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UPPER AND LOWER FREQUENTLY UNIVERSAL SERIES

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Abstract. We introduce the notion of upper and lower frequently universal sequences and see that 'most' of the universal approximations are obtained by sets of indices which have upper density 1 and lower density 0. We also show that a class of universal series related to lower density is of first category.

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1. Introduction. In this paper, motivated by the interesting notion of frequent hypercyclic operators introduced in [**1**] and continued in [**3**], we define and study the notions of frequent universal series and obtain results that allow us to have a better insight of what happens in general in all universal approximations, as we will see in the sequel.

The first universal series was obtained by Fekete [**8**] before 1914. He proved the existence of power series whose partial sums approximate uniformly every continuous real valued function f defined on $[-1, 1]$ with $f(0) = 0$. In 1945, Menchoff proved the existence of universal trigonometric series, that is trigonometric series whose subsequences of the sequence of its partial sums converge almost everywhere (or equivalently in measure) with respect to the Lebesgue measure, to all measurable complexed valued functions defined by $T = \{e^{it} : t \in [0, 2\pi)\}$ (see [**6**] and Example 1.4 in this paper). Since then a lot of results on universal series have appeared. A good amount of references can be found in [**2**].

Typical cases are the universal Taylor series, which exhibit universal approximations with respect to uniform limits. More precisely, a power series $\sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{C}$ with finite radius of convergence, say 1, is universal if its partial sums are dense in the space $\mathbb{X} = \{f : K \to \mathbb{C} \mid f$ is continuous on K and f is holomorphic on $K^0\}$, endowed with the supremum norm, where K is a compact subset of C with connected complement, which is also disjoint from the open unit disc. We note that it is necessary for K to be as above to have such universal approximations (see [**9**]). Also, there are power series which are universal simultaneously for all K (for details see [**7**]).

As in the previous paragraph, in all universal approximations we have the following situation:

There is a topological vector space $\mathbb{X}$ over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ (mainly a topological vector space whose topology is induced by a metric invariant under translations) and a sequence $\{x_n\}$ of elements of $\mathbb{X}$ (x_n is the nth power function, $x_n(z) = z^n$, in the previous paragraph), and by the term universal series we mean a series $\sum_{n=0}^{\infty} a_n x_n$, $a_n \in \mathbb{K}$, $n = 1, 2, \ldots$ such that its partial sums are dense in $\mathbb{X}$.

(In Example 1.4, we study Menchoff's trigonometric series according to the above modelling).

Hence, in order to study the universal series in general, we adopt the following terminology [**2**]:

- $\mathbb{X}$ is a metrizable topological vector space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, for which we also assume that its topology is induced by a translation invariant metric ρ.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\mathbb{A}$ is a vector subspace of $\mathbb{K}^{\mathbb{N}}$, which carries a complete metrizable vector space topology induced by a translation-invariant metric d.

Moreover, we assume the following:

(A.1) The projections $\mathbb{A} \to \mathbb{K}$, $a = (\alpha_j) \mapsto \alpha_m$ are continuous for any $m \in \mathbb{N}$.

(A.2) The set of 'polynomials'

$$G = \{a = (\alpha_j) \in \mathbb{K}^{\mathbb{N}} | \{j : \alpha_j \neq 0\} \text{ is finite}\}$$

is a dense subset of $\mathbb{A}$.

Throughout this paper $(\mathbb{X}, \rho)$ and $(\mathbb{A}, d)$ will be as above. Also, $x_j, j = 0, 1, 2, \ldots$ will be a fixed sequence of elements of $\mathbb{X}$.

Definition 1.1 (**[2]**). (a) A series $\sum_{j=0}^{\infty} \alpha_j x_j$, where $(\alpha_j) \in \mathbb{K}^{\mathbb{N}}$ is called universal iff the sequence $(\sum_{j=0}^{n} \alpha_j x_j)$ of its partial sums is dense in $\mathbb{X}$. We set

$$\mathcal{U} = \left\{(\alpha_j) \in \mathbb{K}^{\mathbb{N}} \middle| \sum_{j=0}^{\infty} \alpha_j x_j \text{ is universal}\right\}.$$

(In other words, $\mathcal{U}$ is the set of unrestricted universal sequences with respect to the fixed sequence $x_0, x_1, \ldots$ in $\mathbb{X}$).

(b) A series $\sum_{j=0}^{\infty} \alpha_j x_j$, where $a = (\alpha_j) \in \mathbb{A}$, is called restricted universal iff $\forall\, x \in \mathbb{X}\ \exists\, \lambda = \{\lambda_1 < \lambda_2 < \ldots\} \subseteq \mathbb{N}$:

(i) $\sum_{j=0}^{\lambda_n} \alpha_j x_j \to x, \quad n \to \infty$,

(ii) $\sum_{j=0}^{\lambda_n} \alpha_j e_j \to \alpha, \quad n \to \infty$,

where as usual $e_n, n = 0, 1, 2, \ldots$, denotes the canonical basis of $\mathbb{K}^{\mathbb{N}}$. We set

$$\mathcal{U}_{\mathbb{A}} = \left\{a = (\alpha_j) \in \mathbb{A} \middle| \sum_{j=0}^{\infty} \alpha_j x_j \text{ is restricted universal in } \mathbb{A}\right\}.$$

(c) Let $\mu = \{\mu_1 < \mu_2 < \cdots\}$ and $a = (\alpha_j) \in \mathbb{A}$. A series $\sum_{j=0}^{\infty} \alpha_j x_j$ is restricted universal in $\mathbb{A}$ with respect to μ iff

$$\forall\, x \in \mathbb{X}\ \exists\, \lambda = \{\lambda_1 < \lambda_2 < \cdots\} \subseteq \mu :$$

(i) and (ii) are satisfied.

We set

$$\mathcal{U}_{\mathbb{A}}^{\mu} = \left\{a = (\alpha_j) \in \mathbb{A} \middle| \sum_{j=0}^{\infty} \alpha_j x_j \text{ is restricted universal in } \mathbb{A} \text{ with respect to} \mu\right\}.$$

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Obviously,

$$\mathcal{U}_{\mathbb{A}}^{\mu} \subseteq \mathcal{U}_{\mathbb{A}} \subseteq \mathcal{U} \cap \mathbb{A}.$$

If for every $a \in \mathbb{A}$ we have $\sum_{j=0}^{n} \alpha_j e_j \to a$, $n \to \infty$. (This happens in Examples 1.3 and 1.4.) Then $\mathcal{U}_{\mathbb{A}} = \mathcal{U} \cap \mathbb{A}$, but in general $\mathcal{U}_{\mathbb{A}} \subsetneqq \mathcal{U} \cap \mathbb{A}$ (see [**2**]). The main theorem regarding these classes is as follows.

THEOREM 1.2 ([**2**]). *The following are equivalent.*

(1) $\mathcal{U}_{\mathbb{A}} \neq \emptyset$.

(2) $\forall\, p \in \mathbb{N}\ \forall\, x \in \mathbb{X}\ \forall\, \varepsilon > 0\ \exists\, n \geq p\ \exists\, \alpha_p, \alpha_{p+1}, \ldots, \alpha_n \in \mathbb{K}$:

$$\rho\left(\sum_{j=p}^{n} \alpha_j x_j, x\right) < \varepsilon \ \ \textit{and}\ \ d\left(\sum_{j=p}^{n} \alpha_j e_j, 0\right) < \varepsilon.$$

(3) $\forall\, x \in \mathbb{X}\ \forall\, \varepsilon > 0\ \ \exists\, n \geq 0\ \exists\, \alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{K}$:

$$\rho\left(\sum_{j=0}^{n} \alpha_j x_j, x\right) < \varepsilon \ \ \textit{and}\ \ d\left(\sum_{j=0}^{n} \alpha_j e_j, 0\right) < \varepsilon.$$

(4) $\forall\, \mu \subseteq \mathbb{N}$, *$\mu$ infinite*:

$$\mathcal{U}_{\mathbb{A}}^{\mu} \textit{ is a dense } G_\delta \textit{ subset of } \mathbb{A}.$$

(5) $\forall\, \mu \subseteq \mathbb{N}$, *$\mu$ infinite*:

$$\mathcal{U}_{\mathbb{A}}^{\mu} \textit{ contains, except 0, a dense subspace of } \mathbb{A}.$$

(For the proof of this theorem see [**2**]).

To make the above notions more concrete, we briefly consider the next examples.

EXAMPLE 1.3 ([**7**]). Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and K be a compact subset of $\mathbb{C}$ such that $K \cap D^c = \emptyset$ and K^c is connected. We set,

- $\mathbb{X} = \{f : K \to \mathbb{C} \mid f \text{ is continuous on } K, \text{ and } f \text{ is holomorphic on } K^0\}$.
- $\rho(f, g) = \sup_{z \in K} |f(z) - g(z)|, \quad f, g \in \mathbb{X}$.
- $x_n(z) = z^n, \quad n \in \mathbb{N}$.
- $$\mathbb{A} = \left\{ a = (\alpha_j) \,\middle|\, f(z) = \sum_{j=0}^{\infty} \alpha_j z^j \in \mathcal{H}(D) \right\} = \left\{ \left(\frac{f^{(j)}(0)}{j\,!} \right) \,\middle|\, f \in \mathcal{H}(D) \right\}.$$
- $d(a, b) = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu} \frac{\|f-g\|_\nu}{1+\|f-g\|_\nu}$, where

$$a = (\alpha_j) = \left(\frac{f^{(j)}(0)}{j!}\right), \quad b = (\beta_j) = \left(\frac{g^{(j)}(0)}{j!}\right), \quad f, g \in \mathcal{H}(D)$$

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and

$$\|f-g\|_\nu = \sup_{|z|\le 1-\frac{1}{\nu+1}} |f(z)-g(z)|, \quad \nu\in\mathbb{N}.$$

It is easy to check that the above spaces $(\mathbb{X},\rho)$ and $(\mathbb{A},d)$ satisfy all requirements of our framework, and that $\sum_{j=0}^{n}\alpha_j e_j \to a,\ n\to\infty$ for all $a\in\mathbb{A}$. It turns out that (2) of Theorem 1.2 is satisfied and hence $\mathcal{U}_{\mathbb{A}}^{\mu}$ is a dense-G_δ subset of $\mathbb{A}$ and contains, except 0, a dense subspace of $\mathbb{A}$ (for details see [**2**, **7**]).

EXAMPLE 1.4 ([**2**, **6**]). Let

$$T=\{e^{it}: t\in[0,2\pi)\}=\mathbb{R}/2\pi\mathbb{Z}.$$

We set,

- $\mathbb{X}=L^0(T)=$ The space of equivalence classes of the 2π-periodic Lebesgue measurable functions ($f\sim g \Leftrightarrow f=g\ \lambda-a\cdot e$, where λ is the Lebesgue measure).
- $\rho(f,g)=\inf\{r+\lambda(\{|f-g|\ge r\}): r>0\}$, or $\rho(f,g)=\int_0^{2\pi}\frac{|f-g|}{1+|f-g|}d\lambda$. (As it is well known, ρ is the metric of convergence in measure).
- $x_n(t)=e^{\mathrm{int}}, \quad n\in\mathbb{N}$.
- $\mathbb{A}=\ell^p, p>2$.
- $d(a,b)=\|a-b\|_p=(\sum_{j=0}^{\infty}|\alpha_j-\beta_j|^p)^{1/p}$, where $a=(\alpha_j)$, $b=(\beta_j)\in\mathbb{A}$. As in the previous example, it turns out that $\mathcal{U}_{\mathbb{A}}$, $\mathcal{U}_{\mathbb{A}}^{\mu}$ are dense subsets of $\mathbb{A}$ and contain, except 0, a dense subspace of $\mathbb{A}$.

Now we recall the definitions of densities of subsets of $\mathbb{N}$ that we will need.

DEFINITION 1.5. Let $\lambda\subseteq\mathbb{N}$. The upper and lower density of λ is defined respectively as

$$\overline{D}(\lambda)=\limsup_N \frac{|\{n\in\lambda: n\le N\}|}{N},$$

$$\underline{D}(\lambda)=\liminf_N \frac{|\{n\in\lambda: n\le N\}|}{N}.$$

If $\overline{D}(\lambda)=\underline{D}(\lambda)$, we define the density of λ to be the limit,

$$D(\lambda)=\lim_{N\to\infty}\frac{|\{n\in\lambda: n\le N\}|}{N},$$

where as usual $|\cdot|$ denotes the cardinality of the corresponding set.

Let $a=(\alpha_j)\in\mathcal{U}_{\mathbb{A}}$ or $a\in\mathcal{U}_{\mathbb{A}}^{\mu}$ or $a\in\mathcal{U}$. To each $x\in\mathbb{X}$, it corresponds the family of those $\lambda\subseteq\mathbb{N}$ which realize the approximations of Definition 1.1. We denote by Λ_x this family. A question that arises naturally is the following: What can be said about the densities of the sets $\lambda\in\Lambda_x$?

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Firstly, in Section 2 we prove that if $\mathcal{U}_{\mathbb{A}} \neq \emptyset$, then the set of those $a \in \mathcal{U}_{\mathbb{A}}$, such that for each $x \in \mathbb{X}$ there exists $\lambda \in \Lambda_x$ with $\overline{D}(\lambda) = 1$, is a G_δ-dense subset of $\mathbb{A}$, which contains, except 0, a dense subspace of $\mathbb{A}$. (In fact, we prove the above result in a more general setting by considering the class $\mathcal{U}_{\mathbb{A}}^{\mu}$ and relative densities in Section 2.) Also, by Example 2.4, we see that it may happen that $\overline{D}(\lambda) < 1$ for all $\lambda \in \Lambda_x$, for all $x \in \mathbb{X}$ and for some $a \in \mathcal{U}_{\mathbb{A}}$.

In Section 3 we see that there is no $a \in \mathcal{U}$ (hence, $a \in \mathcal{U}_{\mathbb{A}}$) such that to each $x \in \mathbb{X}$ there exists $\lambda \in \Lambda_x$ with $\underline{D}(\lambda) > 0$, but it may exist $a = (\alpha_j) \in \mathcal{U}$ such that to each $x \in \mathbb{X}$ and for every neighbourhood V of x we have $\underline{D}(\{n \in \mathbb{N} : \sum_{j=0}^{n} \alpha_j x_j \in V\}) > 0$. The set of these a's is of first category in $(\mathbb{A}, d)$, provided $\mathcal{U}_{\mathbb{A}} \neq \emptyset$.

Summing up, for all $a \in \mathcal{U}_{\mathbb{A}}$ (or $\mathcal{U}_{\mathbb{A}}^{\mu}$), except a set of first category, the approximations (apart at most countable x's in $\mathbb{X}$) are realized with subsets λ of $\mathbb{N}$ with $\overline{D}(\lambda) = 1$ and $\underline{D}(\lambda) = D$.

2. Upper frequently universal series.

We remind that if $\lambda \subseteq \mu \subseteq \mathbb{N}$ are infinite sets, then the relative upper density of λ with respect to μ, $\overline{D}_\mu(\lambda)$, is the following by definition:

$$\overline{D}_\mu(\lambda) = \limsup_N \frac{|\{n \leq N | n \in \lambda\}|}{|\{n \leq N | n \in \mu\}|}.$$

Definition 2.1. Let $\mu \subseteq \mathbb{N}$ with $\overline{D}(\mu) = c \geq 0$. A sequence $a = (\alpha_j)$ is said to be upper frequently universal with respect to μ if and only if

$$\forall\, x \in \mathbb{X}\, \exists\, \lambda = \{\lambda_1 < \lambda_2 < \ldots\} \subseteq \mu \text{ with } \overline{D}(\lambda) = c,\ \ \overline{D}_\mu(\lambda) = 1 :$$
$$\sum_{j=0}^{\lambda_n} \alpha_j x_j \to x, \quad n \to \infty.$$

Moreover, if

$$\sum_{j=0}^{\lambda_n} \alpha_j e_j \to a, \quad n \to \infty,$$

then we say that $a = (\alpha_j)$ is upper frequently restricted universal in $\mathbb{A}$ with respect to μ. (Obviously $a \in \mathbb{A}$). We respectively denote by $\widetilde{\mathcal{U}}^{\mu}$ and $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu}$ these two classes of sequences. If $\mu = \mathbb{N}$, we write $\widetilde{\mathcal{U}}, \widetilde{\mathcal{U}}_{\mathbb{A}}$ for the classes. Of course, $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu} \subseteqq \widetilde{\mathcal{U}}^{\mu} \cap \mathbb{A} \subseteqq \widetilde{\mathcal{U}} \cap \mathbb{A}$.

Remark 2.2. If $0 < c \leq 1$, $\lambda \subset \mu$ and $\overline{D}(\lambda) = c$, then it automatically follows that $\overline{D}_\mu(\lambda) = 1$. Indeed, if $\overline{D}_\mu(\lambda) < 1$ and $\limsup_N \frac{|\{n \leq \mathbb{N} : n \in \lambda\}|}{|\{n \leq \mathbb{N} : n \in \mu\}|} = \xi < 1$, then there exists $N_0 \in \mathbb{N}$ such that $\frac{|\{n \leq N : n \in \lambda\}|}{|\{n \leq N : n \in \mu\}|} < \xi$ for $N \geq N_0$.

Hence,

$$\limsup_N \frac{|\{n \leq N : n \in \lambda\}|}{N} = c \leq \xi \cdot \limsup_N \frac{|\{n \leq N : n \in \mu\}|}{N} = \xi \cdot c,$$

which is a contradiction.

So the requirement $\overline{D}_\mu(\lambda) = 1$ in Definition 2.1 is needed only for the case $c = 0$.

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THEOREM 2.3. *Let μ be an infinite subset of $\mathbb{N}$ with $\overline{D}(\mu) = c \geq 0$. If $\mathcal{U}_{\mathbb{A}} \neq \emptyset$, then $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu}$ is a G_δ-dense subset of $\mathbb{A}$ and contains, except 0, a dense subspace of $\mathbb{A}$.*

Proof. Assume first that $0 < c \leq 1$.

It follows from Theorem 1.2 that $\mathbb{X}$ is separable. Let $\{y_\ell : \ell = 1, 2, \ldots\}$ be a dense subset of $\mathbb{X}$. For each $n \in \mathbb{N}$ and $p, \ell, s \in \mathbb{N} - \{0\}$ we set

$$E_{n,\ell,s,p} = \{a = (\alpha_j) \in \mathbb{A} | \exists \, \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \subseteq \mu \text{ with } \lambda_1 > p$$

such that $\rho(\sum_{j=0}^{\lambda_k} \alpha_j x_j, y_\ell) < \frac{1}{s}$, $d(\sum_{j=0}^{\lambda_k} \alpha_j e_j, \alpha) < \frac{1}{s}$ for $k = 1, 2, \ldots, n$ and $\frac{n}{\lambda_n} > c - \frac{1}{s}\}$.

Firstly, the following holds

$$\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu} = \bigcap_{p,\ell,s=1}^{\infty} \bigcup_{n=0}^{\infty} E_{n,\ell,s,p} \qquad \text{(I)}.$$

[If $a \in \widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu}$ and $p, s, \ell \in \mathbb{N} - \{0\}$, then by Definition 2.1 for $x = y_\ell$ we take that

$$\exists \, \lambda' = \{\lambda'_1 < \lambda'_2 < \cdots < \lambda'_n < \cdots\} \subseteq \mu \text{ with } \overline{D}(\lambda') = c \;\; \exists \, n'_0 \in \mathbb{N} :$$

$$\rho\left(\sum_{j=0}^{\lambda'_k} \alpha_j x_j, y_\ell\right) < \frac{1}{s} \;\text{ for }\; k \geq n'_0$$

and

$$d\left(\left\{\sum_{j=0}^{\lambda'_k} \alpha_j e_j, a\right) < \frac{1}{s}\right\} \;\text{ for }\; k \geq n'_0.$$

Since the upper density of a set does not change if we omit a finite part of it, we have:

$$\overline{D}(\{\lambda'_k \in \lambda' : \lambda'_k \geq \max(n'_0, p)\}) = c.$$

Hence, there exists $\{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \subseteq \lambda'$ with $\lambda_1 > \max(n'_0, p)$ such that the defining conditions of $E_{n,\ell,s,p}$ are all satisfied.

Conversely, let $a \in \bigcap_{\ell,s,p=1}^{\infty} \bigcup_{n=0}^{\infty} E_{n,\ell,s,p}$ and $x \in \mathbb{X}$. We have to construct a subset λ of μ with $\overline{D}(\lambda) = c$ (see Remark 2.2) such that the conditions of Definition 2.1 are satisfied. First, let $\{k_1 < k_2 < \cdots\} \subseteq \mathbb{N} - \{0\}$ with $y_{k_n} \to x$, as $n \to \infty$. By hypothesis for $a = (\alpha_j)$, setting $\ell = s = k_n$, it follows that there exist consecutive blocks $\mathcal{B}_n = \{\lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_{N_{k_n}}^{(n)}\}$ for $n = 1, 2, \ldots$ such that

$$\max \mathcal{B}_n < \min \mathcal{B}_{n+1}, \quad n = 1, 2, \ldots,$$

$$\left.\begin{aligned} &\rho\left(\sum_{j=0}^{\lambda_k^{(n)}} \alpha_j x_j, y_{k_n}\right) < \tfrac{1}{k_n} \\ &d\left(\sum_{j=0}^{\lambda_k^{(n)}} \alpha_j \varepsilon_j, a\right) < \tfrac{1}{k_n} \end{aligned}\right\} \text{ for } k = 1, 2, \ldots, N_{k_n} \tag{2.1}$$

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and

$$\frac{N_{k_n}}{\lambda^{(n)}_{n_{k_n}}} > c - \frac{1}{k_n}, \quad n = 1, 2, \dots . \tag{2.2}$$

We set

$$\lambda = \bigcup_{n=1}^{\infty} \mathcal{B}_n := \{\lambda_1 < \lambda_2 < \dots\}.$$

Then by (2.2) we get that

$$\overline{D}(\lambda) = c,$$

and by (2.1) that,

$$\sum_{j=0}^{\lambda_n} \alpha_j x_j \to x, \ n \to \infty \ \text{and} \ \sum_{j=0}^{\lambda_n} \alpha_j \varepsilon_j \to a, \ n \to \infty.$$

Hence, $a \in \widetilde{\mathcal{U}}^{\mu}_{\mathbb{A}}$].

Secondly, we have that, for each $n \in \mathbb{N}$, $\ell, s, p \in \mathbb{N} - \{0\}$:

$$\text{the set } E_{n,\ell,s,p} \text{ is open} \qquad \text{(II)}.$$

[Indeed, it is easy to check that the sets

$$U_k = \left\{ a \in \mathbb{A} \middle| \rho\left(\sum_{j=0}^{\lambda_k} \alpha_j x_j, y_\ell\right) < \frac{1}{s} \ \text{and} \ d\left(\left\{\sum_{j=0}^{\lambda_k} \alpha_j e_j, a\right) < \frac{1}{s}\right\}$$

are open for $k = 1, 2, \dots, n$. Also,

$$E_{n,\ell,s,p} = \bigcup \left(\bigcap_{k=1}^{n} U_k \right),$$

where the union is taken over all $\{\lambda_1 < \dots < \lambda_n\} \subseteq \mu$ with $\lambda_1 > p$ and $\frac{n}{\lambda_n} > c - \frac{1}{s}$].
Note: If for one particular $n \in \mathbb{N}$ there is no $\{\lambda_1 < \dots < \lambda_n\} \subseteq \mu$ satisfying $\frac{n}{\lambda_n} > c - \frac{1}{s}$, then $E_{n,\ell,s,p}$ is empty. But since $\overline{D}(\mu) = c > 0$, for each $p \in \mathbb{N} - \{0\}$ there is $n \in \mathbb{N}$ and $\{\lambda_1 < \dots < \lambda_n\} \subseteq \mu$ satisfying the above inequality.

Finally, we show that

$$\text{the set } \bigcup_{n=0}^{\infty} E_{n,\ell,s,p} \text{ is dense in } \mathbb{A} \text{ for each } \ell, s, p \in \mathbb{N} - \{0\} \qquad \text{(III)}.$$

Let ℓ, s, p be fixed. Since G is a dense subset of $\mathbb{A}$ (Section 1, A2), it is enough for $b = (\beta_0, \beta_1, \dots, \beta_m, 0, 0, \dots) \in G$ and $\varepsilon > 0$ with $\varepsilon < \frac{1}{s}$ to find $a \in \bigcup_{n=0}^{\infty} \mathbb{E}_{n,\ell,s,p}$ such that $d(a, b) < \varepsilon$.

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By Theorem 1.2 we have that

$$\exists n_0 \in \mathbb{N}\ \exists \alpha'_0, \alpha'_1, \ldots, \alpha'_{n_0} \in \mathbb{K}:$$
$$\rho\Big(\sum_{j=0}^{n_0} \alpha'_j x_j, y_\ell - \sum_{j=0}^{m} \beta_j x_j\Big) < \varepsilon \text{ and } d\Big(\sum_{j=0}^{n_0} \alpha'_j e_j, 0\Big) < \varepsilon.$$

If we set $a = (\alpha'_0, \ldots \alpha'_{n_0}, 0, 0, \ldots) + b := (\alpha_j)$ and since $\overline{D}(\mu) = c > 0$, we can find $\{\lambda_1 < \cdots < \lambda_n\} \subseteq \mu$ with $\lambda_1 > \max(n_0, p, m)$ and $\frac{n}{\lambda_n} > c - \frac{1}{s}$. Then we have

$$\rho\Big(\sum_{j=0}^{\lambda_k} \alpha_j x_j, y_\ell\Big) = \rho\Big(\sum_{j=0}^{n_0} \alpha'_j x_j, y_\ell - \sum_{j=0}^{m} \beta_j x_j\Big) < \varepsilon < \frac{1}{s}$$

and

$$d\Big(\sum_{j=0}^{\lambda_k} \alpha_j e_j, a\Big) = 0$$

for $k = 1, 2, \ldots, n$. Hence, $a \in \bigcup_{n=0}^{\infty} E_{n,\ell,s,p}$ and $d(a, b) = d(\sum_{j=0}^{n_0} \alpha'_j e_j, 0) < \varepsilon$].

Now by (II) and (III) it follows that $\bigcup_{n=0}^{\infty} E_{n,\ell,s,p}$ is a dense open subset of $\mathbb{A}$. Hence, by (I) and the Baire category theorem we take that $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu}$ is a dense G_δ subset in the complete metric space $\mathbb{A}$.

The case where $c = 0$ is treated similarly. We need only to change the inequalities of the form $\frac{n}{\lambda_n} > c - \frac{1}{s}$ by $\frac{n}{|\{m \leq \lambda_n : m \in \mu\}|} > 1 - \frac{1}{s}$ and we replace the upper density $\overline{D}$ by the relative upper density $\overline{D}_\mu$ and c by 1 throughout the proof.

To complete the proof, it remains to show that $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu} \cup \{0\}$ contains a dense subspace of $\mathbb{A}$. Let $\{c^{(\ell)} : \ell = 1, 2, \ldots\}$ be a dense subset of $\mathbb{A}$. We have already proved that $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\nu}$ is dense in $\mathbb{A}$ for each infinite subset ν of $\mathbb{N}$. By induction we get that

$$\exists (\mu^{(\ell)})_{\ell=1,2,\ldots},\ \mu^{(\ell)} = \{\mu_1^{(\ell)} < \mu_2^{(\ell)} < \cdots\} \subseteq \mu\ \exists\, (a^{(\ell)})_{\ell=1,2,\ldots} \text{ in } \mathbb{A}:$$

$$\text{(I)}'\ \ \overline{D}(\mu^{(\ell)}) = c \text{ and } \mu^{(\ell)} \subseteq \mu^{(\ell-1)},\ \ \ell = 1, 2, \ldots$$

and

$$\text{(II)}'\ \ a^{(\ell)} \in \widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu^{(\ell-1)}},\ \ \ell = 1, 2, \ldots$$

$$\text{(III)}' d(a^{(\ell)}, c^{(\ell)}) < \frac{1}{\ell},\ \ \ell = 1, 2, \ldots$$

$$\text{(IV)}'\ \ \sum_{j=0}^{\mu_n^{(\ell)}} \alpha_j^{(\ell)} x_j \to 0,\ \ n \to \infty \text{ and } \sum_{j=0}^{\mu_n^{(\ell)}} \alpha_j^{(\ell)} e_j \to a^{(\ell)},\ \ n \to \infty.$$

(In case that $c = 0$, we set $\overline{D}_\mu(\mu^{(\ell)}) = 1$ in (I)$'$ above. Also $\mu^{(0)} = \mu$).

We set $\mathcal{B}$ to be the linear span generated by $\{a^{(\ell)} : \ell = 1, 2, \ldots\}$, which is dense according to (III)$'$. All we have to prove is that $\mathcal{B} \subseteq \widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu} \cup \{0\}$. Indeed, let $a' = \sum_{k=1}^{m} r_k a^{(k)}$, where $r_m \neq 0$. By (II)$'$, $a^{(m)} \in \widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu^{(m-1)}}$, so for each $x \in \mathbb{X}$,

$\exists\ \lambda = \{\lambda_1 < \lambda_2 < \cdots\} \subseteq \mu^{(m-1)} \subseteq \mu$ with $\overline{D}(\lambda) = c$ (or $\overline{D}_\mu(\lambda) = 1$, in case $c = 0$):

$$r_m \sum_{j=0}^{\lambda_n} \alpha_j^{(m)} x_j \to x, \quad n \to \infty$$

and

$$\sum_{j=0}^{\lambda_n} \alpha_j^{(m)} e_j \to a^{(m)}, \quad n \to \infty. \tag{2.3}$$

(We remind the notation $a^{(m)} = (\alpha_j^{(m)})$, $a' = (\alpha_j)$).

We observe that

$$\sum_{j=0}^{\lambda_n} \alpha'_j x_j = \sum_{j=0}^{\lambda_n} \left(\sum_{k=1}^{m} r_k \alpha_j^{(k)} \right) x_j = \sum_{k=1}^{m} r_k \sum_{j=0}^{\lambda_n} \alpha_j^{(k)} x_j.$$

Because of (IV)$'$ it holds that

$$\sum_{j=0}^{\lambda_n} \alpha_j^{(k)} x_j \to 0, \quad n \to \infty, \quad \text{for } k = 1, 2, \ldots, m-1.$$

Hence,

$$\lim_{n \to \infty} \sum_{j=0}^{\lambda_n} \alpha'_j x_j = \lim_{n \to \infty} r_m \sum_{j=0}^{\lambda_n} \alpha_j^{(m)} x_j = x.$$

Also,

$$\lim_{n \to \infty} \sum_{j=0}^{\lambda_n} \alpha'_j e_j = \lim_{n \to \infty} \sum_{k=1}^{m} r_k \sum_{j=0}^{\lambda_n} \alpha_j^{(k)} e_j = \sum_{k=1}^{m} r_k a^{(k)} = a'.$$

(The last equality follows from (IV)$'$ and (2.3) above). Hence, $a' \in \widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu}$ and the proof is completed. □

A profound question arises: Is $\widetilde{\mathcal{U}}_{\mathbb{A}}^{\mu} = \mathcal{U}_{\mathbb{A}}^{\mu}$? By the next example, we see that this is not the case in general.

EXAMPLE 2.4. Let $\mathbb{X} = \mathbb{R}$, ρ be the usual metric and $x_j = 1, j = 0, 1, 2, \ldots$. Also, let $\mathbb{A} = \mathbb{R}^{\mathbb{N}}$ and d be the product metric,

$$d(a, b) = \sum_{n=0}^{\infty} \frac{|\alpha_n - \beta_n|}{1 + |\alpha_n - \beta_n|}.$$

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We consider the following partition of $\mathbb{N} - \{0\}$. Let $\{A_k : k = 0, 1, 2, \dots\}$, where $A_k = \{2^k \cdot m : m \text{ is odd}\}$. Since the terms of each A_k form an arithmetic progression with difference 2^{k+1}, it holds that

$$D(A_k) = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \dots .$$

Let $\{r_k : k = 0, 1, 2, \dots\}$ be an enumeration of rationals. We set $t_n = r_k$ for $n \in A_k$, $\alpha_j = t_j - t_{j-1}$ for $j = 1, 2, \dots$ $(t_0 = 0)$ and $\alpha_0 = 0$. Then we can easily check that $a = (\alpha_j) \in \mathcal{U}_{\mathbb{A}}$. Let $x \in \mathbb{R}$ and let $\lambda = \{\lambda_1 < \lambda_2 < \dots\} \subseteq \mathbb{N}$ be such that

$$t_{\lambda_n} = \sum_{j=0}^{\lambda_n} \alpha_j x_j \to x, \quad n \to \infty.$$

But there exists $x_0 \in \mathbb{N}$ such that $r_{k_0} \neq x$. Therefore, $\lambda \subseteq \mathbb{N} \setminus A_{k_0}$ (except perhaps of a finite part of λ).

Hence,

$$\overline{D}(\lambda) \leq \overline{D}(\mathbb{N} \setminus A_{k_0}) = D(\mathbb{N} - A_{k_0}) = 1 - \frac{1}{2^{k_0}} < 1.$$

Then, $a \notin \widetilde{\mathcal{U}}_{\mathbb{A}}$ and so $\widetilde{\mathcal{U}}_{\mathbb{A}} \subsetneqq \widetilde{\mathcal{U}}_{\mathbb{A}}$.

3. Lower frequently universal sequences. In this section we study what happens with the lower densities of the sets of indices which realize universal approximations. First by the next proposition we see that we cannot have universal approximations with positive lower density even in the class $\widetilde{\mathcal{U}}$ (which is larger than $\mathcal{U}^{\mu}_{\mathbb{A}}$).

PROPOSITION 3.1. *It holds that*

$$\Big\{a = (\alpha_j) \in \mathcal{U} | \forall\, x \in \mathbb{X}\ \exists\, \lambda = \{\lambda_1 < \lambda_2 < \cdots\} \subseteq \mathbb{N} \text{ with } \underline{D}(\lambda) > 0 : \sum_{j=0}^{\lambda_n} \alpha_j x_j \to x, n \to \infty\Big\} = \emptyset.$$

Proof. Suppose the contrary, and let $a = (\alpha_j)$ be a sequence in this set. If for $n \in \mathbb{N} - \{0\}$,

$$X_n = \Big\{x \in \mathbb{X} | \exists\, \lambda_x = \{\lambda_1 < \lambda_2 < \cdots\} \subseteq \mathbb{N} : \underline{D}(\lambda_x) \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \text{ and } \lim_{n\to\infty} \sum_{j=0}^{\lambda_n} \alpha_j x_j = x\Big\},$$

then, since $\mathbb{X}$ is uncountable, there is $n_0 \in \mathbb{N}$ such that X_{n_0} is uncountable. Obviously, if $x_1, x_2 \in \mathbb{X}$ and $x_1 \neq x_2$ then $\lambda_{x_1} \cap \lambda_{x_2}$ is at most finite and we may assume that $\lambda_{x_1} \cap \lambda_{x_2} = \emptyset$. Then,

$$\frac{|\{n \in \lambda_{x_1} \cup \lambda_{x_2} : n \leq N\}|}{N} = \frac{|\{n \in \lambda_{x_1} : n \leq N\}|}{N} + \frac{|\{n \in \lambda_{x_2} : n \leq N\}|}{N},$$

hence

$$\underline{D}(\lambda_{x_1} \cup \lambda_{x_2}) \geq \underline{D}(\lambda_{x_1}) + \underline{D}(\lambda_{x_2}).$$

Apparently the same holds for $x_1, x_2, \ldots, x_{n_0+1}$ mutually disjoint elements in X_{n_0}, that is,

$$\underline{D}(\lambda_{x_1} \cup \cdots \cup \lambda_{x_{n_0+1}}) \geq \underline{D}(\lambda_{x_1}) + \cdots + \underline{D}(\lambda_{x_{n_0+1}}) > (n_0+1)\frac{1}{n_0+1} = 1,$$

which is a contradiction. $\square$

REMARK 3.2.
(a) It follows from the proof of Proposition 3.1 that for each $a = (\alpha_j) \in \mathcal{U}$ we can have $\underline{D}(\lambda_x) > 0$, for at most countably many $x \in \mathbb{X}$. This may happen, as we see in Example 3.3.

(b) If $\lambda \subseteq \mu \subseteq \mathbb{N}$ are infinite sets and we define the lower relative density of λ with respect to μ, $\underline{D}_\mu(\lambda)$, as follows

$$\underline{D}_\mu(\lambda) = \liminf_N \frac{|\{n \in \lambda : n \leq N\}|}{|\{n \in \mu : n \leq N\}|},$$

then with the same reasoning as in the proof of Proposition 3.1 we can see that

$$\Big\{a = (\alpha_j) \in \mathcal{U} | \forall\, x \in \mathbb{X}\ \exists\, \lambda_x = \{\lambda_1 < \lambda_2 < \cdots\} \subseteq \mu \text{ with } \underline{D}_\mu(\lambda_x) > 0 : \sum_{j=0}^{\lambda_n} \alpha_j x_j \to x,\ n \to \infty\Big\} = \emptyset.$$

(c) For every $\lambda \subseteq \mathbb{N}$ it is easy to see that

$$\underline{D}(\lambda) = 1 - \overline{D}(\mathbb{N} - \lambda).$$

EXAMPLE 3.3. Let $\mathbb{X}$, $\mathbb{A}$ and $a = (\alpha_j) \in \mathcal{U}_{\mathbb{A}}$ be as in Example 2.4. Then for $n \in A_k = \{2^k m : m \text{ is odd}\}$ it holds that

$$\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} \alpha_j x_j = t_n = r_k.$$

Since $D(A_k) = \frac{1}{2^{k+1}}$ $(k = 0, 1, 2, \ldots)$, it follows that for $x = r_k$ and $\lambda_x = \lambda_{r_k} = A_k$ we have $\underline{D}(\lambda_{r_k}) > 0$ for $k = 1, 2, \ldots$ and the corresponding partial sums are equal to r_k. Thus, we have $D(\lambda_k) > 0$ for denumerably many n's.

Although universality with positive lower density has no meaning as we showed in Proposition 3.1, universality with positive lower densities on every neighbourhood of a point $x \in \mathbb{X}$ has a meaning and this is what we are going to examine in the sequel.

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DEFINITION 3.4. A sequence $a = (\alpha_j) \in \mathcal{U}$ is said to be lower frequently universal if and only if $\forall\, x \in \mathbb{X}$, $\forall\, \varepsilon > 0$

$$\underline{D}\left(\left\{n \in \mathbb{N} : \rho\left(\sum_{j=0}^{n} \alpha_j x_j, x\right) < \varepsilon\right\}\right) > 0.$$

We denote by $\underset{\sim}{\mathcal{U}}$ the class of these sequences. Regarding the class $\underset{\sim}{\mathcal{U}}$ we have the following proposition.

PROPOSITION 3.5. *Suppose that* $\mathcal{U}_{\mathbb{A}} \neq \emptyset$. *Then* $\underset{\sim}{\mathcal{U}} \cap \mathbb{A}$ *is of first category in* $(\mathbb{A}, d)$.

Proof. By Theorem 2.3 it is enough to show that $\underset{\sim}{\mathcal{U}} \cap \widetilde{\mathcal{U}} \cap \mathbb{A} = \emptyset$ (as $\widetilde{\mathcal{U}}^{\mu}_{\mathbb{A}} \subseteq \widetilde{\mathcal{U}} \cap \mathbb{A}$). Assume the opposite and let $a = (\alpha_j) \in \underset{\sim}{\mathcal{U}} \cap \widetilde{\mathcal{U}} \cap \mathbb{A}$. If $x \neq 0$, $x \in \mathbb{X}$ and $\varepsilon = \frac{\rho(x,0)}{2}$, then

$$\exists\, \lambda = \{\lambda_1 < \lambda_2 < \cdots\} \subseteq \mathbb{N} \;\text{ with }\; \overline{D}(\lambda) = 1 :$$
$$\sum_{j=0}^{\lambda_n} \alpha_j x_j \;\to\; x,\; n \;\to\; \infty.$$

It follows that the set $\{n \in \mathbb{N} : \rho(\sum_{j=0}^{\lambda_n} \alpha_j x_j, 0) < \varepsilon\}$ is finite. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\lambda \cap [n_0, \infty) \subseteq \mathbb{N} - \left\{n \in \mathbb{N} : \rho\left(\sum_{j=0}^{n} \alpha_j x_j, 0\right) < \varepsilon\right\}. \tag{4}$$

By Remark 3.2(c), we take

$$\underline{D}\left(\left\{n \in \mathbb{N} : \rho\left(\sum_{j=0}^{n} \alpha_j x_j, 0\right) < \varepsilon\right\}\right) = 1 - \overline{D}\left(\mathbb{N} - \left\{n \in \mathbb{N} : \rho\left(\sum_{j=0}^{n} \alpha_j x_j, 0\right) < \varepsilon\right\}\right) = 0$$

(because of (4) and the fact that $\overline{D}(\lambda) = 1$), which is a contradiction since $a \in \underset{\sim}{\mathcal{U}}$. □

QUESTION 3.6. In Example 2.4 if $x \in \mathbb{X}$ and $\varepsilon > 0$, then there exists $k \in \mathbb{N}$ such that $\rho(x, r_k) = |x - r_k| < \varepsilon$. Since $\sum_{j=0}^{n} \alpha_j x_j = t_n$ and $t_n = r_k$ for $n \in A_k$, it follows that

$$A_k \subseteq \left\{n \in \mathbb{N} : \rho\left(\sum_{j=0}^{n} \alpha_i x_j, x\right) < \varepsilon\right\}.$$

Hence, $\underset{\sim}{\mathcal{U}} \cap \mathbb{A} = \underset{\sim}{\mathcal{U}} \neq \emptyset$ ($\underset{\sim}{\mathcal{U}} \cap \mathbb{A} = \underset{\sim}{\mathcal{U}}$ since $\mathbb{A} = \mathbb{R}^N$). But we do not know if there are universal Taylor series (Example 1.3) or universal trigonometric series (Example 1.4) in the class $\underset{\sim}{\mathcal{U}} \cap \mathbb{A}$. An interesting question is whether $\underset{\sim}{\mathcal{U}} \cap \mathbb{A}$ can be empty?

REMARK 3.6. An obvious question is, whether $\mathbb{A} \cap \underset{\sim}{\mathcal{U}}$ is of 1st category in $(\mathbb{A}, d)$ under the weaker assumption $\mathbb{A} \cap \mathcal{U} \neq \emptyset$ and not $\mathcal{U}_{\mathbb{A}} \neq \emptyset$? A combination of the methods of the present paper with the methods of **[4]** yields a positive answer to this question. This is the content of a future paper in preparation **[5]**.

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