 0 & 0 & S_{0,(2,3)} & S_{0,(2,4)} & \cdots \\ 0 & 0 & 0 & S_{0,(3,4)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$(\mathfrak{A}_1^{-1}\mathfrak{A}_0)^2\mathfrak{A}_1^{-1} \cong \begin{bmatrix} 0 & 0 & T_{0,(1,3)} & T_{0,(1,4)} & \cdots \\ 0 & 0 & 0 & T_{0,(2,4)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and so on *ad infinitum*. Since $\lim_{k\to\infty} \|(\mathfrak{A}_1^{-1}\mathfrak{A}_0)^k\|^{1/k} = 0$, we already know that the series for $\mathfrak{R}_\sigma(z)$ converges in the region $\mathcal{U}_{0,\infty}$. However, if we write $\boldsymbol{n}_j = Q_j\boldsymbol{n} \in K_j$ for

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each $j \in \mathbb{N}$ and all $\boldsymbol{n} \in N$, then we can also see that $(\mathfrak{A}_1^{-1}\mathfrak{A}_0)^{k-1}\mathfrak{A}_1^{-1}\boldsymbol{n}_j = \boldsymbol{0}$ when $k > j$. Hence, $\mathfrak{R}_{-k}\boldsymbol{n}_j = \boldsymbol{0}$ when $k > j$. It follows that

$$\mathfrak{R}(z)\boldsymbol{n}_j = \sum_{k\in\mathbb{N}}(\mathfrak{R}_{-k}\boldsymbol{n}_j)z^{-k} = \sum_{k=1}^{j}(\mathfrak{R}_{-k}\boldsymbol{n}_j)z^{-k}$$

is always well defined irrespective of any esoteric arguments about convergence of the series in the operator topology. Thus, in numerical calculations, we may choose to approximate $\boldsymbol{n} \approx \sum_{j=1}^{m} \boldsymbol{n}_j$, where $\boldsymbol{n}_j \in K_j$ for some sufficiently large $m \in \mathbb{N}$, in which case we can calculate

$$\mathfrak{R}(z)\sum_{j=1}^{m}\boldsymbol{n}_j = \sum_{j=1}^{m}\mathfrak{R}(z)\boldsymbol{n}_j = \sum_{j=1}^{m}\sum_{k=1}^{j}(\mathfrak{R}_{-k}\boldsymbol{n}_j)z^{-k}$$

using only finite operator matrices and finite sums. If the decomposition terminates after m steps or if we simply restrict our attention to the projected mappings $\mathfrak{A}_0^{(m)}, \mathfrak{A}_1^{(m)} \in \mathcal{B}(H_1 \times \cdots \times H_m, K_1 \times \cdots \times K_m)$, then we can represent the operators as finite operator matrices

$$\mathfrak{A}_0^{(m)} \cong \mathcal{A}_{0,(1,1)}^{(m)} \cong \begin{bmatrix} 0 & A_{0,(1,2)} & A_{0,(1,3)} & \cdots & A_{0,(1,m-1)} & A_{0,(1,m)} \\ 0 & 0 & A_{0,(2,3)} & \cdots & A_{0,(2,m-1)} & A_{0,(2,m)} \\ 0 & 0 & 0 & \cdots & A_{0,(3,m-1)} & A_{0,(3,m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_{0,(m-1,m)} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\mathfrak{A}_1^{(m)} \cong \mathcal{A}_{0,(1,1)}^{(m)} \cong \begin{bmatrix} A_{1,(1,1)} & A_{1,(1,2)} & A_{1,(1,3)} & \cdots & A_{1,(1,m)} \\ 0 & A_{1,(2,2)} & A_{1,(2,3)} & \cdots & A_{1,(2,m)} \\ 0 & 0 & A_{1,(3,3)} & \cdots & A_{1,(3,m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{1,(m,m)} \end{bmatrix}.$$

It follows that $([\mathfrak{U}_1^{(m)}]^{-1}\mathfrak{U}_0^{(m)})^{k-1}[\mathfrak{U}_1^{(m)}]^{-1} = 0$ for $k > m$. This means that if the decomposition terminates after m steps, then the resolvent has a pole at $z = 0$ of order at most m. If the decomposition continues *ad infinitum*, then it means that we could choose to approximate the resolvent by a projected resolvent with a finite-order pole at $z = 0$.

5.2. The regular part of the resolvent. The regular part of the resolvent is formally represented by a series of nonnegative powers. We define

$$\mathfrak{R}_\rho(z) = (\mathfrak{A}_0^c + \mathfrak{A}_1^c z)^{-1} = \sum_{j\in\mathbb{N}-1}\mathfrak{R}_j^c z^j$$

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for $|z| < r$, where $\mathfrak{R}_j^c = (-1)^j([\mathfrak{A}_0^c]^{-1}\mathfrak{A}_1^c)^j(\mathfrak{A}_0^c)^{-1}$ for all $j \in \mathbb{N} - 1$. We know that $M^c = (I - P)(H) \cong (I - P)(M \times M^\perp)$ and so the general element of $M^c \subseteq M \times M^\perp$ can be written in the form

$$(I - P)\boldsymbol{x} \cong \begin{bmatrix} 0 & -P_{(1,2)} \\ 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{m}^\perp \end{bmatrix} = \begin{bmatrix} -P_{(1,2)}\boldsymbol{m}^\perp \\ \boldsymbol{m}^\perp \end{bmatrix}.$$

Since $\mathfrak{A}_0^c \cong (I - Q)A_0(I - P)$, we can write

$$\mathfrak{A}_0^c \cong \begin{bmatrix} 0 & -Q_{(1,2)}\mathcal{A}_{0,(2,2)} \\ 0 & \mathcal{A}_{0,(2,2)} \end{bmatrix}.$$

We have $N^c = (I - Q)(K) \cong (I - Q)(N \times N^\perp)$ and so the general element of $N^c \subseteq N \times N^\perp$ can be written as

$$(I - Q)\boldsymbol{y} \cong \begin{bmatrix} 0 & -Q_{(1,2)} \\ 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{n}^\perp \end{bmatrix} = \begin{bmatrix} -Q_{(1,2)}\boldsymbol{n}^\perp \\ \boldsymbol{n}^\perp \end{bmatrix}.$$

To show that $\mathfrak{A}_0^c$ is invertible, we need to show that the equation

$$\begin{bmatrix} 0 & -Q_{(1,2)}\mathcal{A}_{0,(2,2)} \\ 0 & \mathcal{A}_{0,(2,2)} \end{bmatrix} \begin{bmatrix} -P_{(1,2)}\boldsymbol{m}^\perp \\ \boldsymbol{m}^\perp \end{bmatrix} = \begin{bmatrix} -Q_{(1,2)}\boldsymbol{n}^\perp \\ \boldsymbol{n}^\perp \end{bmatrix}$$

or the equivalent equation

$$\begin{bmatrix} -Q_{(1,2)}\mathcal{A}_{0,(2,2)}\boldsymbol{m}^\perp \\ \mathcal{A}_{0,(2,2)}\boldsymbol{m}^\perp \end{bmatrix} = \begin{bmatrix} -Q_{(1,2)}\boldsymbol{n}^\perp \\ \boldsymbol{n}^\perp \end{bmatrix}$$

has a unique solution for all $\boldsymbol{n}^\perp \in N^\perp$. Since this equation has a unique solution if and only if $\mathcal{A}_{0,(2,2)}\boldsymbol{m}^\perp = \boldsymbol{n}^\perp$ has a unique solution, and since we have already seen that $[\mathcal{A}_{0,(2,2)}]^{-1} \in \mathcal{B}(N^\perp, M^\perp)$ is well defined, the desired result is true. Hence, $[\mathfrak{A}_0^c]^{-1} \in \mathcal{B}(N^c, M^c)$ is a well-defined bounded linear operator.

6. A particular example

Suppose that $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is a strictly decreasing sequence of real numbers with $1 \geq \lambda_n^{1/n} \downarrow 0$ as $n \uparrow \infty$ and let $\lambda = \sum_{n\in\mathbb{N}} \lambda_n$. Solve the system of differential equations

$$\dot{f}_n(t) - \lambda_n f_{n+1}(t) - \lambda_n g(t) = 0$$

with $f_n(0) = 0$ for all $n \in \mathbb{N}$ and

$$\dot{g}(t) + \lambda g(t) = 0$$

with $g(0) = 1$ on the interval $t \in [0, \infty)$.

The system can be regarded as a speculative model of growth and decay. The mass g of master material $\mathcal{G}$ decays at a rate proportional to its mass. As the master material decays it is converted using additional unspecified ingredients into a collection of mixed materials $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, where the mass f_n of material $\mathcal{F}_n$ for each $n \in \mathbb{N}$ grows at

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a rate proportional to the total of the masses f_{n+1} and g of the materials $\mathcal{F}_{n+1}$ and $\mathcal{G}$, respectively.

We begin by taking a Laplace transform of the system. For $s \in \mathbb{C}$ with real part $\mathfrak{R}(s) > 0$, define

$$\boldsymbol{F}(s) = \begin{bmatrix} F_1(s) \\ F_2(s) \\ \vdots \end{bmatrix},$$

where $F_n(s) = \int_{[0,\infty)} f_n(t)e^{-st}\,dt$ for all $n \in \mathbb{N}$ and $G(s) = \int_{[0,\infty)} g(t)e^{-st}\,dt$. Now we can write the Laplace transform system in augmented operator matrix form as

$$\left[\begin{array}{c|c} sI - L & -\boldsymbol{\lambda} \\ \hline \boldsymbol{0}^* & s + \lambda \end{array}\right] \left[\begin{array}{c} \boldsymbol{F}(s) \\ \hline G(s) \end{array}\right] = \left[\begin{array}{c} \boldsymbol{0} \\ \hline 1 \end{array}\right], \tag{6.1}$$

where we have written

$$L = \begin{bmatrix} 0 & \lambda_1 & 0 & \cdots \\ 0 & 0 & \lambda_2 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \end{bmatrix} \quad \text{and} \quad \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

The space $\ell^2 \times \mathbb{C}$ becomes a Hilbert space if we define

$$\langle (\boldsymbol{x}, \xi), (\boldsymbol{y}, \eta) \rangle = \sum_{j \in \mathbb{N}} x_j \overline{y_j} + \xi \overline{\eta}$$

for each $(\boldsymbol{x}, \xi), (\boldsymbol{y}, \eta) \in \ell^2 \times \mathbb{C}$. Define $A_0, A_1 \in \mathcal{B}(\ell^2 \times \mathbb{C})$ by

$$A_0 = \left[\begin{array}{c|c} -L & -\boldsymbol{\lambda} \\ \hline \boldsymbol{0}^* & \lambda \end{array}\right] \quad \text{and} \quad A_1 = \left[\begin{array}{c|c} I & \boldsymbol{0} \\ \hline \boldsymbol{0}^* & 1 \end{array}\right]$$

and write

$$A(z) = A_0 + A_1 z = \begin{bmatrix} zI - L & -\boldsymbol{\lambda} \\ \boldsymbol{0}^* & z + \lambda \end{bmatrix} \in \mathcal{B}(\ell^2 \times \mathbb{C}) \tag{6.2}$$

for all $z \in \mathbb{C}$. To solve (6.1), we need to calculate the resolvent $R(z) = A(z)^{-1}$. It is clear that $A_0^{-1}(\{\boldsymbol{0}\}) \neq \{\boldsymbol{0}\}$ and that the proposed recursive projection procedure continues *ad infinitum*. Nevertheless, the projections do not exhaust the entire space. For each $(\boldsymbol{x}, \xi) \in \ell^2 \times \mathbb{C}$ and each $n \in \mathbb{N}$, we have $S_n(\boldsymbol{x}, \xi) = T_n(\boldsymbol{x}, \xi) = \sum_{j=1}^n x_j \boldsymbol{e}_j$. It follows that the maximal orthogonal projections are given by $S(\boldsymbol{x}, \xi) = T(\boldsymbol{x}, \xi) = \boldsymbol{x}$ for each $(\boldsymbol{x}, \xi) \in \ell^2 \times \mathbb{C}$. Therefore, the orthogonal partition defined by the maximal projections coincides with the natural partition of the space $\ell^2 \times \mathbb{C}$. It follows from Theorem 4.2 that the basic solution $R_{-1}, R_0 \in \mathcal{B}(\ell^2 \times \mathbb{C})$ takes the form

$$R_{-1} = \left[\begin{array}{c|c} I & \boldsymbol{\alpha} \\ \hline \boldsymbol{0}^* & 0 \end{array}\right] \quad \text{and} \quad R_0 = \left[\begin{array}{c|c} 0 & \boldsymbol{\beta} \\ \hline \boldsymbol{0}^* & 1/\lambda \end{array}\right]$$

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on some region $\mathcal{U}_{0,r}$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{B}(\mathbb{C}, \ell^2)$ are defined by $\boldsymbol{\alpha}(\xi) = \xi\boldsymbol{\alpha}$ and $\boldsymbol{\beta}(\xi) = \xi\boldsymbol{\beta}$ for all $\xi \in \mathbb{C}$. By substituting in the fundamental equations $R_{-1}A_1 + R_0A_0 = I$ and $A_1R_{-1} + A_0R_0 = I$, we have $\boldsymbol{\alpha} = (\lambda I + L)^{-1}\boldsymbol{\lambda}$ and $\boldsymbol{\beta} = (-1)(\lambda I + L)^{-1}\boldsymbol{\lambda}/\lambda$. Therefore,

$$R_{-1} = \begin{bmatrix} I & (\lambda I + L)^{-1}\boldsymbol{\lambda} \\ \mathbf{0}^* & 0 \end{bmatrix} \quad \text{and} \quad R_0 = \begin{bmatrix} 0 & -(\lambda I + L)^{-1}\boldsymbol{\lambda}/\lambda \\ \mathbf{0}^* & 1/\lambda \end{bmatrix}.$$

Hence, we can calculate the key projections $P = R_{-1}A_1$ and $Q = A_1R_{-1}$ and use the general formulæ (3.5) and (3.6) to find the Laurent series representation for the resolvent $R(z)$. Although we can find $R(z)$ by using these formal structures, it is quite straightforward in this particular example—because the maximal orthogonal decomposition for the space $\ell^2 \times \mathbb{C}$ coincides with the natural decomposition—to simply apply elementary block row operations to $A(z)$ in (6.2). This gives

$$R(z) = \begin{bmatrix} (zI - L)^{-1} & (z+\lambda)^{-1}(zI - L)^{-1}\boldsymbol{\lambda} \\ \mathbf{0}^* & (z+\lambda)^{-1} \end{bmatrix} \in \mathcal{B}(\ell^2 \times \mathbb{C}) \tag{6.3}$$

for all $z \neq 0, -\lambda$, where we have used the Laurent series expansion

$$(zI - L)^{-1} = (1/z)[I + (L/z) + (L/z)^2 + \cdots] \tag{6.4}$$

to define the inverse operator $(zI - L)^{-1}$. In this regard, we note that the individual elements of L^k are given by

$$L^k_{ij} = \begin{cases} \lambda_n\lambda_{n+1}\cdots\lambda_{n+k-1} & \text{when } (i,j) = (n, n+k) \text{ for each } n \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

for all $(i, j) \in \mathbb{N}^2$ and $k \in \mathbb{N}$. It follows that $\|L^k\|^{1/k} = (\lambda_1\lambda_2\cdots\lambda_k)^{1/k} \leq \lambda_k^{1/k} \downarrow 0$ as $k \uparrow \infty$. Hence, the Laurent series (6.4) converges for all $z \neq 0$. From (6.3), we also deduce that the solution to the Laplace transform system (6.1) is

$$\begin{bmatrix} \boldsymbol{F}(s) \\ G(s) \end{bmatrix} = \begin{bmatrix} (sI - L)^{-1} & (s+\lambda)^{-1}(sI - L)^{-1}\boldsymbol{\lambda} \\ \mathbf{0}^* & (s+\lambda)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} (s+\lambda)^{-1}(sI - L)^{-1}\boldsymbol{\lambda} \\ (s+\lambda)^{-1} \end{bmatrix}$$

for all s with $\Re(s) > 0$. An elementary inversion gives $g(t) = e^{-\lambda t}$ for all $t \in [0, \infty)$. For the remaining inversions,

$$\boldsymbol{e}_n^* L^k \boldsymbol{\lambda} = \lambda_n\lambda_{n+1}\cdots\lambda_{n+k}$$

for each $n, k \in \mathbb{N}$ and so

$$F_n(s) = \frac{1}{s+\lambda}\left[\frac{\lambda_n}{s} + \frac{\lambda_n\lambda_{n+1}}{s^2} + \frac{\lambda_n\lambda_{n+1}\lambda_{n+2}}{s^3} + \cdots\right],$$

from which it follows that

$$f_n(t) = e^{-\lambda t}\int_{[0,t]} e^{\lambda\tau}\left[\lambda_n + \lambda_n\lambda_{n+1}\tau + \frac{\lambda_n\lambda_{n+1}\lambda_{n+2}\tau^2}{2!} + \cdots\right] d\tau$$

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for all $t \in [0, \infty)$ and all $n \in \mathbb{N}$. Note that the series under the integral sign is uniformly convergent on $[0, t]$ for all $t \in [0, \infty)$. If we wish to extract a particular material F_n from the mixture at any given time, the mass of extractable material is given by

$$g_n(t) = f_n(t) - \lambda_n \int_0^t f_{n+1}(\tau)\, d\tau$$

for all $n \in \mathbb{N}$. Note that $g + \sum_{n \in \mathbb{N}} g_n = 1$.

REMARK 6.1. Although the Laplace transform inversion only requires the Laurent series for $R(z)$ on some region $\mathcal{U}_{0,r}$, in this particular example we note that

$$(z + \lambda)^{-1} = \begin{cases} (1/\lambda)[1 - (z/\lambda) + (z/\lambda)^2 - \cdots] & \text{for } 0 \leq |z| < \lambda, \\ (1/z)[1 - (\lambda/z) + (\lambda/z)^2 - \cdots] & \text{for } \lambda < |z| < \infty. \end{cases} \tag{6.5}$$

Now we can use (6.3), (6.4) and the alternative expansions in (6.5) to write down the full series for $R(z)$. By extracting the relevant coefficients, we can see that the basic solution to the fundamental equations is given by

$$R_{-1} = \begin{bmatrix} I & (\lambda I + L)^{-1}\boldsymbol{\lambda} \\ \mathbf{0}^* & 0 \end{bmatrix} \quad \text{and} \quad R_0 = \begin{bmatrix} 0 & -(\lambda I + L)^{-1}\boldsymbol{\lambda}/\lambda \\ \mathbf{0}^* & 1/\lambda \end{bmatrix}$$

for the region $0 < |z| < \lambda$ and by

$$R_{-1} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^* & 1 \end{bmatrix} = I \quad \text{and} \quad R_0 = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0}^* & 0 \end{bmatrix} = 0$$

for $\lambda < |z| < \infty$. In each case the coefficients of the Laurent series are given in terms of $\{R_{-1}, R_0\}$ by the general formulæ (3.5) and (3.6). □

7. Numerical calculations

More work is needed to develop effective numerical algorithms for computation of the resolvent operator. The main challenge is to calculate the maximal projection operators. For the moment we simply wish to highlight two important points.

7.1. Calculation of projection operators for the unitary transformations. The proposed unitary transformations rely on calculation of associated projection operators. At each stage we have Hilbert spaces H and K and an operator $A_0 \in \mathcal{B}(H, K)$ such that $A_0(H) = K$ and $M = A_0{}^{-1}(\{\mathbf{0}\}) \neq \{\mathbf{0}\}$. Since A_0 is a 1–1 mapping of $M^\perp$ onto $K = A_0(H)$, it follows that $A_0{}^*$ is a 1–1 mapping of K onto $M^\perp$. Thus, $A_0 A_0{}^* \in \mathcal{B}(K)$ is invertible. The natural orthogonal projection $P_M \in \mathcal{B}(H)$ which maps H onto M is given by

$$P_M = I - A_0{}^*(A_0 A_0{}^*)^{-1} A_0.$$

The operator $A_0 A_0{}^*$ is a bounded self-adjoint operator. There are several standard algorithms that can be used for numerical calculation of $(A_0 A_0{}^*)^{-1}$.

At each stage we also have $(A_1 P_M) \in \mathcal{B}(M, K)$ with $(A_1 P_M)(M) = N$. It follows that $(A_1 P_M)^* \in \mathcal{B}(K, M)$ with $(A_1 P_M)^*(K) = M$. Since $(A_1 P_M)^*$ is a 1–1 mapping of N onto

M and (A_1P_M) is a 1–1 mapping of M onto N, we can argue that $(A_1P_M)^*(A_1P_M) \in \mathcal{B}(M)$ is invertible. Therefore, the natural orthogonal projection $Q_N \in \mathcal{B}(K)$ which maps K onto N can be defined by the formula

$$Q_N = (A_1P_M)[(A_1P_M)^*(A_1P_M)]^{-1}(A_1P_M)^*.$$

The operator $(A_1P_M)^*(A_1P_M)$ is also a bounded self-adjoint operator and so, once again, we note that standard algorithms can be used for numerical calculation of $[(A_1P_M)^*(A_1P_M)]^{-1}$. If we regard $(A_1P_M)^*(A_1P_M) \in \mathcal{B}(H)$, then the inverse operator in the definition of Q_N is the Moore–Penrose inverse $[(A_1P_M)^*(A_1P_M)]^\dagger$.

7.2. Calculation of the maximal orthogonal projections. If the decomposition terminates after m steps, then calculation of the maximal projections $S = S_m \in \mathcal{B}(H)$ and $T = T_m \in \mathcal{B}(K)$ is simply the standard projection calculation described above. Thus, we need only consider the case where the decomposition continues *ad infinitum*. Let $\boldsymbol{x} \in H$ and $\boldsymbol{y} \in K$. Define $\boldsymbol{m} = S\boldsymbol{x} \in S(H) \cong H_1 \times H_2 \times \cdots$ and $\boldsymbol{n} = T\boldsymbol{y} \in T(K) \cong K_1 \times K_2 \times \cdots$ and write $\boldsymbol{m} = \sum_{j\in\mathbb{N}} \boldsymbol{m}_j$ and $\boldsymbol{n} = \sum_{j\in\mathbb{N}} \boldsymbol{n}_j$, where $\boldsymbol{m}_j \in H_j$ and $\boldsymbol{n}_j \in K_j$ for each $j \in \mathbb{N}$. Now we have $S_n\boldsymbol{x} = \sum_{j=1}^n \boldsymbol{m}_j$ and $T_n\boldsymbol{y} = \sum_{j=1}^n \boldsymbol{n}_j$ and so

$$\|S\boldsymbol{x} - S_n\boldsymbol{x}\|^2 = \sum_{j=n+1}^{\infty} \|\boldsymbol{m}_j\|^2 \to 0 \quad \text{and} \quad \|T\boldsymbol{y} - T_n\boldsymbol{y}\|^2 = \sum_{j=n+1}^{\infty} \|\boldsymbol{n}_j\|^2 \to 0$$

as $n \to \infty$. Therefore, $S_n\boldsymbol{x} \to S\boldsymbol{x}$ for all $\boldsymbol{x} \in H$ and $T_n\boldsymbol{y} \to T\boldsymbol{y}$ for all $\boldsymbol{y} \in K$. Thus, we can see that $\{S_n\}_{n\in\mathbb{N}}$ converges weakly to S and $\{T_n\}_{n\in\mathbb{N}}$ converges weakly to T. This means that for each given $\boldsymbol{x} \in H$ and $\boldsymbol{y} \in K$, we can calculate $\boldsymbol{m} = S\boldsymbol{x}$ and $\boldsymbol{n} = T\boldsymbol{y}$ as accurately as we please. However, if $\boldsymbol{m}_{n+1} \in H_{n+1}$, then $(S - S_n)\boldsymbol{m}_{n+1} = \boldsymbol{m}_{n+1} \Rightarrow \|S - S_n\| \geq 1$ for all $n \in \mathbb{N}$ and, if $\boldsymbol{n}_{n+1} \in K_{n+1}$, then $(T - T_n)\boldsymbol{n}_{n+1} = \boldsymbol{n}_{n+1} \Rightarrow \|T - T_n\| \geq 1$ for all $n \in \mathbb{N}$. Thus, $\{S_n\}_{n\in\mathbb{N}}$ does not converge strongly to S and $\{T_n\}_{n\in\mathbb{N}}$ does not converge strongly to T.

8. Conclusions

We have proposed a sequence of unitary transformations that allow us to define maximal orthogonal projections and thereby find general expressions for the Laurent series coefficients of the resolvent and for the key (nonorthogonal) projection operators that separate the original doubly-infinite sets of left and right fundamental equations into two singly-infinite sets. The separated sets can then be solved recursively. Although we can calculate the action of the maximal orthogonal projections on each fixed element as accurately as we please, the question of valid operator approximations to these projections remains open. An important component of future research will be the development of efficient numerical computation routines.

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