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THE WEAK WEAK CATEGORY OF A SPACE

BY

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1. Let X be a topological space. We say that cat $X \leq n$ if there exists a map $\phi: X \to T_1(X, \ldots, X)$ such that $j\phi \simeq \Delta: X \to X^{n+1}$, where $T_1(X, \ldots, X)$ is the "fat wedge", j is the inclusion and Δ is the diagonal map. This is an example of a right structure system. This right structure system leads to an associated weak structure system, namely weak category in this particular case. We recall that w cat $X \leq n$ if $q\Delta \simeq *$, where $q: X^{n+1} \to \bigwedge_{i=1}^{n+1} X$ is the projection, $\bigwedge_{j=1}^{n+1} X$ being the smashed product of $(n+1)$ copies of X. This weak category is again a right structure system. We can iterate this process and thus consider statements of the form w^m cat $X \leq n$ where $m \geq 1$. In [3], Peterson made the following statement: "The author doubts that w^m cat $\leq n$ is interesting if $m > 1$." The purpose of this note is to show that in fact w cat $X \leq n$ if and only if w^m cat $X \leq n$ for all $m \geq 1$. We shall show this more generally for all right structure systems.

2. We shall follow the terminology and notation of [1] and [2]. For convenience, we recall some definitions here. We work in the category $\mathscr{T}$ of spaces with base point and having the homotopy type of countable CW complexes. All maps and homotopies shall respect base points. The maps of our category $\mathscr{T}$ shall be homotopy classes of maps, but for simplicity we shall use the same symbol for a map and its homotopy class.

Let $\mathscr{C}$ be a category. By a right structure system over $\mathscr{C}$ we mean $\mathscr{R} = (R, P, T; d, j)$ where $R, P, T: \mathscr{C} \to \mathscr{T}$ are covariant functors and $d: R \to P$, $j: T \to P$ are natural transformations. If $X \in \mathscr{C}$, we say that X is $\mathscr{R}$-structured if there is a map $\phi: RX \to TX$ such that $j(X)\phi \simeq d(X)$. We may assume that j is a natural cofibration. Let $q: P \to Q$ be the cofibre of j, and let $j_w: T_w \to P$ be the fibre of q. Then $\mathscr{R}_w = (R, P, T_w; d, j_w)$ is a right structure system over $\mathscr{C}$, and is called the associated weak structure system. If $X \in \mathscr{C}$ is $\mathscr{R}_w$-structured, we shall say that it is weakly $\mathscr{R}$-structured.

EXAMPLE. The right structure system $\mathscr{K}_n = (1, \prod_{i=1}^{n+1}, T_1; \Delta, j)$ over $\mathscr{T}$ is the cat $\leq n$ structure system, where T_1 is the fat wedge, Δ is the diagonal and $j: T_1 \to \prod_{i=1}^{n+1}$ is the natural inclusion. The associated weak structure system in this case is the w cat $\leq n$ structure, that is the weak category structure system.

Given two right structure systems over $\mathscr{C}$, $\mathscr{R}_i = (R_i, P_i, T_i; d_i, j_i)$, $i = 1, 2$, a map $\mathscr{F}: \mathscr{R}_1 \to \mathscr{R}_2$ means a triple $\mathscr{F} = (\rho, \pi, \tau)$ of natural transformations $\rho: R_1 \to R_2$, $\pi: P_1 \to P_2$, $\tau: T_1 \to T_2$ such that $d_2\rho = \pi d_1$, $j_2\tau = \pi j_1$. If we have a map $(1, \pi, \tau)$:

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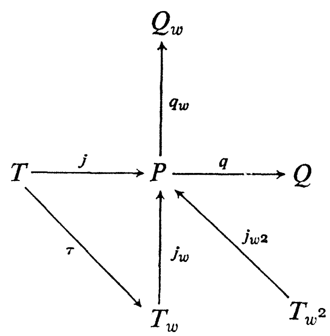
$\mathscr{R}_1 \to \mathscr{R}_2$, then if $X \in \mathscr{C}$ is $\mathscr{R}_1$-structured by $\phi: RX \to T_1X$, clearly $\tau(X)\phi: RX \to T_2X$ is an $\mathscr{R}_2$-structure on X. Thus if $\mathscr{R}=(R, P, T; d, j)$ is a right structure system over a category $\mathscr{C}$, and if $\mathscr{R}_w=(R, P, T_w; d, j_w)$ is its associated weak structure system, then we have a natural transformation $\tau: T \to T_w$ such that $j_w\tau = j$, and hence a map $(1, 1, \tau): \mathscr{R} \to \mathscr{R}_w$. This means that if $X \in \mathscr{C}$ may be $\mathscr{R}$-structured, it may also be weakly $\mathscr{R}$-structured.

3. We are now ready to consider our problem. Let $\mathscr{R}=(R, P, T; d, j)$ be a right structure system over a category $\mathscr{C}$. We have the associated weak structure system $\mathscr{R}_w=(R, P, T_w; d, j_w)$ and a map $(1, 1, \tau): \mathscr{R} \to \mathscr{R}_w$ where $j_w\tau = j$. We repeat this process and obtain the associated weak weak structure system $\mathscr{R}_{w2}=(R, P, T_{w2}; d, j_{w2})$ and also a map $(1, 1, \tau_1): \mathscr{R}_w \to \mathscr{R}_{w2}$ where $j_{w2}\tau_1 = j_w$. We shall construct a map $(1, 1, \mu): \mathscr{R}_{w2} \to \mathscr{R}_w$.

As above, let $q: P \to Q$ be the cofibre of j, j_w the fibre of q, $q_w: P \to Q_w$ the cofibre of j_w and j_{w2} the fibre of q_w. We have the following diagram:



We see that $q_wj = q_wj_w\tau \simeq *$. Since q is the cofibre of j, we can find a natural transformation $s: Q \to Q_w$ such that $sq = q_w$. Also since $qj_w \simeq *$ and q_w is the cofibre of j_w, we can find a natural transformation $t: Q_w \to Q$ such that $tq_w = q$. Hence $qj_{w2} = tq_wj_{w2} \simeq *$ since j_{w2} is the fibre of q_w. Finally, since j_w is the fibre of q, this means that there exists a natural transformation $\mu: T_{w2} \to T_w$ such that $j_w\mu = j_{w2}$. Hence we have a map $(1, 1, \mu): \mathscr{R}_{w2} \to \mathscr{R}_w$. This, together with the map $\mathscr{R}_w \to \mathscr{R}_{w2}$, gives us the following result.

THEOREM. *Let $\mathscr{R}=(R, P, T; d, j)$ be a right structure system over a category $\mathscr{C}$. Let $X \in \mathscr{C}$. Then X may be $\mathscr{R}_w$-structured if and only if it may be $\mathscr{R}_{w2}$-structured.*

COROLLARY. *If X is a topological space, then w cat $X \leq n$ if and only if w^m cat $X \leq n$ for all $m \geq 1$.*

REMARK. The corollary settles Peterson's remark. The above results may be dualized to left structure systems.

References

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