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THE RELATION BETWEEN STABLE OPERATIONS FOR CONNECTIVE AND NON-CONNECTIVE p-LOCAL COMPLEX K-THEORY

BY

KEITH JOHNSON

ABSTRACT. The question of which degree 0 stable cohomology operations for connective K-theory localized at a prime p arise from operations for non-connective K-theory is investigated. A necessary and sufficient condition is established, and an example of a connective operation not arising in this way is constructed.

Let k, K be the spectra representing connective and nonconnective K-theory respectively localized at a prime p. It was shown in [4] that the algebra K^0K of stable operations of degree 0 for nonconnective K-theory is unexpectedly large, in fact uncountable. The corresponding algebra for k was analyzed in [1]. Since there is a canonical map $k \rightarrow K$, which induces an isomorphism in homotopy groups in non-negative dimensions, there is also an induced map $K^0K \rightarrow k^0k$. In this paper we will show:

THEOREM. *The natural map from the algebra of stable operations of degree* 0 *of non-connective K-theory to the corresponding algebra for connective K-theory is injective but not surjective.*

We will produce a set of necessary and sufficient conditions which a connective operation must satisfy in order that it arise from a non-connective operation. Using them we will explicitly construct an operation which doesn't arise in this way.

§1. To begin we will briefly recall the facts about K-theory which we will require, expressing them as results about rings of rational polynomials defined by various integrality conditions. This approach originated in [5].

DEFINITION 1.
(i) $B = \{f \in Q[w] | f(k) \in Z_{(p)}$ *if* $k \in Z$ *and* $(k, p) = 1\}$
(ii) $C = \{f \in Q[w, w^{-1}] | f(k) \in Z_{(p)}$ *if* $k \in Z$ *and* $(k, p) = 1\}$

THEOREM 2 ([1]).
(i) $k_0(k) \cong B$
(ii) $K_0(K) \cong C$

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DEFINITION 3 ([6]). *Let $a_1, a_2, a_3, \ldots$ denote the integers prime to p in increasing order. Let*

$$\gamma_p(n) = \nu_p((n + [n/p - 1])!)$$

where $[x]$ denotes the largest integer less than or equal to x, and $\nu_p(x)$ denotes the p-adic valuation of x, i.e. the largest integer m for which p^m divides x. Also let:

(i) $q_n(w) = \prod_{i=1}^{n}(w - a_i)/p^{\gamma_p(n)}$

(ii) $r_n(w) = w^{-[n/2]} \prod_{i=1}^{n}(w - a_i)/p^{\gamma_p(n)}$

THEOREM 4([1], [6]).

(i) $\{q_n(w) | n = 0, 1, 2, 3, \ldots\}$ *is a* $Z_{(p)}$ *basis for* B.

(ii) $\{r_n(w) | n = 0, 1, 2, 3, \ldots\}$ *is a* $Z_{(p)}$ *basis for* C.

COROLLARY 5.

(i) $k^0(k) \cong Hom(B, Z_{(p)}) = B^*$.

(ii) $K^0(K) \cong Hom(C, Z_{(p)}) = C^*$.

COROLLARY 6.

(i) The algebra of stable degree 0 *operations in connective complex K-theory is isomorphic to* B^*.

(ii) The algebra of stable degree 0 *operations in non-connective complex K-theory is isomorphic to* C^*.

DEFINITION 7. *Define integers* $T(n, i)$ *by:*

$$\prod_{i=1}^{n} (w - a_i) = \sum_{i=0}^{n} T(n, i) w^i$$

COROLLARY 8 ([6]).

(i) If the action of a stable degree 0 *connective K-theory operation on* $k^0(S^{2i}) \cong Z_{(p)}$ *is multiplication by* λ_i*, then these numbers satisfy the congruences:*

$$\sum_{i=0}^{n} T(n, i)\lambda_i \equiv 0 \qquad (\text{mod } p^{\gamma_p(n)})$$

and, conversely, any sequence $\{\lambda_i | i = 0, 1, 2, 3, \ldots\}$ *of numbers satisfying these congruences arises in this way from a unique operation.*

(ii) If the action of a stable degree 0 *K-theory operation on* $K^0(S^{2i}) \cong Z_{(p)}$ *is multiplication by* λ_i*, then these numbers satisfy the congruences:*

$$\sum_{i=0}^{n} T(n, i)\lambda_{i-j} \equiv 0 \quad (\text{mod } p^{\gamma_p(n)})$$

for any integer j, and conversely, any sequence $\{\lambda_i | i = 1, 2, 3, \ldots\}$ *of numbers satisfying these congruences for the special case of* $j = [n/2]$ *arises in this way from a unique operation.*

We next turn to the result of Adams ([2]) concerning the decomposition of K into simpler pieces.

THEOREM 9 ([2]).
(i) There exists a spectrum E such that

$$K \cong \bigvee_{i=0}^{p-2} \Sigma^{2i} E$$

The homotopy groups of E are:

$$\pi_i E = \begin{cases} Z_{(p)} & \text{if } i = 0 \pmod{2(p-1)} \\ 0 & \text{otherwise} \end{cases}$$

(ii) $B = k_0(k) = \oplus_{i=0}^{p-2} B_i$ *where* $B_0 = B \cap Q[w^{(p-1)}]$ *and* $B_i = w^i B_0$.
(iii) $C = K_0(K) = \oplus_{i=0}^{p-2} C_i$ *where* $C_0 = C \cap Q[w^{(p-1)}, w^{-(p-1)}]$ *and* $C_i = w^i C_0$.

REMARK 10. If we let $x = w^{p-1}$ then we may describe the new algebras B_0 and C_0 by:

$$B_0 = \{f \in Q[x] | f(1 + kp) \in Z_{(p)} \text{ if } k \in Z\}$$

$$C_0 = \{f \in Q[x, x^{-1}] | f(1 + kp) \in Z_{(p)} \text{ if } k \in Z\}$$

If we let $q_{0,n}(x) = \Pi_{i=0}^{n-1}(x - (1 + ip))/n!p^n$ and $r_{0,n}(x) = x^{-[n/2]}q_{0,n}(x)$ then in the same way as in theorem 4 $\{q_{0,n}(x)|n = 0, 1, 2, 3 \ldots\}$ is a basis for B_0 and $\{r_{0,n}(x)|n = 0, 1, 2, 3 \ldots\}$ is a basis for C_0.

The final result we will require is the existence of certain specific K-theory operations Ψ^i called the Adams operations.

THEOREM 11 ([3]). *There exist degree* 0 *K-theory operations* Ψ^i *characterized by the properties:*
(i) $\Psi^k(x + y) = \Psi^k(x) + \Psi^k(y)$.
(ii) $\Psi^k(L) = L^k$ *if L is a line bundle.*
(iii) $\Psi^k(xy) = \Psi^k(x)\,\Psi^k(y)$ *and* $\Psi^k(1) = 1$.

If $(k, p) = 1$ then Ψ^k is a stable operation, and, under our identification of stable operations with elements of B^* and C^* it corresponds to the homomorphism:

$$\Psi^k(f(w)) = f(k)$$

These operations also induce operations on the summands of the Adams splitting of K. As elements of B_0^* or C_0^* they can be described as:

$$\Psi^k(f(x)) = f(k)$$

where in this case $k \equiv 1 \pmod{p}$.

Using corollary 8 we may construct inverse operations, $\Psi^{1/k}$, for the stable Adams operations via $\Psi^{1/k}(f(w)) = f(1/k)$. The subalgebras of B^*, C^*, B_0^*, and C_0^* generated by the Adams operations are the group algebras:

$$Z_{(p)}[Z^*_{(p)}] \subset B^*, C^*$$

$$Z_{(p)}[Z^{**}_{(p)}] \subset B^*_0, C^*_0$$

where $Z^*_{(p)}$ denotes the group of units in $Z_{(p)}$ and $Z^{**}_{(p)}$ denotes the units congruent to 1 mod p.

§2. The subalgebras of operations generated by the Adams operations can only form a small portion of B^* or C^* since the later are both uncountable. The Adams operations do, however, provide a great deal of information about B^* and C^* because they are dense in the following sense. As in [4] we will use the notation $B(n, m)$ to denote $B \cap span(w^n, w^{n+1}, \ldots, w^m)$ and similarly for C, C_0, etc.

THEOREM 12.

(i) Given $\alpha \in, B^*$ *or* C^* *and* $m > n$ (≥ 0 *in the case of* B^*) *there exists a linear combination* $\alpha' = \sum_{i=0}^{m-n} c_i \Psi^{a_i}$ *such that:*

$$\alpha|_{B(n,m)} = \alpha'|_{B(n,m)}$$

and similarly for C^*.

(ii) Given $\alpha \in B^*_0$ *or* C^*_0 *and* $m > n$ (≥ 0 *in the case of* B^*_0) *there exists a linear combination* $\alpha' = \sum_{i=0}^{m-n} c_i \Psi^{1+ip}$ *such that:*

$$\alpha|_{B_0(n,m)} = \alpha'|_{B_0(n,m)}$$

and similarly for C^*_0.

PROOF: (i) By theorem 4 above and lemma 6 of [6] $\{w^n q_i(w) | i = 1, 2, \ldots m - n\}$ forms a basis for $C(n, m)$ (and for $B(n, m)$ if $n > 0$). As in (i) above it suffices to show that the linear system

$$[\Psi_i^a(w^n q_j(w))] \begin{bmatrix} c_0 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \alpha(w^n q_0(w)) \\ \vdots \\ \alpha(w^n q_m(w)) \end{bmatrix}$$

is solvable. This however is clear since

$$\Psi^{a_i}(w^n q_j(w)) = \begin{cases} 0 & \text{if } j < i \\ a_i^n q_i(a_i) & \text{if } i = j \end{cases}$$

and the result follows since $q_i(a_i) \in Z^*_{(p)}$ according to proposition 8 of [6].

(ii) As in (i) above we can show that $\{w^n q_{0,i}(x) | i = 0, 1, 2, \ldots m - n\}$ is a basis for $C_0(n, m)$ (and for $B_0(n, m)$ if $n \geq 0$). Also, as in (i)

$$\Psi^{1+ip}(q_{0,j}(x)) = \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j = i \end{cases}$$

and the result follows.

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This result suggests that elements of B^* etc. should be expressable as infinite sums of Adams operations. We make this explicit in the following representation theorem.

DEFINITION 13.
(ii) define $\phi_{0,n} \in B_0^*$ *by*

$$\phi_{0,n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \Psi^{1+ip}$$

(iii) define $\phi'_{0,n} \in C_0^*$ *by*

$$\phi'_{0,n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1 + ip)^{[n-1/2]} \Psi^{1+ip}$$

We may also define $\psi_{j,n} = \psi_{0,n} \circ e_j$ and $\psi'_{j,n} = \psi'_{0,n} \circ e_j$ where $e_j: B_j \to B_0$ is the map $e_j(f) = w^{-j} f$ and similarly for C.

LEMMA 14. *The series* $\Sigma_{i=0}^{\infty} c_n \phi_{0,n}$ *and* $\Sigma_{i=0}^{\infty} c_n \phi'_{0,n}$ *are well defined elements of* B_0^* *and* C_0^* *respectively.*

PROOF: The proof rests on the identity:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i i^m = \begin{cases} 0 & \text{if } m < n \\ (-1)^n n! & \text{if } m = n \end{cases}$$

which can easily be established by induction.

Once we note that:

$$\phi_{0,n}(w^m) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1 + ip)^m$$

$$= \sum_{j=0}^{m} \binom{m}{j} p^j \sum_{i=0}^{n} \binom{n}{i} (-1)^i i^j$$

$$= \begin{cases} 0 & \text{if } m < n \\ (-1)^n n! p^n & \text{if } m = n \end{cases}$$

it follows directly that $\phi_{0,n}(w^m) = 0$ if $n > m$ and so that for any polynomial, f, $\Sigma_{n=0}^{\infty} c_n \phi_{0,n}(f)$ is actually a finite sum and so certainly convergent.

For the second series we need only add that:

$$\phi'_{0,n}(w^m) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1 + ip)^{m+[n-1/2]}$$

$$= 0, \textit{ if } -[n - 1/2] \le m < n - [n - 1/2]$$

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THEOREM 15.

(i_0) *Any* $\alpha \in B_0^*$ *has a unique representation in the form*

$$\alpha = \sum_{n=0}^{\infty} c_n \phi_{0,n}$$

(i) *Any* $\alpha \in B^*$ *has a unique representation in the form*

$$\alpha = \sum_{j=0}^{p-2} \sum_{n=0}^{\infty} c_{n,j} \phi_{j,n}$$

(ii_0) *Any* $\alpha \in C_0^*$ *has a unique representation in the form*

$$\alpha = \sum_{n=0}^{\infty} c_n \phi'_{0,n}$$

(ii) *Any* $\alpha \in C^*$ *has a unique representation in the form*

$$\alpha = \sum_{j=0}^{p-2} \sum_{n=0}^{\infty} c_{n,j} \phi'_{j,n}$$

PROOF: Since $\{q_{0,n} | n = 0, 1, 2, \ldots\}$ is a basis for B_0, it suffices to note that

$$\phi_{0,n}(q_{0,m}) = 0 \text{ if } m < n$$

$$\phi_{0,n}(q_{0,n}) = \phi_{0,r}(w^n)/n!p^n = \pm 1$$

and so that the coefficients c_n can be chosen successively in such a way that

$$\sum_{n=0}^{m} c_n \phi_{0,n}|_{B_0(0,m)} = \alpha|_{B_0(0,n)}$$

(i) follows from (i_0) and the fact that $\Sigma_{j=0}^{p-2} e_j = 1$.

For (ii_0) we must compute $\phi'_{0,n}(r_m)$:

$$\phi'_{0,n} = 0 \text{ if } m < n$$

$$\phi'_{0,n} = \begin{cases} \frac{1}{n!p^n}\phi_{0,n}(w^n) = 1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{1}{n!p^n}\phi_{0,n}(w^{-1}) & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$\frac{1}{n!p^n}\phi_{0,n}(w^{-1}) = \frac{1}{n!p^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i(1+ip)^{-1}$$

$$= \frac{1}{n!p^n}\sum_{j=0}^{\infty}(-1)^j p^j \sum_{i=0}^{i} i^j$$

$$= \pm 1 + \text{terms divisible by } p$$

$$\in Z_{(p)}^*$$

(ii) follows from (ii_0)

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§3. In view of the direct sum decompositions of theorem 9 it suffices for us to restrict our attention to one of the Adams summands. The map we wish to consider is $C_0^* \to B_0^*$ which is the dual of the inclusion $B_0 \hookrightarrow C_0$.

LEMMA 16. *The composition* $C_0 \hookrightarrow Q[x, x^{-1}] \to Q[x^{-1}]$ *induces an isomorphism:*

$$C_0/B_0 \cong x^{-1}Q[x^{-1}]$$

PROOF: It is sufficient for us to show that, given k and n, x^{-k}/p^m is in the image of our map. For this we need only choose m such that $m \geq n - 1$ and $p^m > k$. This ensures that $x^{-k}(1 - x^{p^m})/p^m)/p^{m+1} \in C_0$ and maps to x^{-k}/p^m.

COROLLARY 17. *The map* $C_0^* \to B_0^*$ *is injective but not surjective.*

PROOF: Using the *Hom-Ext* sequence associated to the short exact sequence:

$$0 \to B_0 \to C_0 \to C_0/B_0 \to 0$$

we see that it suffices to prove that:

$$Hom(C_0/B_0, Z_{(p)}) = 0$$

$$Ext(C_0/B_0, Z_{(p)}) \neq 0$$

Both these formulas follow from the previous lemma and the fact that:

$$Hom(Q, Z_{(p)}) = 0$$

$$Ext(Q, Z_{(p)}) = Z_p/\hat{Z}_{(p)} \neq 0$$

We would like now to describe which elements of B_0^* are in the image of this map or, equivalently, which connective degree 0 K-theory operations extend to non-connective operations. Our answer to this question is based on two observations. First, that since

$$C_0 = \bigcup_{n=0}^{\infty} x^{-n}B_0$$

if $\alpha \in B_0^*$ extends to an element of C_0^*, it must extend to an element of $Hom(x^{-1}B_0, Z_{(p)})$ first. Moreover, an obstruction developed for this situation can be used repeatedly, since $x^{-1}B_0 \cong B_0$ by lemma 6 of [6].

The second observation is that if we replace $Z_{(p)}$ by $\hat{Z}_p$ in our considerations above, then any connective operation extends. This can be proved just as in corollary 17 above, using:

$$Hom(Q, \hat{Z}_p) = 0$$

$$Ext(Q, \hat{Z}_p) = 0$$

DEFINITION 18. *Let us define* $\bar{\phi}_{0,n}(x) \in Hom(x^{-1}B_0, Z_{(p)})$ *by*

$$\bar{\phi}_{0,n}(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1 + ip)^{-1} \Psi^{1+ip}$$

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Just as in theorem 15 we can show that any $\alpha \in Hom(x^{-1}B_0, Z_{(p)})$ has a unique representation in the form

$$\alpha(x) = \sum_{n=0}^{\infty} b_n \bar{\phi}_{0,n}(x)$$

with $b_n \in Z_{(p)}$ and any $\alpha \in Hom(x^{-1}B_0, \hat{Z}_p)$ has a similar representation with $b_n \in \hat{Z}_p$.

Given $\alpha \in B_0^*$ with representation:

$$\alpha(x) = \sum_{n=0}^{\infty} b_n \phi_{0,n}(x)$$

we know by our second observation above that α can be extended uniquely to an element of $Hom(x^{-1}B_0, \hat{Z}_p)$ with representation

$$\alpha(x) = \sum_{n=0}^{\infty} \bar{b}_n \bar{\phi}_{0,n}(x)$$

with $\bar{b}_n \in \hat{Z}_p$. This extension will be an element of $Hom(x^{-1}B_0, Z_{(p)})$ if and only if the coefficients of the representation satisfy $\bar{b}_n \in Z_{(p)}$ for all n.

It is clear now how we should proceed. We need only express the coefficients $\bar{b}_n$ in terms of b_n and find conditions insuring that they are in $Z_{(p)}$. To do this we first consider the special case of $\alpha = \phi_{0,n}$.

LEMMA 19.

$$\bar{\phi}_{0,n} = (1 + np)\phi_{0,n} - np\phi_{0,n-1}$$

PROOF: This follows easily from the definitions and the identity

$$(np + 1)\binom{n}{i} - np\binom{n-1}{i} = (ip + 1)\binom{n}{i}$$

Inverting this relation we obtain:

LEMMA 20.

$$\phi_{0,n} = \sum_{i=0}^{n} c_{j,n} \bar{\phi}_{0,j}$$

where

$$c_{j,n} = \frac{1}{jp + 1} \prod_{k=j+1}^{n} \frac{kp}{kp + 1}$$

PROOF: This follows from the previous lemma and the recurrence relation:

$$c_{j,n} = \frac{np}{np + 1} c_{j,n-1} \qquad c_{n,n} = \frac{1}{np + 1}$$

THEOREM 21. $\alpha = \sum_{n=0}^{\infty} b_n \phi_{0,n} \in B_0^*$ *extends to an element of* $Hom(x^{-1}B_0, Z_{(p)})$ *if and only if for each n the p-adic integer*

$$\sum_{i=0}^{\infty} b_k c_{k,n}$$

is an element of $Z_{(p)}$, *in which case*

$$\alpha = \sum_{n=0}^{\infty} \bar{b}_n \bar{\phi}_{0,n}$$

with

$$\bar{b}_n = \sum_{k=0}^{\infty} b_n c_{k,n}$$

Using this theorem we may now give a necessary and sufficient condition for $\alpha \in B_0^*$ to be in the image of the map $C_0^* \rightarrow B_0^*$. If α has the representation

$$\alpha = \sum_{n=0}^{\infty} b_n \phi_{o,n}$$

let us define inductively:

$$b_n^0 = b_n$$

$$b_n^{i+1} = \sum_{k=0}^{\infty} b_k^i c_{k,n}$$

COROLLARY 22. α *is in the image of the map above if and only if for every* n, *and* i $b_n^i \in Z_{(p)}$.

To make use of these results it is necessary to describe the image of the inclusion $Z_{(p)} \hookrightarrow \hat{Z}_p$. Recall that any p-adic integer has a unique representation in the form:

$$z = \sum_{i=0}^{\infty} z_i p^i$$

where $z_i \in \{0, 1, \ldots, p-1\}$.

LEMMA 23. *A p-adic integer z lies in the image of* $Z_{(p)}$ *if and only if its infinite sum representation is periodic i.e. there exist* n, k *such that* $z_{i+k} = z_i$ *for all* $i > n$.

COROLLARY 24. *If*

$$b_n = p^{\nu_p(n!)+n} \prod_{k=1}^{n} (kp+1)/n!$$

then

$$\alpha = \sum_{n=0}^{\infty} b_n \phi_{0,n} \in B_0^*$$

is not in the image of the map $C_0^* \rightarrow B_0^*$.

REFERENCES

1. J. F. Adams, *Stable Homotopy and Generalized Homology*, The University of Chicago Press, Chicago, 1974.

2. J. F. Adams, *Lectures on Generalized Cohomology*, Lecture notes in Mathematics **99** (1969), pp. 1–138, Springer-Verlag, Berlin-Heidelberg-New York.

3. J. F. Adams, *Vector Fields on Spheres,* Ann. of Math. **75** (1962), pp. 603–632.

4. J. F. Adams and F. Clarke, *Stable Operations on Complex K-theory*, Ill. Jour. Math. **21** (1977), pp. 826–829.

5. J. F. Adams, A. S. Harris, and R. M. Switzer, *Hopf Algebras of Cooperations for Real and Complex K-theory*, J. London Math. Soc. **23** (1971), pp. 385–408.

6. K. Johnson, *The Action of the Stable Operations of Complex K-theory on Coefficient Groups,* Ill. Jour. Math. **28** (1984), pp. 57–63.

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA