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# Christoffel Functions and Universality in the Bulk for Multivariate Orthogonal Polynomials

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*Abstract.* We establish asymptotics for Christoffel functions associated with multivariate orthogonal polynomials. The underlying measures are assumed to be regular on a suitable domain. In particular, this is true if they are positive a.e. on a compact set that admits analytic parametrization. As a consequence, we obtain asymptotics for Christoffel functions for measures on the ball and simplex under far more general conditions than previously known. As another consequence, we establish universality type limits in the bulk in a variety of settings.

## 1 Introduction

Let $\mu$ be a positive measure on the real line with infinitely many points in its support, and let $\int x^j d\mu(x)$ be finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

The $n$-th *reproducing kernel* is

$$K_n(x,t) = \sum_{j=0}^{n-1} p_j(x) p_j(t),$$

and the $n$-th *Christoffel function* is

$$\lambda_n(\mu, x) = \frac{1}{K_n(x,x)} = \inf_{\deg(P)<n} \frac{\int P(t)^2 d\mu(t)}{P^2(x)}.$$

Asymptotics for Christoffel functions play a crucial role in analysis of orthogonal polynomials and in weighted approximation [12]. The most general asymptotics for

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the case where the support $\operatorname{supp}[\mu]$ is compact, require $\mu$ to be *regular in the sense of Stahl, Totik, and Ullman* , or just *regular*. This requires [15, p. 66] that

$$\limsup_{n\to\infty}\Big(\sup_{\deg(P)\le n}\frac{|P(x)|^2}{\int|P|^2d\mu}\Big)^{1/n}\le 1,\quad q.e.\ \text{in}\ \operatorname{supp}[\mu] \tag{1.1}$$

Here q.e. (quasi-everywhere) means except on a set of logarithmic capacity 0. This type of regularity should not be confused with the notion of a regular Borel measure. When the complement of $\operatorname{supp}[\mu]$ is regular in the sense of the Dirichlet problem (yet another notion of regularity!) [15, p. 68], one may replace (1.1) by

$$\limsup_{n\to\infty}\Big(\sup_{\deg(P)\le n}\frac{\|P\|_{L_\infty(\operatorname{supp}[\mu])}}{\int|P|^2d\mu}\Big)^{1/n}\le 1.$$

If $\operatorname{supp}[\mu]$ consists of finitely many intervals, a sufficient condition for regularity, called the Erdős–Turán criterion, is that $\mu'>0$ a.e. in $\operatorname{supp}[\mu]$. See the comprehensive monograph [15].

When $\mu$ is regular and in some subinterval $I$ of the support, we have

$$\int_I\log\mu'>-\infty. \tag{1.2}$$

Totik [17] proved that for a.e. $x\in I$,

$$\lim_{n\to\infty}n\lambda_n(\mu,x)=\frac{\mu'(x)}{\omega(x)}.$$

Here $\omega$ is the equilibrium density for $\operatorname{supp}[\mu]$, in the sense of potential theory. In the special case that the support of $\mu$ is an interval, this result was established earlier by Maté, Nevai, and Totik [10].

One application of asymptotics for Christoffel functions is to universality limits. These arise in analysis of random matrices associated with unitary ensembles and, for compactly supported $\mu$, may be reduced to the limit

$$\lim_{n\to\infty}\frac{K_n\big(x+\frac{a\mu'(x)}{n\omega(x)},x+\frac{b\mu'(x)}{n\omega(x)}\big)}{K_n(x,x)}=\frac{\sin\pi(a-b)}{\pi(a-b)},$$

uniformly for $a,b$ in compact subsets of the real line. For the case of compactly supported $\mu$ that are regular and satisfy (1.2), Totik [18] showed that universality holds for a.e. $x\in I$. The second author showed that universality holds in measure without assuming regularity [9]. We emphasize that there is a vast literature on this topic, and varying measures are of more interest to physicists than fixed measures with compact support. See [4, 5, 13, 14].

In this paper, we shall analyze asymptotics of Christoffel functions for multivariate orthogonal polynomials and apply these to universality type limits. Let $d\ge 2$ and let

$\Pi_n^d$ denote the space of polynomials in $d$ variables of degree at most $n$. Let $N_n^d$ denote its dimension, so

$$N_n^d = \binom{n+d}{n}.$$

Let $\mu$ be a positive measure on $\mathbb{R}^d$ with compact support and $\{\mathbf{x} \in \mathbb{R}^d : \mu'(\mathbf{x}) > 0\}$ having non-empty interior. This ensures that

$$\int P^2 d\mu > 0$$

for every non-trivial polynomial $P$.

We let $K_n(\mu, \mathbf{x}, \mathbf{y})$ denote the reproducing kernel for $\mu$ and $\Pi_n^d$, so that for all $P \in \Pi_n^d$, and all $\mathbf{x} \in \mathbb{R}^d$,

$$P(\mathbf{x}) = \int K_n(\mu, \mathbf{y}, \mathbf{x}) P(y) d\mu(\mathbf{y}).$$

Note that this notation is different to the one-dimensional case, where we assumed exactness for polynomials of degree $\leq n-1$, in accordance with the standard univariate notation. We adopt this difference to be consistent with the most common multivariate convention.

One of the convenient features of the reproducing kernel is that it is independent of how we order the monomials and generate orthonormal polynomials. The $n$-th Christoffel function for $\mu$ is

$$\lambda_n(\mu, \mathbf{x}) = \frac{1}{K_n(\mu, \mathbf{x}, \mathbf{x})}.$$

It admits the extremal property

$$\lambda_n(\mu, \mathbf{x}) = \inf_{P \in \Pi_n^d} \frac{\int P(\mathbf{t})^2 d\mu(\mathbf{t})}{P^2(\mathbf{x})}.$$

When $\mu$ is absolutely continuous with respect to $d$ dimensional Lebesgue measure and $\mu' = W$, we shall write $\lambda_n(W, \mathbf{x})$.

Asymptotics for these multivariate Christoffel functions have been established in a number of papers [1–3, 19, 20, 23] for Jacobi weights and weights that satisfy some structural restriction, such as being radially or centrally symmetric. For our purposes, the most general result is due to Bos, Della Vecchia, and Mastroianni [2]. They showed that for a centrally symmetric weight $W(\mathbf{x})$ on the $d$ dimensional ball, for which $W(\mathbf{x})\sqrt{1 - \|\mathbf{x}\|^2}$ satisfies a centrally symmetric Lipschitz condition of some positive order on the unit ball in $\mathbb{R}^d$,

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(W, \mathbf{x}) = \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \sqrt{1 - \|\mathbf{x}\|^2} W(\mathbf{x}), \quad \|\mathbf{x}\| < 1.$$

Here and in the sequel the norm is the Euclidean norm. Xu [20] established one-sided asymptotics under more general conditions.

By using "needle polynomials", we shall generalize the above quoted results to a much larger class of measures. We first extend the notion of regularity to $d$ dimensions. A compactly supported measure $\mu$ on $\mathbb{R}^d$ is said to be *regular* if

$$\lim_{n\to\infty}\left(\sup_{P\in\Pi_n^d}\frac{\|P\|^2_{L_\infty(\operatorname{supp}[\mu])}}{\int|P|^2d\mu}\right)^{1/n}=1. \tag{1.3}$$

This is often called the Bernstein–Markov condition [3], but we prefer the term regularity.

Our most general ratio asymptotic is the following theorem.

***Theorem 1.1*** *Let $\mu,\nu$ be positive measures whose support is a compact set $\mathcal{K}\subset\mathbb{R}^d$ and both are regular. Let $D\subset D_1\subset\mathcal{K}$, where $D$ is compact and $D_1$ is open. Assume that $\nu$ and that $\mu$ are mutually absolutely continuous in $D$, and the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$ is positive and continuous in $D$, while uniformly in $\overline{D_1}$,*

$$\lim_{\varepsilon\to0+}\left(\limsup_{n\to\infty}\frac{\lambda_{[n(1-\varepsilon)]}(\mu,\mathbf{x})}{\lambda_n(\mu,\mathbf{x})}\right)=1. \tag{1.4}$$

*Then uniformly for $\mathbf{x}\in D$, and $\mathbf{y}\in B(\mathbf{x},\frac{1}{\sqrt{n}})$, we have*

$$\lim_{n\to\infty}\frac{\lambda_n(\nu,\mathbf{y})}{\lambda_n(\mu,\mathbf{y})}=\frac{d\nu}{d\mu}(\mathbf{x}). \tag{1.5}$$

***Remark*** (a) In the statement, $[s]$ denotes the greatest integer $\le s$, while $B(\mathbf{x},r)$ is the $d$-dimensional open ball with center $\mathbf{x}$, radius $r$.

(b) Note that (1.4) is satisfied if for some $\beta>0$ and positive continuous function $F$,

$$\lim_{n\to\infty}n^\beta\lambda_n(\mu,\mathbf{x})=F(\mathbf{x})$$

uniformly in $\overline{D_1}$.

(c) In Section 7, we shall present a version of Theorem 1.1 in which we replace the continuity of $\frac{d\nu}{d\mu}$ with a Lebesgue point condition.

A sufficient condition for the regularity (1.3) involves a compact set $\mathcal{K}$ with *analytic parametrization*. This means that for any $\mathbf{x}\in\mathcal{K}$, there exists a curve $\gamma(t)\in\mathbb{R}^d, t\in[0,1]$, analytic and bounded in an open set $\Omega\subset\mathbb{C}$ that contains $[0,1]$ and such that $\gamma(0)=\mathbf{x}$, while for all $0<t<1$,

$$B\big(\gamma(t),\phi(t)\big)\subset\mathcal{K}.$$

Here $\Omega$ and the bound on $\gamma$ depend only on $\mathcal{K}$, while $\phi$ is a positive continuous function tending to 0 as $t\to0$ and that also depends only on $\mathcal{K}$. In particular, any polygon or convex set with non-empty interior has analytic parametrization. In fact local convexity also suffices; see [6] for details.

***Theorem 1.2*** *Assume that $\mathcal{K} \subset \mathbb{R}^d$ is a compact set with analytic parametrization, and that $\mu, \nu$ are positive measures on $\mathcal{K}$ such that $\mu', \nu' > 0$ a.e. on $\mathcal{K}$. Let $D \subset D_1 \subset \mathcal{K}$, where $D$ is compact and $D_1$ is open. Assume that the Radon–Nikodym derivative $\frac{\nu'}{\mu'}$ is positive and continuous in $D$, while (1.4) holds uniformly in $\overline{D_1}$. Then we have (1.5) uniformly for $\mathbf{x} \in D$ and $\mathbf{y} \in B(\mathbf{x}, \frac{1}{\sqrt{n}})$.*

The proof of Theorem 1.2 also shows that we may replace the condition that $\mathcal{K}$ has analytic parametrization with a condition that it admits a Remez inequality.

As a consequence, we can deduce asymptotics for Christoffel functions associated with regular measures on the ball and the simplex.

***Theorem 1.3*** *Let $\bar{B} = \overline{B(\mathbf{0}, 1)} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\}$. Let $\nu$ be a regular measure on $\bar{B}$ and assume that $D$ is a compact subset of the interior of $\bar{B}$, such that $\nu'$ is positive and continuous in $D$. Then, uniformly for $\mathbf{x} \in D$, and for $\mathbf{y} \in B(\mathbf{x}, \frac{1}{\sqrt{n}})$,*

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\nu, \mathbf{y}) = \frac{\nu'(\mathbf{x})}{W_0^{ball}(\mathbf{x})},$$

*where*

$$W_0^{ball}(\mathbf{x}) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} (1 - \|\mathbf{x}\|^2)^{-1/2}.$$

***Theorem 1.4*** *Let*

$$\Sigma^d = \left\{ \mathbf{x} \in \mathbb{R}^d : x_1, x_2, \ldots, x_d \geq 0; 1 - \sum_{j=1}^{d} x_j \geq 0 \right\}$$

*denote the $d$-dimensional simplex. Let $\nu$ be a regular measure on $\Sigma^d$ and assume that $D$ is a compact subset of the interior of $\Sigma^d$, such that $\nu'$ is positive and continuous in $D$. Then, uniformly for $\mathbf{x} \in D$ and for $\mathbf{y} \in B(\mathbf{x}, \frac{1}{\sqrt{n}})$,*

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\nu, \mathbf{y}) = \frac{\nu'(\mathbf{x})}{W_0^{simplex}(\mathbf{x})},$$

*where*

$$W_0^{simplex}(\mathbf{x}) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} x_1^{-1/2} x_2^{-1/2} \cdots x_d^{-1/2} x_{d+1}^{-1/2}$$

*and*

$$x_{d+1} = 1 - \sum_{j=1}^{d} x_j. \tag{1.6}$$

*In the second last line, of course $\mathbf{x} = (x_1, x_2, \ldots, x_d)$.*

We next turn to universality limits. We start with a general ratio result.

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***Theorem 1.5*** *Let $\mu, \nu$ be positive measures and let $\mathcal{K}, D, D_1$ be sets satisfying the hypotheses of Theorem 1.1. Assume moreover, that $\mu$ and $\nu$ are absolutely continuous in $D_1$, while $\mu'$ and $\nu'$ are bounded above and below by positive constants in $D_1$. Then, uniformly for $\mathbf{x} \in D$, and for $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{K_n(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}) - \frac{d\nu}{d\mu}(\mathbf{x}) K_n(\nu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n})}{K_n(\mu, \mathbf{x}, \mathbf{x})} = 0. \tag{1.7}$$

As a consequence, we can prove the following theorem.

***Theorem 1.6*** *Let $\mu, \nu$ be positive measures and $\mathcal{K}, D, D_1$ be sets satisfying the hypotheses of Theorem 1.2. Assume moreover, that $\mu$ and $\nu$ are absolutely continuous in $D_1$, while $\mu'$ and $\nu'$ are bounded above and below by positive constants in $D_1$. Assume that for some function $F\colon \mathbb{R}^2 \to \mathbb{R}$, uniformly for $\mathbf{x} \in D$ and for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,*

$$\lim_{n\to\infty} \frac{K_n(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n})}{K_n(\mu, \mathbf{x}, \mathbf{x})} = F(\mathbf{u}, \mathbf{v}).$$

*Then, uniformly for $\mathbf{x} \in D$, and for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,*

$$\lim_{n\to\infty} \frac{K_n(\nu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n})}{K_n(\nu, \mathbf{x}, \mathbf{x})} = F(\mathbf{u}, \mathbf{v}).$$

It is straightforward to compute universality limits for the Chebyshev weight on the ball and simplex from known representations due to Y. Xu for the reproducing kernel for the Chebyshev weight on the ball and simplex. Using these and Theorem 1.5 we can obtain general universality results on the ball and simplex. Somewhat surprisingly, the bulk of the concrete formulations involve the Bessel function

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{j=0}^{\infty} \frac{(-\frac{1}{4}z^2)^j}{j!\Gamma(j+\alpha+1)}$$

rather than the usual sinc kernel. Of course, on the real line the Bessel function and its associated Bessel kernel arise primarily at the edge of the spectrum. We shall find it convenient to also use

$$J_\alpha^*(z) = z^{-\alpha} J_\alpha(z) = \left(\frac{1}{2}\right)^\alpha \sum_{j=0}^{\infty} \frac{(-\frac{1}{4}z^2)^j}{j!\Gamma(j+\alpha+1)}.$$

This has the advantage of being entire, and in particular, non-zero at 0, with

$$J_\alpha^*(0) = \frac{1}{2^\alpha \Gamma(\alpha+1)}.$$

In the literature, $j_\alpha(x)$ is sometimes used to denote $J_\alpha^*(x)/J_\alpha^*(0)$, but we shall not use this.

***Theorem 1.7*** *Let $\mu$ be a regular measure on $\bar{B}$ and assume that $D$ is a compact subset of the interior of $\bar{B}$, such that $\mu'$ is positive and continuous in $D$. Then, uniformly for $\mathbf{x} \in D$, and for $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{K_n(\mu, \mathbf{x}+\frac{\mathbf{u}}{n}, \mathbf{x}+\frac{\mathbf{v}}{n})}{K_n(\mu,\mathbf{x},\mathbf{x})} = \frac{J^*_{d/2}\big(\sqrt{G(\mathbf{x},\mathbf{u},\mathbf{v})}\big)}{J^*_{d/2}(0)}, \tag{1.8}$$

*where, if $\cdot$ denotes the standard Euclidean inner product, then*

$$G(\mathbf{x},\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|^2 + \frac{(\mathbf{x}\cdot(\mathbf{u}-\mathbf{v}))^2}{1-\|\mathbf{x}\|^2}. \tag{1.9}$$

For the simplex, we have the following theorem.

***Theorem 1.8*** *Let $\mu$ be a regular measure on $\Sigma^d$, and assume that $D$ is a compact subset of the interior of $\Sigma^d$ such that $\mu'$ is positive and continuous in $D$. Then, uniformly for $\mathbf{x} \in D$, and for $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{K_n(\mu, \mathbf{x}+\frac{\mathbf{u}}{n}, \mathbf{x}+\frac{\mathbf{v}}{n})}{K_n(\mu,\mathbf{x},\mathbf{x})} = \frac{J^*_{d/2}\big(\sqrt{H(\mathbf{x},\mathbf{u},\mathbf{v})}\big)}{J^*_{d/2}(0)}, \tag{1.10}$$

*where, with the notation (1.6),*

$$H(\mathbf{x},\mathbf{u},\mathbf{v}) = \sum_{j=1}^{d+1} \frac{(u_j - v_j)^2}{x_j}. \tag{1.11}$$

*Of course, in the last line, $x_j$ is the $j$-th component of $\mathbf{x}$, and so on, while $u_{d+1}, v_{d+1}$ are given by (1.6) for the vectors $\mathbf{u}, \mathbf{v}$.*

This paper is organized as follows: in Section 2, we prove Theorem 1.1. We prove Theorem 1.2 in Section 3, and Theorems 1.3 and 1.4 in Section 4. Theorems 1.5 and 1.6 are proved in Section 5, and Theorems 1.7 and 1.8 are proved in Section 6. We prove an extension of Theorem 1.1 in Section 7.

Throughout, $c, C, C_1, C_2, \dots$ denote positive constants independent of $n$, and vectors $\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$, as well as polynomials $p$. A given constant does not necessarily denote the same constant in different occurrences.

## 2 Proof of Theorem 1.1

We shall use the "needle" polynomials constructed by Kroó and Swetits [7]. We could also have used the fast decreasing polynomials of Ivanov and Totik.

***Lemma 2.1*** *Let $n \ge 1$, $\delta \in (0,1)$, and $\mathbf{x} \in \bar{B}$. There exists $q_n \in \Pi_n^d$ such that*

(i) $q_n(\mathbf{x}) = 1$;
(ii) $0 \le q_n \le 1$ *in* $\bar{B}$;

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(iii) $|q_n(\mathbf{y})| \leq e^{-cn\delta}$, $\mathbf{y} \in B \backslash B(\mathbf{x}, \delta)$
*(here c is an absolute constant).*

***Remark*** We emphasize that $q_n$ depends on $\mathbf{x}$ and $\delta$. The theory of fast decreasing polynomials implies that one cannot choose $q_n$ independent of $\delta$.

**Proof** We follow the construction of Lemma 3 and Corollary 2 in [7, pp. 92–93]. Consider the polynomial

$$r_m(t) = \left( \frac{T_m(1+\delta^2 - t^2)}{T_m(1+\delta^2)} \right)^2 \in \Pi^1_{4m},$$

where $T_m$ is the usual Chebyshev polynomial. Here $r_m(0) = 1$. For $t \in [-1, 1]$, we have $0 \leq 1 + \delta^2 - t^2 \leq 1 + \delta^2$, and $T_n$ is increasing on $[1, \infty)$, so in $[-1, 1]$,

$$0 \leq r_m \leq 1.$$

Finally, for $|t| \in [\delta, 1]$, we have $1 + \delta^2 - t^2 \leq 1$, so

$$r_m(t) \leq \frac{1}{(T_m(1+\delta^2))^2} \leq e^{-Cm\delta},$$

an easy consequence of the identity

$$T_m(t) = \frac{1}{2}\Big( \big(t + \sqrt{t^2-1}\big)^m + \big(t + \sqrt{t^2-1}\big)^{-m} \Big).$$

Now we set $m = [n/4]$, and

$$q_n(\mathbf{y}) = r_m\Big( \frac{\|\mathbf{y} - \mathbf{x}\|}{4} \Big).$$

Then clearly $q_n \in \Pi^d_n$. Properties (i) and (ii) follow directly from those of $r_m$, while if $\mathbf{y} \in B$ and $\|\mathbf{y} - \mathbf{x}\| \geq 4\delta$,

$$q_n(\mathbf{y}) \leq e^{-Cm\delta} \leq e^{-c_1 n\delta}.$$

Now simply replace $\delta$ by $\frac{\delta}{4}$. ■

**Proof of Theorem 1.1** As the measure $\mu$ is regular, with support $\mathcal{K}$, there exists a sequence $\{\delta_n\}$ with limit 0 such that for $n \geq 1$,

$$\sup_{P \in \Pi^d_n} \frac{\|P\|^2_{L_\infty(\mathrm{supp}[\mu])}}{\int |P|^2 d\mu} \leq e^{n\delta_n^2}. \tag{2.1}$$

We may assume that

$$\lim_{n\to\infty} n\delta_n^2 = \infty. \tag{2.2}$$

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Since $h = \frac{d\nu}{d\mu}$ is uniformly continuous on $D$,

$$\varepsilon_n = \sup\left\{ |h(\mathbf{x}) - h(\mathbf{y})| : \mathbf{x} \in D, \mathbf{y} \in \mathcal{K}, \|\mathbf{x} - \mathbf{y}\| \le \delta_n \right\} \to 0, \quad n \to \infty. \tag{2.3}$$

Let us set $m = m(n) = n - [\frac{2\delta_n n}{c}] - 1$, where $c$ is the absolute constant in Lemma 2.1. We may assume, by a translation and dilation of the support, that $\mathcal{K} \subset B$. Now choose any $\mathbf{x}_0 \in D$ and any $\mathbf{y} \in B(\mathbf{x}_0, \frac{\delta_n}{2})$. Choose $p_m \in \Pi_n^d$ that is extremal for $\lambda_m(\mu, \mathbf{y})$, so that

$$\lambda_m(\mu, \mathbf{y}) = \int p_m^2 d\mu \quad \text{and} \quad p_m(\mathbf{y}) = 1.$$

Choose $q_{n-m}$ as in Lemma 2.1, with the properties $q_{n-m}(\mathbf{y}) = 1$, $0 \le q_{n-m} \le 1$ in $B$, and

$$|q_{n-m}(\mathbf{x})| \le e^{-c(n-m)\frac{\delta_n}{2}}, \mathbf{x} \in B \backslash B\left(\mathbf{y}, \frac{\delta_n}{2}\right).$$

Set

$$S_n = p_m q_{n-m} \in \Pi_n^d.$$

We have $S_n(\mathbf{y}) = 1$, and so the extremal property of $\lambda_n$, followed by the properties of $q_{n-m}$, give

$$\begin{aligned} \lambda_n(\nu, \mathbf{y}) &\le \int_{\mathcal{K}} S_n^2 d\nu \\ &\le \int_{B(\mathbf{x}_0,\delta_n)} p_m^2 h d\mu + e^{-c(n-m)\delta_n} \|p_m\|^2_{L_\infty(\mathcal{K})} \int_{\mathcal{K}\backslash B(\mathbf{x}_0,\delta_n)} d\nu \\ &\le \left(h(\mathbf{x}_0) + \varepsilon_n\right) \int_{B(\mathbf{x}_0,\delta_n)} p_m^2 d\mu + e^{-c(n-m)\delta_n} e^{n\delta_n^2} \left(\int_{\mathcal{K}} p_m^2 d\mu\right)\left(\int_{\mathcal{K}} d\nu\right), \end{aligned}$$

by (2.1) and (2.3). Using our choice of $m$, we continue this as

$$\begin{aligned} \lambda_n(\nu, \mathbf{y}) &\le \left(\int_{\mathcal{K}} p_m^2 d\mu\right)\left(h(\mathbf{x}_0) + \varepsilon_n + e^{-2n\delta_n^2 + n\delta_n^2} \int_{\mathcal{K}} d\nu\right) \\ &= \lambda_m(\mu, \mathbf{y})\left(h(\mathbf{x}_0) + \varepsilon_n + e^{-n\delta_n^2} \int_{\mathcal{K}} d\nu\right). \end{aligned}$$

Since $\delta_n$ and $\varepsilon_n$ are independent of $\mathbf{x}_0 \in D, \mathbf{y} \in B(\mathbf{x}_0, \frac{\delta_n}{2})$, we have

$$\frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \le \frac{\lambda_m(\mu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})}\left(h(\mathbf{x}_0) + \varepsilon_n + e^{-n\delta_n^2} \int_{\mathcal{K}} d\nu\right) \le h(\mathbf{x}_0) + o(1),$$

uniformly for $\mathbf{x}_0 \in D, \mathbf{y} \in B(\mathbf{x}_0, \frac{\delta_n}{2})$, because $\frac{m}{n} = 1 + o(1)$ and by our hypothesis (1.4). Thus we have shown that uniformly for such $\mathbf{x}_0, \mathbf{y}$,

$$\limsup_{n\to\infty} \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \le h(\mathbf{x}_0). \tag{2.4}$$

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Since (2.2) holds, $B(\mathbf{x}_0, \frac{\delta_n}{2}) \supset B(\mathbf{x}_0, \frac{1}{\sqrt{n}})$, for large enough $n$. For the converse inequality, we note that with $m_1 = m_1(n) = n + [\frac{2\delta_n n}{c}]$, we obtain, by swapping the roles of $\mu$ and $\nu$ in the above,

$$\lambda_{m_1}(\mu, \mathbf{y}) \leq \lambda_n(\nu, \mathbf{y}) \Bigl( h^{-1}(\mathbf{x}_0) + o(1) + e^{-n\delta_n^2} \int_{\mathcal{K}} d\mu \Bigr),$$

and hence

$$\frac{\lambda_{m_1}(\mu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \leq \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \Bigl( h^{-1}(\mathbf{x}_0) + o(1) + e^{-n\delta_n^2} \int_{\mathcal{K}} d\mu \Bigr).$$

Here the left-hand side is $1 + o(1)$ by our hypothesis (1.4), and, as $\frac{m_1}{n} = 1 + o(1)$,

$$1 \leq \liminf_{n\to\infty} \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} h^{-1}(\mathbf{x}_0),$$

which, together with (2.4), gives the result. ∎

## 3 Proof of Theorem 1.2

Recall that we defined the notion of analytic parametrization before Theorem 1.2. It suffices to prove the following theorem, and then apply Theorem 1.1.

***Theorem 3.1*** *Assume that $\mathcal{K}$ is a compact set with analytic parametrization and that $\mu$ is a positive measure on $\mathcal{K}$ such that $\mu' > 0$ a.e. on $K$. Then $\mu$ is regular.*

**Proof** Let $m$ denote Lebesgue measure on $\mathbb{R}^d$. The first fact we need is a Remez inequality, proved in [6, Thm. 5, p. 30]. There exists a positive continuous function $\phi^*$ defined on $[0, \infty)$, such that

$$\lim_{\varepsilon\to 0+} \phi^*(\varepsilon) = 0 \tag{3.1}$$

and if $n \geq 1, P \in \Pi_n^d$, and $\mathcal{F} \subset \mathcal{K}$ with $m(\mathcal{F}) \leq \varepsilon$, then

$$\|P\|_{L_\infty(\mathcal{K})} \leq e^{n\phi^*(\varepsilon)} \|P\|_{L_\infty(\mathcal{K}\setminus\mathcal{F})}. \tag{3.2}$$

Next, for $\delta > 0$, we set

$$\psi(\delta) = \inf\bigl\{ \mu(L) : L \subset \mathcal{K}, m(L) \geq \delta \bigr\}.$$

Since $\mu' > 0$ a.e. on $\mathcal{K}$, $\psi(\delta) > 0$ for $\delta > 0$, and $\psi$ is monotonically increasing, with limit 0 at 0.

We proceed to prove regularity. Let $\varepsilon \in (0, \min\{1, m(\mathcal{K})\})$ and $P \in \Pi_n^d$ be a non-constant polynomial. Choose $a = a(\varepsilon) \in (0, 1)$ such that

$$Q_a = \bigl\{ \mathbf{x} \in K : |P(\mathbf{x})| > a\|P\|_{L_\infty(\mathcal{K})} \bigr\}$$

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has $m(Q_a) = \varepsilon$. Note that $m(Q_a)$ is a strictly decreasing continuous function of $a$, with limit 0 at 1 and limit $m(K)$ at 0, so we can choose such an $a$. Now we have

$$|P| \leq a\|P\|_{L_\infty(\mathcal{K})} \text{ in } \mathcal{K}\backslash Q_a.$$

Moreover, since $m(Q_a) = \varepsilon$, the Remez inequality (3.2) gives

$$\|P\|_{L_\infty(\mathcal{K})} \leq e^{n\phi^*(\varepsilon)}\|P\|_{L_\infty(\mathcal{K}\backslash Q_a)}.$$

Combining these two inequalities, yields $ae^{n\phi^*(\varepsilon)} \geq 1$. In addition, as $m(Q_a) = \varepsilon$, $\mu(Q_a) \geq \psi(\varepsilon)$. These last two inequalities give

$$\int_{\mathcal{K}} |P|^2 d\mu \geq \int_{Q_a} |P|^2 d\mu \geq a^2\|P\|^2_{L_\infty(\mathcal{K})}\mu(Q_a) \geq e^{-2n\phi^*(\varepsilon)}\|P\|^2_{L_\infty(\mathcal{K})}\psi(\varepsilon).$$

As $\varepsilon > 0$ , $\psi$ and $\phi^*$ are independent of $P$, we obtain

$$\sup_{P\in\Pi_n^d} \frac{\|P\|^2_{L_\infty(\mathcal{K})}}{\int_{\mathcal{K}} |P|^2 d\mu} \leq \frac{1}{\psi(\varepsilon)} e^{2n\phi^*(\varepsilon)}.$$

Taking $n$-th roots and lim sup's gives

$$\limsup_{n\to\infty} \left( \sup_{P\in\Pi_n^d} \frac{\|P\|^2_{L_\infty(\mathcal{K})}}{\int_{\mathcal{K}} |P|^2 d\mu} \right)^{1/n} \leq e^{2\phi^*(\varepsilon)}.$$

Finally, as $\varepsilon$ is arbitrary, and (3.1) holds, we have the result. ∎

## 4 Proofs of Theorems 1.3 and 1.4

It is easily seen that the ball and simplex admit analytic parametrization, and we can take the curve $\gamma$ to be just a straight line segment. All we need are asymptotics for the Christoffel functions for the Chebyshev weight on the ball or simplex, and these have been established by Bos and Xu.

**Proof of Theorem 1.3** Bos [1, p. 100] and Xu [20, Theorem 4.1, p. 266] proved that for $\|\mathbf{x}\| < 1$,

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(W_0^{\text{ball}}, \mathbf{x}) = 1.$$

Bos states the uniform convergence in compact sets, while Xu omits it from his statement, but it is clear from his proof. Our normalization of $W_0^{\text{ball}}$ is that of Xu. Then Theorem 1.1, with $d\mu(\mathbf{x}) = W_0^{\text{ball}}(\mathbf{x})d\mathbf{x}$, gives the result. Note that this $\mu$ satisfies (1.4) uniformly in compact subsets of $B$. ∎

**Proof of Theorem 1.4** Xu [23, Cor. 2.4, p. 127] proved that for $\mathbf{x}$ in the interior of $\Sigma^d$,

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(W_0^{\text{simplex}}, \mathbf{x}) = 1.$$

Again, the uniform convergence in compact sets is obvious from the proof. We note a minor misprint in [23, p. 123] in the definition of $W_0^{\text{simplex}}$: the normalization constant should be replaced by its reciprocal. Again, we can apply Theorem 1.1. ∎

## 5 Proofs of Theorems 1.5 and 1.6

The method follows that in [8]. We begin with the following lemma.

***Lemma 5.1*** *Assume that $\mu, \mu^*$ are measures with compact support $\mathcal{K} \subset \mathbb{R}^d$, and for some $\rho > 0$, $d\mu \le \rho \quad d\mu^*$ in $\mathcal{K}$. Then for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,*

$$\left| K_n(\mu, \mathbf{x}, \mathbf{y}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{y}) \right| / K_n(\mu, \mathbf{x}, \mathbf{x}) \le \left( \frac{K_n(\mu, \mathbf{y}, \mathbf{y})}{K_n(\mu, \mathbf{x}, \mathbf{x})} \right)^{1/2} \left[ 1 - \frac{K_n(\mu^*, \mathbf{x}, \mathbf{x})}{\rho K_n(\mu, \mathbf{x}, \mathbf{x})} \right]^{1/2}. \tag{5.1}$$

**Proof** This is essentially the same as in [8], but we include the details because of the different setting. Now

$$\begin{aligned}
&\int_{\mathcal{K}} \left( K_n(\mu, \mathbf{x}, \mathbf{t}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{t}) \right)^2 d\mu(\mathbf{t}) \\
&= \int_{\mathcal{K}} K_n^2(\mu, \mathbf{x}, \mathbf{t}) d\mu(\mathbf{t}) - \frac{2}{\rho} \int_{\mathcal{K}} K_n(\mu, \mathbf{x}, \mathbf{t}) K_n(\mu^*, \mathbf{x}, \mathbf{t}) d\mu(\mathbf{t}) \\
&\quad + \frac{1}{\rho^2} \int_{\mathcal{K}} K_n^2(\mu^*, \mathbf{x}, \mathbf{t}) d\mu(\mathbf{t}) \\
&= K_n(\mu, \mathbf{x}, \mathbf{x}) - \frac{2}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{x}) + \frac{1}{\rho^2} \int_{\mathcal{K}} K_n^2(\mu^*, \mathbf{x}, \mathbf{t}) d\mu(\mathbf{t}),
\end{aligned}$$

by the reproducing kernel property. As $d\mu \le \rho \, d\mu^*$, we also have

$$\int_{\mathcal{K}} K_n^2(\mu^*, \mathbf{x}, \mathbf{t}) d\mu(\mathbf{t}) \le \rho \int_{\mathcal{K}} K_n^2(\mu^*, \mathbf{x}, \mathbf{t}) d\mu^*(\mathbf{t}) = \rho K_n(\mu^*, \mathbf{x}, \mathbf{x}).$$

So

$$\int_{\mathcal{K}} \left( K_n(\mu, \mathbf{x}, \mathbf{t}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{t}) \right)^2 d\mu(\mathbf{t}) \le K_n(\mu, \mathbf{x}, \mathbf{x}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{x}). \tag{5.2}$$

Next, for any polynomial $P \in \Pi_n^d$, we have the Christoffel function estimate

$$|P(\mathbf{y})| \le K_n(\mu, \mathbf{y}, \mathbf{y})^{1/2} \left( \int_{\mathcal{K}} P^2 d\mu \right)^{1/2}.$$

Applying this to $P(\mathbf{t}) = K_n(\mu, \mathbf{x}, \mathbf{t}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{t})$ and using (5.2) gives, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\left| K_n(\mu, \mathbf{x}, \mathbf{y}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{y}) \right| \le K_n(\mu, \mathbf{y}, \mathbf{y})^{1/2} \left[ K_n(\mu, \mathbf{x}, \mathbf{x}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{x}) \right]^{1/2}. \quad \blacksquare$$

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Next, we establish an elementary bound on Christoffel functions.

***Lemma 5.2*** *Let $\mu$ be a measure with compact support $\mathcal{K}$, and let $B(\mathbf{x}_0, \delta)$ be a ball inside that support. Assume that $\mu$ is absolutely continuous in $B(\mathbf{x}_0, \delta)$, satisfying a.e. there*

$$0 < C_1 \leq \mu' \leq C_2 < \infty.$$

*Then, given $0 < \eta < \delta$, there exist $C_3, C_4 > 0$, such that for $n \geq 1$,*

$$C_3 \leq \binom{n+d}{d} \lambda_n(\mu, \mathbf{x}) \leq C_4 \text{ in } B(\mathbf{x}_0, \eta). \tag{5.3}$$

**Proof** The lower bound follows from Theorem 1.3 and monotonicity of Christoffel functions in the measure. Indeed, let $\nu_1' = 1$ in $B(\mathbf{x}_0, \delta)$ and 0 elsewhere. By a scaled form of Theorem 1.3,

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\nu_1, \mathbf{x}) = C\sqrt{1 - \Big(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{\delta}\Big)^2}, \quad \mathbf{x} \in B(\mathbf{x}_0, \delta),$$

where $C$ depends only on $d$ and $\delta$. The convergence is uniform in compact subsets of $B(\mathbf{x}_0, \delta)$. Since $\lambda_n(\mu, \mathbf{x}) \geq C_1 \lambda_n(\nu_1, \mathbf{x})$, the lower bound in (5.3) follows. Next, choose $r > 0$ so large that $B(\mathbf{x}_0, r)$ contains $\mathcal{K}$, and let $d\nu_2 = d\mu_{|\mathcal{K}\setminus B(\mathbf{x}_0,\delta)} + C_2 d\mathbf{x}_{|B(\mathbf{x}_0,r)}$. Thus $d\nu_2$ is the sum of $\mu$ restricted to $\mathcal{K}\setminus B(\mathbf{x}_0, \delta)$ and a multiple of the Legendre weight for the ball $B(\mathbf{x}_0, r)$. Since $d\nu_2 \geq C_2 d\mathbf{x}_{|B(\mathbf{x}_0,r)}$, and the latter is regular on $B(\mathbf{x}_0, r)$, it is easily seen that $\nu_2$ is also regular. Moreover, $\nu_2'$ is positive and continuous on $B(\mathbf{x}_0, \delta)$, so Theorem 1.3 gives uniformly on $B(\mathbf{x}_0, \eta)$,

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\nu_2, \mathbf{x}) = C\sqrt{1 - \Big(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{r}\Big)^2},$$

for some $C$ depending on $r$. But also $\mu \leq \nu_2$, so $\lambda_n(\mu, \cdot) \leq \lambda_n(\nu_2, \cdot)$, and the upper bound in (5.3) follows. ■

**Proof of Theorem 1.5** Let $\mathbf{x}_0 \in D$, $\varepsilon \in (0, 1)$ and choose $\delta > 0$ such that $h = \frac{d\nu}{d\mu}$ (which is positive and continuous on $D$) satisfies

$$1 - \varepsilon \leq h(\mathbf{y})/h(\mathbf{x}) \leq 1 + \varepsilon \text{ for } \mathbf{x}, \mathbf{y} \in B(\mathbf{x}_0, \delta). \tag{5.4}$$

Set $\rho = h(\mathbf{x}_0)(1 + \varepsilon)$. We shall apply Lemma 5.1 twice. Define a measure $\mu^*$ by $d\mu^* = d\mu$ in $B(\mathbf{x}_0, \delta)$, and

$$d\mu^* = \max\Big\{1, \frac{1}{\rho}\Big\}(d\mu + d\nu) \text{ in } \mathcal{K}\setminus B(\mathbf{x}_0, \delta).$$

**Step 1: $\mu$ and $\mu^*$**
Since $\mu^* \geq \mu$, we have the inequality (5.1). Moreover, since $\mu$ is regular and $\mu^* \geq \mu$, $\mu^*$ is also regular. Next, from Theorem 1.1,

$$\lim_{n\to\infty} \frac{K_n(\mu^*, \mathbf{x}, \mathbf{x})}{K_n(\mu, \mathbf{x}, \mathbf{x})} = 1$$

uniformly in $B(\mathbf{x}_0,\eta)$, for any $\eta<\delta$, while from Lemma 5.2, for $\mathbf{x},\mathbf{y}\in B(\mathbf{x}_0,\eta)$,

$$\frac{K_n(\mu,\mathbf{y},\mathbf{y})}{K_n(\mu,\mathbf{x},\mathbf{x})}\le C.$$

It is only here that we need the extra hypothesis in Theorem 1.5 that $\mu'$ is bounded above and below by positive constants on $D_1$. Then Lemma 5.1, with $\rho=1$ there, shows that for $\mathbf{x}\in B(\mathbf{x}_0,\eta)$, and uniformly for $\mathbf{u},\mathbf{v}$ in compact subsets of $\mathbb{R}^n$,

$$\lim_{n\to\infty}\frac{K_n(\mu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n})-K_n(\mu^*,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n})}{K_n(\mu,\mathbf{x},\mathbf{x})}=0. \tag{5.5}$$

**Step 2: $\nu$ and $\mu^*$**
Now $d\nu\le\rho\,d\mu^*$ in $\mathcal{K}\backslash B(\mathbf{x}_0,\delta)$. Also, in $B(\mathbf{x}_0,\delta)$,

$$d\nu=\frac{d\nu}{d\mu}d\mu\le\rho\,d\mu=\rho\,d\mu^*.$$

So in $\mathcal{K}$, $d\nu\le\rho\,d\mu^*$. By Theorem 1.1 and (5.4),

$$\lim_{n\to\infty}\frac{K_n(\mu^*,\mathbf{x},\mathbf{x})}{\rho K_n(\nu,\mathbf{x},\mathbf{x})}=\frac{1}{\rho}\frac{d\nu}{d\mu^*}(\mathbf{x})=\frac{1}{\rho}\frac{d\nu}{d\mu}(\mathbf{x})\ge\frac{1-\varepsilon}{1+\varepsilon},$$

uniformly in $B(\mathbf{x}_0,\eta)$, for any $\eta<\delta$. Note that (1.4) holds for $\nu$ and $\mathbf{x}$ in $B(x_0,\delta)$; indeed, this follows from the ratio limit in Theorem 1.1. Furthermore, from Lemma 5.2, for $\mathbf{x},\mathbf{y}\in B(\mathbf{x}_0,\eta)$,

$$\frac{K_n(\nu,\mathbf{y},\mathbf{y})}{K_n(\nu,\mathbf{x},\mathbf{x})}\le C.$$

Here again we need the extra hypothesis in Theorem 1.5 that $\nu'$ is bounded above and below by positive constants on $D_1$. Then Lemma 5.1 gives

$$\begin{aligned}\frac{|K_n(\nu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n})-\frac{1}{\rho}K_n(\mu^*,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n})|}{K_n(\nu,\mathbf{x},\mathbf{x})}&\le C\Big[1-\frac{K_n(\mu^*,\mathbf{x},\mathbf{x})}{\rho K_n(\nu,\mathbf{x},\mathbf{x})}\Big]^{1/2}\\&\le C_1(2\varepsilon)^{1/2}\end{aligned}$$

for $n\ge n_0$, $\mathbf{x}\in B(\mathbf{x}_0,\eta)$, and $\mathbf{u},\mathbf{v}$ in compact subsets of $\mathbb{R}^d$. Since $C$ and $C_1$ are independent of $\mathbf{u},\mathbf{v},\mathbf{x},n$, we obtain from this, equation (5.5), and the bound on the Christoffel functions in Lemma 5.2,

$$\frac{\Big|\rho K_n\Big(\nu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n}\Big)-K_n\Big(\mu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n}\Big)\Big|}{K_n(\mu,\mathbf{x},\mathbf{x})}\le C_1\varepsilon^{1/2},$$

and hence for large enough $n$, and uniformly for $\mathbf{x}\in B(\mathbf{x}_0,\eta)$, and $\mathbf{u},\mathbf{v}$ in compact subsets of $\mathbb{R}^n$,

$$\frac{|\frac{d\nu}{d\mu}(\mathbf{x}_0)K_n(\nu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n})-K_n(\mu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n})|}{K_n(\mu,\mathbf{x},\mathbf{x})}\le C_1\varepsilon^{1/2},$$

where $C_1$ is independent of $\mathbf{u},\mathbf{v},\mathbf{x},n$. Then using (5.4) again, we obtain (1.7). The uniformity in $B(\mathbf{x}_0,\eta)$ was also established above. As $D$ is compact, uniformity in $D$ follows. ■

**Proof of Theorem 1.6** This follows directly from Theorems 1.5 and 3.1. ■

## 6 Proofs of Theorems 1.7 and 1.8

We need only compute the universality limit for the Chebyshev weight on the ball or simplex and then apply Theorem 1.5. We make essential use of known representations for the reproducing kernel. These involve the standard Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree $n$ that satisfies the orthogonality relation

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)x^j(1-x)^\alpha(1+x)^\beta dx = 0, \quad 0 \le j \le n-1,$$

and normalized by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

***Lemma 6.1*** (i) *Let*

$$c_n = \frac{\Gamma(\frac{d+2}{2})}{\Gamma(d+1)}\frac{\Gamma(n+d)}{\Gamma(n+\frac{d}{2})}.$$

*Then*

$$K_n(W_0^{ball},\mathbf{x},\mathbf{y}) = c_n\Big\{P_n^{(d/2,d/2-1)}(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}) + P_n^{(d/2,d/2-1)}(\mathbf{x}\cdot\mathbf{y}-\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2})\Big\}. \tag{6.1}$$

(ii) *Let*

$$d_n = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+1)}\frac{\Gamma(2n+d+1)}{\Gamma(2n+\frac{d}{2}+1)}.$$

*Then*

$$K_n(W_0^{simplex},\mathbf{x},\mathbf{y}) = \frac{d_n}{2^{2d+2}}\sum_{\varepsilon_i=\pm1} P_{2n}^{(d/2,d/2)}\Big(\sum_{j=1}^{d+1}\sqrt{x_j y_j}\varepsilon_j\Big), \tag{6.2}$$

*where*

$$x_{d+1} = 1-\sum_{j=1}^{d} x_j;\ y_{d+1} = 1-\sum_{j=1}^{d} y_j.$$

**Proof** (i) See [22, eqn. (3.8), Thm 3.3, p. 2449].

(ii) See [21, Theorem 2.3, p. 3032]. We have also taken account of Xu's convention of replacing integrals by sums in the confluent case "$\alpha_i = 0$" when the Jacobi parameters reduce to those of the Chebyshev weight. Note that Xu uses the ultraspherical polynomial $C_n^{(\lambda)}$, defined [16, eqn. (4.7.1), p. 80] by

$$C_n^{(\lambda)}(x) = d_{n,\lambda}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x),$$

where in our case $\lambda = \frac{d+1}{2}$, and

$$d_{n,\lambda} = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(2n + 2\lambda)}{\Gamma(2n + \lambda + \frac{1}{2})}. \qquad \blacksquare$$

Next, we turn to asymptotics of Jacobi polynomials.

***Lemma 6.2*** (i) *Let $\alpha > 0, \beta > -1$. Uniformly for $s$ in bounded subsets of $[0, \infty)$, we have*

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\Big(1 - \frac{s}{2n^2}\Big) = 2^\alpha J_\alpha^*(\sqrt{s}). \tag{6.3}$$

(ii) *Let $\alpha, \beta \geq -\frac{1}{2}$. Then*

$$\sup_{n\geq 1} n^{1/2} \sup_{x\in(-1,1)} |P_n^{(\alpha,\beta)}(x)|(1-x)^{\frac{\alpha+1/2}{2}}(1+x)^{\frac{\beta+1/2}{2}} \leq C < \infty. \tag{6.4}$$

**Proof** (i) This is Mehler–Heine's asymptotic formula [16, Thm. 8.1.1, p. 192].
(ii) See, for example, [11, Lemma 29, p. 170], and use the fact that if $\rho_n$ is the constant such that $\rho_n P_n^{(\alpha,b)}$ is an orthonormal polynomial, then $\rho_n$ grows like $Cn^{1/2}(1 + o(1))$ for some positive $C$. See, for example, [16, p. 68]. $\blacksquare$

Next, we compute asymptotics for the arguments in the kernel in (6.1) and (6.2):

***Lemma 6.3*** (i) *Assume that $\xi \in B$, and*

$$\mathbf{x} = \xi + \frac{1}{n}\mathbf{u} \quad \textit{and} \quad \mathbf{y} = \xi + \frac{1}{n}\mathbf{v}. \tag{6.5}$$

*Then*

$$\mathbf{x}\cdot\mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2} = 1 - \frac{G}{2n^2} + O\Big(\frac{1}{n^3}\Big), \tag{6.6}$$

*where $G = G(\xi, \mathbf{u}, \mathbf{v})$ is defined by (1.9).*
(ii) *Let $\xi$ lie in the interior of $\Sigma^d$, and let $\mathbf{x}, \mathbf{y}$ be given by (6.5). Then*

$$\sum_{j=1}^{d+1} \sqrt{x_j y_j} = 1 - \frac{H}{8n^2} + O\Big(\frac{1}{n^3}\Big), \tag{6.7}$$

*where $H = H(\xi, \mathbf{u}, \mathbf{v})$ is given by (1.11).*

**Proof** (i) We see that

$$\mathbf{x}\cdot\mathbf{y} = \|\xi\|^2 + \frac{1}{n}\xi\cdot(\mathbf{u}+\mathbf{v}) + \frac{1}{n^2}\mathbf{u}\cdot\mathbf{v}.$$

Also,

$$\sqrt{1-\|\mathbf{x}\|^2}=\sqrt{1-\|\xi\|^2-\frac{2}{n}\xi\cdot\mathbf{u}-\frac{1}{n^2}\|\mathbf{u}\|^2}.$$

A straightforward computation, using the Maclaurin series $\sqrt{1+t}=1+\frac{t}{2}-\frac{t^2}{8}+O(t^3)$, gives

$$\begin{aligned}&\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\\&=1-\|\xi\|^2-\frac{1}{n}\xi\cdot(\mathbf{u}+\mathbf{v})-\frac{1}{2n^2}\left(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2\right)\\&\quad+\frac{2(\xi\cdot\mathbf{u})(\xi\cdot\mathbf{v})}{n^2(1-\|\xi\|^2)}-\frac{(\xi\cdot(\mathbf{u}+\mathbf{v}))^2}{2n^2(1-\|\xi\|^2)}+O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{x}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{x}\|^2}\\&=1+\frac{1}{n^2}\mathbf{u}\cdot\mathbf{v}-\frac{1}{2n^2}\left(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2\right)-\frac{1}{2n^2(1-\|\xi\|^2)}\left(\xi\cdot(\mathbf{u}-\mathbf{v})\right)^2+O\Big(\frac{1}{n^3}\Big)\\&=1-\frac{G}{2n^2}+O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

(ii) For $1\le j\le d$,

$$\begin{aligned}(6.8)\qquad\sqrt{x_jy_j}&=\sqrt{\Big(\xi_j+\frac{u_j}{n}\Big)\Big(\xi_j+\frac{v_j}{n}\Big)}\\&=\xi_j+\frac{1}{2n}(u_j+v_j)+\frac{1}{2n^2\xi_j}u_jv_j-\frac{1}{8n^2\xi_j}(u_j+v_j)^2+O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Also,

$$\begin{aligned}(6.9)\qquad&\sqrt{x_{d+1}y_{d+1}}=\\&\sqrt{\Big(1-\sum_{j=1}^dx_j\Big)\Big(1-\sum_{j=1}^dy_j\Big)}\\&=1-\sum_{j=1}^d\xi_j-\frac{1}{2n}\sum_{j=1}^d(u_j+v_j)+\frac{1}{2n^2\big(1-\sum_{j=1}^d\xi_j\big)}\Big(\sum_{j=1}^du_j\Big)\Big(\sum_{j=1}^dv_j\Big)\\&\quad-\frac{1}{8n^2\big(1-\sum_{j=1}^d\xi_j\big)}\Big(\sum_{j=1}^d(u_j+v_j)\Big)^2+O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Combining (6.8) and (6.9) gives

$$\sum_{j=1}^{d+1} \sqrt{x_j y_j} =$$

$$1 + \frac{1}{2n^2} \sum_{j=1}^{d} \frac{u_j v_j}{\xi_j} - \frac{1}{8n^2} \sum_{j=1}^{d} \frac{(u_j + v_j)^2}{\xi_j}$$

$$+ \frac{1}{2n^2 \xi_{d+1}} \Big( \sum_{j=1}^{d} u_j \Big) \Big( \sum_{j=1}^{d} v_j \Big) - \frac{1}{8n^2 \xi_{d+1}} \Big( \sum_{j=1}^{d} (u_j + v_j) \Big)^2 + O\Big( \frac{1}{n^3} \Big)$$

$$= 1 - \frac{H}{8n^2} + O\Big( \frac{1}{n^3} \Big).$$

■

**Proof of Theorem 1.7** As remarked before, we need only establish the limit (1.8) for the Chebyshev weight $W_0^{\text{ball}}$ and then can apply Theorem 1.5. Let $\mathbf{x}, \mathbf{y}$ be given by (6.5). Now the dominant term in the right-hand side in (6.1) is the term with argument $\mathbf{x} \cdot \mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2} \sqrt{1 - \|\mathbf{y}\|^2}$. Since $\mathbf{x} \cdot \mathbf{y} - \sqrt{1 - \|\mathbf{x}\|^2} \sqrt{1 - \|\mathbf{y}\|^2}$ remains in a compact subset of $(-1, 1)$ as $n \to \infty$, the bound (6.4) shows that it is of essentially smaller size than the other term. We then have, using (6.6),

$$\begin{aligned} n^{-d/2} K_n(W_0^{\text{ball}}, \mathbf{x}, \mathbf{y}) &= c_n n^{-d/2} \Big\{ P_n^{(d/2, d/2-1)} \Big( \mathbf{x} \cdot \mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2} \sqrt{1 - \|\mathbf{y}\|^2} \Big) \\ &\quad + P_n^{(d/2, d/2-1)} \Big( \mathbf{x} \cdot \mathbf{y} - \sqrt{1 - \|\mathbf{x}\|^2} \sqrt{1 - \|\mathbf{y}\|^2} \Big) \Big\} \\ &= c_n \Big\{ n^{-d/2} P_n^{(d/2, d/2-1)} \Big( 1 - \frac{G(\xi, \mathbf{u}, \mathbf{v})}{2n^2} + O\Big( \frac{1}{n^3} \Big) \Big) + o(1) \Big\} \\ &= c_n \Big\{ 2^{d/2} J_{d/2}^{*} \Big( \sqrt{G(\xi, \mathbf{u}, \mathbf{v})} \Big) + o(1) \Big\}, \end{aligned}$$

by (6.3). Using this with $\mathbf{u} = \mathbf{v} = \mathbf{0}$ gives

$$n^{-d/2} K_n(W_0^{\text{ball}}, \xi, \xi) = c_n \big\{ 2^{d/2} J_{d/2}^{*}(0) + o(1) \big\}.$$

Then (1.8) follows on changing $\xi$ to $\mathbf{x}$. ■

**Proof of Theorem 1.8** The dominant terms in the right-hand side in (6.2) are the two terms with all $\varepsilon_j = 1$ or with all $\varepsilon_j = -1$. For all other choices of $\{\varepsilon_j\}$, the argument $\sum_{j=1}^{d+1} \sqrt{x_j y_j} \varepsilon_j$ remains in a compact subset of $(-1, 1)$ as $n \to \infty$, so the bound (6.4) shows that the corresponding terms in (6.2) are of essentially smaller size

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than the above two terms. Because of the evenness of $P_{2n}^{(d/2,d/2)}$, we have, using (6.7),

$$\begin{aligned}
&(2n)^{-d/2}K_n(W_0^{\text{simplex}},\mathbf{x},\mathbf{y})\\
&=\frac{d_n}{2^{2d+1}}(2n)^{-d/2}P_{2n}^{(d/2,d/2)}\Big(\sum_{j=1}^{d+1}\sqrt{x_jy_j}\Big)\big(1+o(1)\big)\\
&=\frac{d_n}{2^{2d+1}}(2n)^{-d/2}P_{2n}^{(d/2,d/2)}\Big(1-\frac{H(\xi,\mathbf{u},\mathbf{v})}{2(2n)^2}+O\Big(\frac{1}{n^3}\Big)\Big)\big(1+o(1)\big)\\
&=\frac{d_n}{2^{2d+1}}2^{d/2}J_{d/2}^{*}\big(\sqrt{H(\xi,\mathbf{u},\mathbf{v})}\big)\big(1+o(1)\big),
\end{aligned}$$

by (6.3). Using this with $\mathbf{u}=\mathbf{v}=\mathbf{0}$ gives

$$(2n)^{-d/2}K_n\big(W_0^{\text{simplex}},\mathbf{x},\mathbf{y}\big)=\frac{d_n}{2^{2d+1}}2^{d/2}J_{d/2}^{*}(0)\big(1+o(1)\big).$$

Then (1.10) for the Chebyshev weight $W_0^{\text{simplex}}$ follows. Now apply Theorem 1.5. ■

## 7 An Extension of Theorem 1.1

We can weaken the continuity of $\frac{d\nu}{d\mu}$ in Theorem 1.1 to a Lebesgue point condition. Recall that $\mathbf{x}_0$ is a Lebesgue point of a function $h$ of $d$ variables if

$$\lim_{r\to 0+}\frac{\int_{B(\mathbf{x}_0,r)}|h(\mathbf{x}_0)-h(\mathbf{t})|d\mathbf{t}}{m(B(\mathbf{x}_0,r))}=0,$$

where $m$ denotes Lebesgue measure on $\mathbb{R}^d$.

***Theorem 7.1*** *Let $\mu,\nu$ be positive measures, whose support is a compact set $\mathcal{K}\subset\mathbb{R}^d$, and both are regular. Let $\mathbf{x}_0\in\mathcal{K}$ and assume that $\mu,\nu$ are absolutely continuous in some ball $B(\mathbf{x}_0,\delta)\subset\mathcal{K}$, while $\mu',\nu'$ are bounded above and below a.e. by positive constants there, and (1.4) holds uniformly in $B(\mathbf{x}_0,\delta)\subset\mathcal{K}$. Assume that $\mathbf{x}_0$ is a Lebesgue point of $\frac{\nu'}{\mu'}$. Then, given $r>0$, we have, uniformly for $\mathbf{y}\in B(\mathbf{x}_0,\frac{r}{n})$,*

$$\lim_{n\to\infty}\frac{\lambda_n(\nu,\mathbf{y})}{\lambda_n(\mu,\mathbf{y})}=\frac{d\nu}{d\mu}(\mathbf{x}_0).$$

**Proof** We may assume that $\mathcal{K}\subset B=B(\mathbf{0},1)$. Fix $r>0$ and let $\mathbf{y}\in B(\mathbf{x}_0,\frac{r}{n})$. Let $\tau\geq 2r$ and $\varepsilon\in(0,1)$. Let $\ell_n=[\frac{\varepsilon n}{2}]$ and $m=m(n)=n-2\ell_n$. Choose $p_m\in\Pi_m^d$ that is extremal for $\lambda_m(\mu,\mathbf{y})$, so that

$$\lambda_m(\mu,\mathbf{y})=\int p_m^2d\mu \text{ and } p_m(\mathbf{y})=1.$$

Choose $q_{\ell_n}^{(k)}$, $k=1,2$, as in Lemma 2.1, with the properties $q_{\ell_n}^{(k)}(\mathbf{y})=1$, $0\leq q_{\ell_n}^{(k)}\leq 1$ in $B$, and

$$\big|q_{\ell_n}^{(1)}(\mathbf{x})\big|\leq e^{-c\ell_n\frac{\tau}{n}}\leq e^{-C\varepsilon\tau},\ \mathbf{x}\in B\backslash B\Big(\mathbf{y},\frac{\tau}{n}\Big), \tag{7.1}$$

$$\big|q_{\ell_n}^{(2)}(\mathbf{x})\big|\leq e^{-c\ell_n\delta}\leq e^{-C\varepsilon n},\ \mathbf{x}\in B\backslash B(\mathbf{y},\delta). \tag{7.2}$$

As above, let $h = \frac{d\nu}{d\mu}$ and $S_n = p_m q^{(1)}_{\ell_n} q^{(2)}_{\ell_n} \in \Pi^d_n$. We have $S_n(\mathbf{y}) = 1$, so

$$\begin{aligned}
&\lambda_n(\nu,\mathbf{y})\\
&\quad\le \int_{\mathcal{K}} S_n^2 d\nu\\
&\quad\le h(\mathbf{x}_0)\int_{B(\mathbf{y},\frac{\tau}{n})} p_m^2 d\mu + \|p_m\|^2_{L_\infty(B(\mathbf{y},\frac{\tau}{n}))}\int_{B(\mathbf{y},\frac{\tau}{n})} |h(\mathbf{x}_0)-h(t)|d\mu(t)\\
&\qquad + e^{-C\varepsilon\tau}\int_{B(\mathbf{x}_0,2\delta)\setminus B(\mathbf{y},\frac{\tau}{n})} p_m^2 h\, d\mu + \|p_m\|^2_{L_\infty(\mathcal{K})} e^{-C\varepsilon n}\int_{\mathcal{K}\setminus B(\mathbf{x}_0,2\delta)} d\nu\\
&\quad =: T_1+T_2+T_3+T_4,
\end{aligned}$$

by (7.1), (7.2). First, $T_1 \le h(\mathbf{x}_0)\lambda_m(\mu,\mathbf{y})$. Next, Lemma 5.2 gives

$$\begin{aligned}
\|p_m\|^2_{L_\infty(B(\mathbf{y},\frac{\tau}{n}))} &\le \|\lambda_m^{-1}(\mu,\cdot)\|_{L_\infty(B(\mathbf{y},\frac{\tau}{n}))}\int p_m^2 d\mu\\
&\le Cn^d\lambda_m(\mu,\mathbf{y}),
\end{aligned}$$

so as $\tau > r$ and $\|\mathbf{y}-\mathbf{x}_0\| < \frac{r}{n}$,

$$T_2 \le Cn^d\lambda_m(\mu,\mathbf{y})\|\mu'\|_{L_\infty(B(\mathbf{x}_0,\delta))}\int_{B(\mathbf{x}_0,\frac{2\tau}{n})} |h(x_0)-h(t)|d\mathbf{t} = o(\lambda_m(\mu,\mathbf{y})),$$

as $\mathbf{x}_0$ is a Lebesgue point of $h$. Next, by (7.1),

$$T_3 \le e^{-C\varepsilon\tau}\|h\|_{L_\infty(B(\mathbf{x}_0,\delta))}\int_{B(\mathbf{x}_0,\delta)\setminus B(\mathbf{y},\frac{\tau}{n})} p_m^2\, d\mu \le C_1 e^{-C\varepsilon\tau}\lambda_m(\mu,\mathbf{y}).$$

Finally, using the regularity of $\mu$,

$$T_4 \le (1+o(1))^n\left(\int p_m^2\, d\mu\right)e^{-C\varepsilon n}\int_{\mathcal{K}\setminus B(\mathbf{x}_0,\delta_n)} d\nu = o(\lambda_m(\mu,\mathbf{y})).$$

Combining all the above estimates gives

$$\frac{\lambda_n(\nu,\mathbf{y})}{\lambda_n(\mu,\mathbf{y})} \le \frac{\lambda_m(\mu,\mathbf{y})}{\lambda_n(\mu,\mathbf{y})}\left(h(\mathbf{x}_0)+C_1e^{-C\varepsilon\tau}+o(1)\right).$$

Here $\tau$ is independent of $\varepsilon$ and may be chosen as large as we please. We deduce that

$$\limsup_{n\to\infty}\frac{\lambda_n(\nu,\mathbf{y})}{\lambda_n(\mu,\mathbf{y})} \le h(\mathbf{x}_0)\limsup_{n\to\infty}\frac{\lambda_{n-2[\frac{\varepsilon n}{2}]}(\mu,\mathbf{y})}{\lambda_n(\mu,\mathbf{y})}.$$

The proof may now be completed as in Theorem 1.1. ∎

We note that the absolute continuity of $\mu$ and boundedness below of $\mu'$, were needed only for the upper bound for $\|\lambda_m^{-1}(\mu,\cdot)\|_{L_\infty(B(\mathbf{y},\frac{\tau}{n}))}$. If we assumed a suitable bound for this, we could allow $\mu$ to have a singular part.

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