LAGUERRE AND MEIXNER POLYNOMIALS IN DUALITY

FRANCISZEK HUGON SZAFRANIEC

Instytut Matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, PL-30059 Kraków
e-mail: fhszafra@im.uj.edu.pl

Abstract. A closer look at Laguerre and Meixner polynomials shows that they interact, with a more active role played by the latter. We intend here to expound on this development of the story up to some level of abstraction.

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Both Laguerre and Meixner polynomials belong to the classical part of the theory of orthogonal polynomials. While Laguerre polynomials have found a firm position in theoretical physics (*exemplum*: the hydrogen atom), the Meixner ones have rather been put aside. It turns out that if one passes from Laguerre polynomials to Laguerre functions (they are orthonormal and complete in $\mathcal{L}^2(\mathbb{R}_+)$) on the one hand and from Meixner polynomials to Meixner sequences (which are orthonormal and complete in ℓ^2) on the other, the two *meet* in perfect harmony (if not to say, coincide, at least from the point of view of the Hilbert space category); this theme is developed in [**9**]. Here we intend to show how they *interplay*. This kind of interplay was discovered in [**5**] as a property of Hilbert space operators and then settled in the proper environment of special functions in [**6**] and [**7**].

The differential operators. The Laguerre polynomials $L_n^{(\alpha)}$, $n = 0, 1, \ldots, \alpha > -1$, can be given[1] as

$$L_n^{(\alpha)}(x) \stackrel{\mathrm{df}}{=} \frac{(\alpha)_n}{n!} \, {}_1F_1 \left(\begin{matrix} -n \\ \alpha + 1 \end{matrix} \,\middle|\, x \right).$$

They satisfy[2] [**10**]

$$(n + 1)L_{n+1}^{(\alpha)}(x) = (-xD^2 - (\alpha + 1 - 2x)D + \alpha + 1 - x)L_n^{(\alpha)}(x), \ n = 0, 1, \ldots \quad (1)$$

which is the *raising formula* for them[3]. The other, the *lowering formula* [**8**] is[4] (with convention $L_{-1}^{(\alpha)} = 0$)

$$(n + \alpha)L_{n-1}^{(\alpha)}(x) = (-xD^2 - (\alpha + 1)D)L_n^{(\alpha)}(x), \quad n = 0, 1, \ldots$$

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[1] $(a)_n$ defined as $(a)_0 \stackrel{\mathrm{df}}{=} 1$ and $(a)_n \stackrel{\mathrm{df}}{=} (a)_{n-1}(a + n - 1)$ stands for the shifted factorial (the Pochhammer symbol).

[2] Occasionally, depending on the context, we use the shorthand notation $D \stackrel{\mathrm{df}}{=} \frac{\mathrm{d}}{\mathrm{d}x}$.

[3] We distinguish between *operational formulae* and Hilbert space *operators*; this is to avoid messy exposition.

[4] There is a misprint in formula (2) (as wel as in (4)) in [**8**]; +1 has to be dropped in both.

Recalling the orthogonality relation

$$\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x}\mathrm{d}x = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{m,n}, \quad m,n = 0,1,\dots$$

and defining the Laguerre functions

$$l_n^{(\alpha)}(x) \stackrel{\mathrm{df}}{=} \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}x^{\alpha/2}e^{-x/2}L_n^{(\alpha)}(x), \quad x \geq 0, \quad n = 0,1,\dots$$

we get from (1) for $n = 0, 1, \dots$ the following formula

$$\sqrt{(n+1)(n+\alpha+1)}l_{n+1}^{(\alpha)}(x) = \left(-xD^2 + (x-1)D - \frac{x^2 - 2x - \alpha^2}{4x}\right)l_n^{(\alpha)}(x). \quad (2)$$

Since the sequence $\{L_n^{(\alpha)}\}_{n=0}^\infty$ is complete in $\mathcal{L}^2(\mathbb{R}_+, x^\alpha e^{-x}\,\mathrm{d}x)$, the sequence $\{l_n^{(\alpha)}\}_{n=0}^\infty$ forms an orthonormal basis in $\mathcal{L}^2(\mathbb{R}_+)$ (with $\mathbb{R}_+$ for $[0, +\infty)$). Thus we come from (2) to a family S_α, $\alpha > -1$, of densely defined operators in $\mathcal{L}^2(\mathbb{R}_+)$ defined as[5]

$$(S_\alpha f)(x) \stackrel{\mathrm{df}}{=} \left(-x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x-1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{-x^2 + 2x + \alpha^2}{4x}\right)f(x),$$

$$f \in \mathcal{D}(S_\alpha) \stackrel{\mathrm{df}}{=} \mathrm{lin}\{l_n^{(\alpha)}; n = 0, 1, \dots\}, \quad (3)$$

which act as forward weighted shift operators (with respect to the basis $\{l_n^{(\alpha)}\}_{n=0}^\infty$), that is

$$S_\alpha l_n^{(\alpha)} = \sqrt{(n+1)(n+\alpha+1)}l_{n+1}^{(\alpha)}, n = 0, 1, \dots \quad (4)$$

The Hilbert space adjoint S_α^* of S_α, acts for $f \in \mathrm{lin}\{l_n^{(\alpha)}; n = 0, 1, \dots\}$ as

$$(S_\alpha^* f)(x) \stackrel{\mathrm{df}}{=} \left(-x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (x+1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{-x^2 - 2x + \alpha^2}{4x}\right)f(x).$$

It is a backward weighted shift with respect to the Laguerre functions

$$S_\alpha^* l_n^{(\alpha)} = \begin{cases} \sqrt{n(n+\alpha)}l_{n+1}^{(\alpha)} & \text{if } n = 1, 2, \dots \\ 0 & \text{if } n = 0 \end{cases}$$

Both S_α and S_α^* are the *ladder operators* for the Laguerre polynomials.

The finite difference operators. The Meixner polynomials $M_n^{(\beta,c)}$, $n = 0, 1, \dots,$ with parameters $\beta > 0$ and $0 < c < 1$, can be defined (cf. [3]) as

$$M_n^{(\beta,c)}(x) \stackrel{\mathrm{df}}{=} {}_2F_1\left(\begin{array}{c} -n, -x \\ \beta \end{array} \middle| 1 - \tfrac{1}{c}\right) \quad (5)$$

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[5] $\mathcal{D}(A)$ stands for the domain of an operator, lin denotes the linear span.

They are related to their generating function as follows

$$\left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta} = \sum_{n=0}^{\infty} (\beta)_n M_n^{(\beta,c)}(x) \frac{t^n}{n!}, \tag{6}$$

Write (6) for $x+1$ instead of x and put these two together so as to get

$$\left(1 - \frac{t}{c}\right) \sum_{n=0}^{\infty} (\beta)_n M_n^{(\beta,c)}(x) \frac{t^n}{n!} = (1-t) \sum_{n=0}^{\infty} (\beta)_n M_n^{(\beta,c)}(x+1) \frac{t^n}{n!}.$$

Comparing the coefficients at t^n we arrive at

$$(\beta+n-1)\left[M_{n+1}^{(\beta,c)}(x) - M_{n+1}^{(\beta,c)}(x+1)\right] = n\left(\frac{1}{c}M_n^{(\beta,c)}(x) - M_n^{(\beta,c)}(x+1)\right). \tag{7}$$

On the other hand, the difference relation for the Meixner polynomials is

$$-nM_n^{(\beta,c)}(x) = \frac{c}{1-c}(x+\beta)M_n^{(\beta,c)}(x+1)$$
$$- \frac{x+(x+\beta)c}{1-c}M_n^{(\beta,c)}(x) + \frac{x}{1-c}M_n^{(\beta,c)}(x-1).$$

Changing n for $n+1$ we can rewrite the above difference relation as

$$-(n+1)M_{n+1}^{(\beta,c)}(x) = \frac{c}{1-c}(x+\beta)\left(M_{n+1}^{(\beta,c)}(x+1) - M_{n+1}^{(\beta,c)}(x)\right)$$
$$- \frac{x}{1-c}\left(M_{n+1}^{(\beta,c)}(x) - M_{n+1}^{(\beta,c)}(x-1)\right) \tag{8}$$

and inserting (7) into the brackets of (8) we get

$$-(n+1)M_{n+1}^{(\beta,c)}(x) = \frac{c}{1-c}(x+\beta)\frac{n+1}{\beta+n}\left(M_n^{(\beta,c)}(x+1) - c^{-1}M_n^{(\beta,c)}(x)\right)$$
$$- \frac{x}{1-c}\frac{n+1}{\beta+n}\left(M_n^{(\beta,c)}(x) - c^{-1}M_n^{(\beta,c)}(x-1)\right).$$

Finally, the *raising formula* for Meixner polynomials is

$$(\beta+n)M_{n+1}^{(\beta,c)}(x) = \frac{1}{1-c}((2x+\beta)M_n^{(\beta,c)}(x) - c(x+\beta)M_n^{(\beta,c)}(x+1)$$
$$+ -c^{-1}xM_n^{(\beta,c)}(x-1)), \quad n = 0, 1, \dots \tag{9}$$

Recall that the orthogonality relation for Meixner polynomials is of the form

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_k^{(\beta,c)}(x) M_l^{(\beta,c)}(x) = \frac{c^{-k}k!}{(\beta)_k(1-c)^\beta}\delta_{kl}. \tag{10}$$

Define the *n-th Meixner sequence* as

$$m_n^{(\beta,c)}(x) \stackrel{\mathrm{df}}{=} (1-c)^{\beta/2}\sqrt{\frac{(\beta)_n c^n}{n!}}\sqrt{\frac{(\beta)_x c^x}{x!}} M_n^{(\beta,c)}(x), \quad x = 0, 1, \dots \tag{11}$$

Then (9) takes the form (notice here $m_n^{(\beta,c)}(-1)$ or, rather, the whole expression in the second bracket is irrelevant)

$$\sqrt{(n+1)(\beta+n)}m_{n+1}^{(\beta,c)}(x) = \frac{1}{1-c}\left((2x+\beta)\sqrt{c}m_n^{(\beta,c)}(x) - c\sqrt{(x+1)(x+\beta)}m_n^{(\beta,c)}(x+1)\right.$$
$$\left. - \sqrt{x(x+\beta-1)}m_n^{(\beta,c)}(x-1)\right). \tag{12}$$

Because, due to (10) each $m_n^{(\beta,c)}$ is in ℓ^2, we can define an operator $S_{\beta,c}$ in ℓ^2 as follows: $\mathcal{D}(S_{\beta,c}) \overset{\text{df}}{=} \text{lin}\{m_n^{(\beta,c)}; \ n = 0, 1, \ldots\}$ and for $f \in \mathcal{D}(S_{\beta,c})$,

$$(S_{\beta,c}f)(x) \overset{\text{df}}{=} \frac{1}{1-c}\left(\sqrt{c}(2x+\beta)f(x) - c\sqrt{(x+1)(x+\beta)}f(x+1)\right.$$
$$\left. - \sqrt{x(x+\beta-1)}f(x-1)\right), \quad x = 1, 2, \ldots,$$
$$(S_{\beta,c}f)(0) \overset{\text{df}}{=} \frac{1}{1-c}\left(c\beta f(0) - c\sqrt{\beta}f(1)\right).$$

A look at (11) and (5) convinces us that

$$m_n^{(\beta,c)}(x) = m_x^{(\beta,c)}(n), \quad n, x = 0, 1, \ldots \tag{13}$$

Because of this selfduality, by [2], the sequence $\{m_n^{(\beta,c)}\}_{n=0}^{\infty}$ forms a basis, that is an orthonormal and complete set, in ℓ^2. Thus $S_{\beta,c}$ is a densely defined operator in ℓ^2 and

$$S_{\beta,c}m_n^{(\beta,c)} = \sqrt{(n+1)(n+\beta)}m_{n+1}^{(\beta,c)}, \quad n = 0, 1, \ldots$$

Consequently, its adjoint $S_{\beta,c}^*$ acts on $\text{lin}\{m_n^{(\beta,c)}; \ n = 0, 1, \ldots\}$ as (cf. [4])

$$S_{\beta,c}^*m_n^{(\beta,c)} = \begin{cases} \sqrt{n(n+\beta-1)}m_{n-1}^{(\beta,c)} & \text{if } n = 1, 2, \ldots \\ 0 & \text{if } n = 0 \end{cases} \tag{14}$$

and its finite difference form is

$$(S_{\beta,c}^*f)(x) = \frac{1}{1-c}\left(\sqrt{c}(2x+\beta)f(x) - c\sqrt{x(x+\beta-1)}f(x-1)\right.$$
$$\left. - \sqrt{(x+1)(x+\beta)}f(x+1)\right), \quad x = 1, 2, \ldots,$$
$$(S_{\beta,c}^*f)(0) = \frac{1}{1-c}\left(\sqrt{c}\beta f(0) - \sqrt{\beta}f(1)\right).$$

Notice that this formula extends to the whole of $\mathcal{D}(S_{\beta,c}^*)$ and that $\mathcal{D}(\bar{S}_{\beta,c}) = \mathcal{D}(S_{\beta,c}^*)$, where $\bar{S}_{\beta,c}$ is the closure of $S_{\beta,c}$.

Invoking the defining formula (11) we get from the above and (14)

$$nM_{n-1}^{(\beta,c)}(x) = \frac{c}{1-c}\left((2x+\beta)M_n^{(\beta,c)}(x) - (x+\beta)M_n^{(\beta,c)}(x+1)\right.$$
$$\left. - xM_n^{(\beta,c)}(x-1)\right), \quad n = 1, 2, \ldots$$

and this is the *lowering formula*[6] for Meixner polynomials.

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[6] Notice that we have used here a, rather simple, argument concerning Hilbert space operators.

Let $\{\epsilon_n\}_{n=0}^\infty$ be the canonical zero-one basis in ℓ^2, that is $\epsilon_n = \{\delta_{n,i}\}_{i=0}^\infty$. Because $\{\epsilon_n\}_{n=0}^\infty \subset \mathcal{D}(\bar{S}_{\beta,c})$ we have

$$\bar{S}_{\beta,c}\epsilon_n = \frac{1}{1-c}(\sqrt{c}(2n+\beta)\epsilon_n - c\sqrt{n(n+\beta-1)}\epsilon_{n-1}$$
$$- \sqrt{(n+1)(n+\beta)}\epsilon_{n+1}), \quad n = 1, 2, \ldots,$$
$$\bar{S}_{\beta,c}\epsilon_0 = \frac{1}{1-c}(c\beta\epsilon_0 - \sqrt{\beta}\epsilon_1).$$

and

$$S_{\beta,c}^*\epsilon_n = \frac{1}{1-c}(\sqrt{c}(2n+\beta)\epsilon_n - c\sqrt{(n+1)(n+\beta)}\epsilon_{n+1}$$
$$- \sqrt{n(n+\beta-1)}\epsilon_{n-1}), \quad x = 1, 2, \ldots,$$
$$S_{\beta,c}^*\epsilon_0 = \frac{1}{1-c}(\sqrt{c}\beta\epsilon_0 - \sqrt{\beta}\epsilon_1).$$

The abstract setup. Let $\mathcal{H}$ be a separable Hilbert space with the inner product $\langle\,\cdot\,, -\rangle$ and let $\{e_n\}_{n=0}^\infty$ be a basis in it. Set

$$e_n^{(\beta,c)} \overset{\text{df}}{=} \sum_{k=0}^\infty m_k^{(\beta,c)}(n)e_k; \; n = 0, 1, \ldots \tag{15}$$

Then, by (13) and according to Proposition 2 of [6], $e^{(\beta,c)} \overset{\text{df}}{=} \{e_n^{(\beta,c)}\}_{n=0}^\infty$ is another basis in $\mathcal{H}$ and the reciprocity

$$e_n = \sum_{k=0}^\infty m_n^{(\beta,c)}(k)e_k^{(\beta,c)}; \; n = 0, 1, \ldots$$

holds.

A densely defined closable operator S in $\mathcal{H}$ is said to be a (*forward*) *weighted shift* with respect to e and with the weights $\{\sigma_n\}_{n=0}^\infty$ if $\text{lin}\{e_n; \, n = 0, 1, \ldots\}$ is a core[7] of S, $Se_n = \sigma_n e_{n+1}$ for $n = 0, 1, \ldots$ and all σ_n's are positive numbers. In like manner, an operator T is called a *backward weighted shift* with respect to e and with the weights $\{\tau_n\}_{n=0}^\infty$ if $\text{lin}\{e_n; \, n = 0, 1, \ldots\}$ is a core of T, $Te_n = \tau_{n-1}e_{n-1}$ for $n = 0, 1, \ldots$ and all τ_n's are positive numbers. If all $\sigma_n = \tau_n$ the operators S and T are formally adjoint each to the other, that is $\langle Sf, g \rangle = \langle f, Tg \rangle$, for $f, g \in \text{lin}\{e_n; \, n = 0, 1, \ldots\}$; in fact we always have $\mathcal{D}(S) \subset \mathcal{D}(S^*)$ and the inclusion may be strict in general, cf. [4].

Prompted by the presentation so far we are going to distinguish two weighted shifts, both with the same weights $\sigma_n = \sqrt{(n+1)(n+\beta)}$ with $\beta > 0$: the forward weighted shift with these weights $S_{\beta,c}$ and the related backward shift which is nothing, as we already know, but the adjoint $S_{\beta,c}^*$ restricted to $\text{lin}\{e_n; \, n = 0, 1, \ldots\}$. They are a kind of *creation* and *annihilation* operators in our circumstances and we would like to hold on these names here. According to what we have said it is enough to consider one of them, the creation operator say. Fortunately in this case $\mathcal{D}(\bar{S}_{\beta,c}) = \mathcal{D}(S_{\beta,c}^*)$.

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[7] $\mathcal{D} \subset \mathcal{D}(A)$ is a *core* of a closable operator A if $\overline{A|_{\mathcal{D}}} = \bar{A}$ with ‾ standing for the closure.

Set

$$M \overset{\mathrm{df}}{=} \operatorname{diag}(n)_{n=0}^{\infty} \text{ with respect to the basis } \boldsymbol{e}. \tag{16}$$

Using closability of M, by (15), we infer that $\boldsymbol{e}^{(\beta,c)} \subset \mathcal{D}(\bar{M})$. Also $f_n \overset{\mathrm{df}}{=} \sqrt{(N+1)(N+\beta)}$
$\times [m_N^{(\beta,c)}(n)e_{N+1} - cm_{N+1}^{(\beta,c)}(n)e_N]$. By (12) we have[8] (with $e_{-1} = 0$ and $m_k^{(\beta,c)}(-1) \overset{\mathrm{df}}{=} 0$)

$$[\boldsymbol{S}_{\beta,c} + c\boldsymbol{S}_{\beta,c}^* - \beta\sqrt{c}I - 2\sqrt{c}M] \sum_{k=0}^{N} m_k^{(\beta,c)}(n)e_k$$

$$\overset{(12)}{=} \sum_{k=0}^{N} m_k^{(\beta,c)}(n)[\sqrt{(k+1)(k+\beta)}e_{k+1} + c\sqrt{k(k+\beta-1)}e_{k-1} - \sqrt{c}(2k+\beta)e_k]$$

$$= \sum_{i=0}^{N} \big[\sqrt{i(i+\beta-1)}m_n^{(\beta,c)}(i-1) + c\sqrt{(i+1)(i+\beta)}m_n^{(\beta,c)}(i+1)$$

$$-\sqrt{c}(2i+\beta)m_n^{(\beta,c)}(i)\big]e_i + f_N$$

$$\overset{(12)}{=} -(1-c)\sqrt{(n+1)(n+\beta)} \sum_{i=0}^{N} m_{n+1}^{(\beta,c)}(i)e_i + f_N. \tag{17}$$

Because $f_N \to 0$, $\sum_{k=0}^{N} m_k^{(\beta,c)}(n)e_k \to e_n^{(\beta,c)}$ and $\boldsymbol{S}_{\beta,c} \sum_{k=0}^{N} m_k^{(\beta,c)}(n)e_k \to \sqrt{c}(2n+\beta)e_n^{(\beta,c)} - c\sqrt{n(n+\beta-1)}e_{n-1}^{(\beta,c)} - \sqrt{(n+1)(n+\beta)}e_{n+1}^{(\beta,c)}$ as $N \to \infty$ we get therefrom $e_n^{(\beta,c)} \in \mathcal{D}(\bar{\boldsymbol{S}}_{\beta,c})$ and

$$(\bar{\boldsymbol{S}}_{\beta,c} + c\boldsymbol{S}_{\beta,c}^* - \beta\sqrt{c}I - 2\sqrt{c}\bar{M})e_n^{(\beta,c)} = -(1-c)\sqrt{(n+1)(n+\beta)}e_{n+1}^{(\beta,c)}. \tag{18}$$

Let $U_{\beta,c}$ be a unitary operator sending e_n's to $e_n^{(\beta,c)}$'s. Then (1) can be read as

$$\boldsymbol{S}_{\beta,c} = -\frac{1}{1-c}U_{\beta,c}^*(\bar{\boldsymbol{S}}_{\beta,c} + c\boldsymbol{S}_{\beta,c}^* - \beta\sqrt{c}I - 2\sqrt{c}\bar{M})U_{\beta,c} \tag{19}$$

Let us record this as

THEOREM 1. *The creation operator $\boldsymbol{S}_{\beta,c}$ satisfies* (1) *or, equivalently,* (19).

The one before last and the last component of (17) can be now stated as

THEOREM 2. *The creation operators $\boldsymbol{S}_{\beta,c}$ in $\mathcal{H}$ and $S_{\beta,c}$ in ℓ^2 satisfy*

$$\boldsymbol{S}_{\beta,c}\, e_n^{(\beta,c)} = -\frac{1}{1-c} \sum_{k=0}^{\infty} \big(S_{\beta,c}m_n^{(\beta,c)}\big)(k)\, e_k, \quad n = 0, 1, \dots \tag{20}$$

As a next step we work out a kind of converse to these Theorems.

The punchline. Now we take delight in convincing ourselves that the operators $\boldsymbol{S}_{\beta,c}$ and $S_{\beta,c}$, appearing in Theorems 1 and 2, are the only possible solutions of (19) and (20) within the class of weighted shifts.

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[8] Here and in the sequel we make constant use of (13) without mentioning it.

Fix c and β and suppose we are given

  1° two bases $e = \{e_n\}_{n=0}^{\infty}$ and $f = \{f_n\}_{n=0}^{\infty}$
  2° two closable operators S and T such that $\lim e$ is a core for S and so is $\lim e^{(\beta,c)}$ for T, and such that $e^{(\beta,c)} \subset \mathcal{D}(S) \cap \mathcal{D}(S^*)$ and $e, f \subset \mathcal{D}(T)$.

By analogy with (19) suppose (with M defined by (3) and the remark afterwards)

$$Te_n = -\frac{1}{1-c}(S + cS^* - \beta\sqrt{c}I - 2\sqrt{c}\bar{M})e_n, \quad n = 0, 1, \ldots \tag{21}$$

THEOREM 3. *Suppose for some $\beta > 0$ and $0 < c < 1$ S is a weighted shift with respect to e with some weights $\{\sigma_n\}_{n=0}^{\infty}$ and so is T with respect to f with some other weights $\{\tau_n\}_{n=0}^{\infty}$. Then $f = e^{(\beta,c)}$ and S is a creation operator with respect to e and at the same time so is T with respect to $e^{(\beta,c)}$, all this provided $\sigma_0 = \tau_0 = 1$ and $f_0 = e_0^{(\beta,c)}$.*

Proof. Making use of closability of the operators involved and the very specific way they act, as we have done it so far, we get (suppose the weights for S and T are σ_n's and τ_n's resp.)

$$\text{LHS of (4)} = \sum_{i=0}^{\infty}\langle e_n, f_i\rangle\tau_i f_i = \sum_{i,j=0}^{\infty}\langle e_n, f_i\rangle\tau_i\langle f_i, e_j\rangle e_j,$$

$$\text{RHS of (3)} = -(1-c)^{-1}(\sigma_n e_{n+1} + \sigma_{n-1}e_{n-1} - \sqrt{c}(2n+\beta)e_n).$$

Taking into account the fact that, due to (13), $\langle e_k^{(\beta,c)}, e_l\rangle = m_k^{(\beta,c)}(l) = m_l^{(\beta,c)}(k)$ we get from these two for the inner product with $e_k^{(\beta,c)}$

$$\sum_{i,j=0}^{\infty}\langle e_n, f_i\rangle\tau_i\langle f_i, e_j\rangle m_k^{(\beta,c)}(j)$$
$$= -(1-c)^{-1}(\sigma_n m_k^{(\beta,c)}(n+1) + \sigma_{n-1}m_k^{(\beta,c)}(n-1) - \sqrt{c}(2n+\beta)m_k^{(\beta,c)}(n)).$$

Comparing this with (12) and starting with the initial conditions $\sigma_0 = \tau_0 = 1$ and $f_0 = e_0^{(\beta,c)}$ we come to the final conclusion. $\qquad\square$

The next result refers to (20) we state one of the possibilities, the proof as well as other versions can be pattern after [7]. Consider the following two conditions

$$Se_n^{(\beta,c)} = -\frac{1}{1-c}\sum_{k=0}^{\infty}\left(S m_n^{(\beta,c)}\right)(k)e_k, \quad n = 0, 1, \ldots, \tag{22}$$

$$Se_n = -\frac{1}{1-c}\sum_{k=0}^{\infty}\left(S m_n^{(\beta,c)}\right)(k)e_k^{(\beta,c)}, \quad n = 0, 1, \ldots \tag{23}$$

for S to be a closable operator in ℓ^2 for which $\lim\{m_n^{(\beta,c)}; n = 0, 1, \ldots\}$ is a core and S an operator in $\mathcal{H}$ for which either $\lim e$ or $\lim e^{(\beta,c)}$ is a core, depending on which of (22) and (23) is chosen.

THEOREM 4. *Suppose S is a weighted shift with respect $\{m_n^{(\beta,c)}\}_{n=0}^{\infty}$ with weights $\{\sigma_n\}_{n=0}^{\infty}$ and S is a weighted shift with respect to either e or $e^{(\beta,c)}$ depending on choice of between (22) and (23). Then $S = S_{\beta,c}$ and $S = S_{\beta,c}$, that is they are respective creation operators, provided $\sigma_0 = \tau_0 = 1$.*

"Meixner" plays with "Laguerre".   For the reason touched upon at the very beginning of this paper the Laguerre polynomials are specially adapted to Meixner ones. They provide us with a specific example of a Hilbert space $\mathcal{H}$ in which the creation operator $S_{\beta,c}$ manifests itself in a natural way: according to (4), $S_{\alpha+1}$ defined by (3) acts on the Laguerre functions $l_n^{\alpha+1}$ as a creation operator in our abstract sense. Thus one can write down *mutatis mutandis* our Theorems corresponding to this concrete situation. In particular the new basis (or rather a three parameter family of) is explicitly written as

$$\sum_{k=0}^{\infty} m_k^{(\beta,c)}(n) l_k^{\alpha+1}, \quad n = 0, 1, \ldots$$

It might be interesting to notice that (20), after passing to source polynomials, which can be done with ease, according to Theorem 4, determines the couple "Laguerre – Meixner" uniquely.

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