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# DIRECTED GRAPHS AND THE JACOBI-TRUDI IDENTITY

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**1. Introduction.** Let $|a_{ij}|_{n \times n}$ denote the $n \times n$ determinant with $(i, j)$-entry $a_{ij}$, and $h_k = h_k(x_1, \ldots, x_n)$ denote the $k^{th}$-*homogeneous symmetric function* of $x_1, \ldots, x_n$ defined by

$$h_k = \sum x_1^{m_1} \ldots x_n^{m_n}$$

where the summation is over all $m_1, \ldots, m_n \geqq 0$ such that $m_1 + \ldots + m_n = k$. We adopt the convention that $h_k = 0$ for $k < 0$. For integers $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n \geqq 0$, the *Jacobi-Trudi identity* (see **[6]**, **[7]** ) states that

$$|h_{\alpha_i - i + j}|_{n \times n} = |x_i^{\alpha_j + n - j}|_{n \times n} / |x_i^{n-j}|_{n \times n}.$$

In this paper we give a combinatorial proof of an equivalent identity, Theorem 1.1, obtained by moving the denominator on the RHS to the numerator on the LHS.

THEOREM 1.1. *For* $\alpha_1 \geqq \ldots \geqq \alpha_n \geqq 0$, *we have*

$$|x_i^{n-j}|_{n \times n} |h_{\alpha_i - i + j}|_{n \times n} = |x_i^{\alpha_j + n - j}|_{n \times n}.$$

Our proof is obtained by adopting the following strategy for proving an identity, say $f = g$. We define a set $\mathscr{D}$ of combinatorial objects and a weight function for the elements of $\mathscr{D}$, such that the generating function for $\mathscr{D}$ with respect to this weight is $f$ (with no cancellation). We find a subset $\mathscr{A}$ of $\mathscr{D}$ for which the generating function is $g$ (again, no cancellation of terms). We then find an involution on $\mathscr{D} - \mathscr{A}$ such that the weight of each element of $\mathscr{D} - \mathscr{A}$ is equal to the negative of the weight of its image under the involution (so we say the involution is *weight-reversing*). This immediately proves that the generating function for $\mathscr{D} - \mathscr{A}$ is equal to 0, and so the identity follows.

By considering tournaments as the combinatorial objects, this "involutionary" method has been applied by Gessel **[3]** to prove the Vandermonde determinant formula. Bressoud **[1]** has generalized this to prove the Weyl denominator formulae for the root systems $B_n$, $C_n$ and $D_n$. In this case the combinatorial objects are tournaments in which the edges are coloured and the vertices are "handicapped". An involution for $n$-tuples of

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lattice paths is used by Gessel and Viennot [**4**] (see also [**5**], Section 5.4) to prove that the LHS of the Jacobi-Trudi identity equals the generating function for column-strict plane partitions with shape $(\alpha_1, \ldots, \alpha_n)$. In the present paper this involutionary method is applied to a set of directed graphs.

All of the above applications of the involutionary method give elegant and insightful proofs of results for which algebraic proofs are known. It is worth mentioning that Zeilberger and Bressoud [**8**] (see also [**2**] for a generalization) have obtained an involutionary proof of the $q$-Dyson Theorem, for which no other proof is known.

In Section 2 of this paper we define a set of directed graphs $\mathscr{D}$, a weight function wt, and a subset $\mathscr{A}$ of $\mathscr{D}$ such that the generating functions for $\mathscr{D}$ and $\mathscr{A}$, respectively, are the LHS and RHS of Theorem 1.1. We also partition $\mathscr{D} - \mathscr{A}$ into two sets, $\mathscr{B}$ and $\mathscr{C}$. Section 3 gives weight-reversing involutions for $\mathscr{B}$ and $\mathscr{C}$ separately, finishing a proof of Theorem 1.1 by the involutionary method.

Finally, we define some terms. A *tournament* on $n$ vertices is a loopless directed graph with labelled vertices $1, \ldots, n$ in which every pair of vertices $(i, j)$ is joined by exactly one edge, either directed from $i$ to $j$, or from $j$ to $i$. A *transitive tournament* is a tournament with no directed cycles. For any transitive tournament there exists a permutation $\sigma \in S_n$ such that vertex $\sigma(i)$ has out-degree $n - i$, for $i = 1, \ldots, n$. We say that $\sigma$ is the *winner permutation* of the transitive tournament, and

$$\rho = \sigma(n)\sigma(n - 1) \ldots \sigma(1)$$

is the *loser permutation* (this is motivated by supposing that an edge directed from $i$ to $j$ means that person $i$ "beats" person $j$). We denote the sign of the permutation $\sigma$ by $\text{sgn}(\sigma)$.

**2. Directed graphs.** Consider directed loopless graphs on $2n$ labelled vertices, which consist of $n$ "black" vertices, labelled $1, \ldots, n$, and $n$ "white" vertices, labelled $1, \ldots, n$. For fixed $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n \geqq 0$, let $\mathscr{D}$ consist of all such graphs in which:

(i) the edges incident with black vertices only form a transitive tournament on $n$ vertices (called the black tournament).

(ii) the edges incident with white vertices only form a transitive tournament on $n$ vertices (called the white tournament).

(iii) the edges incident with both a black and a white vertex are all directed from the black vertex to the white vertex, with multiple edges allowed.

(iv) the only restriction is that in-degree (white vertex $i$) $= \alpha_i + n - i$, for $i = 1, \ldots, n$.

For $D \in \mathscr{D}$, let $a_i =$ out-degree (black vertex $i$), for $i = 1, \ldots, n$, let $\sigma$ be the winner permutation of $D$'s black tournament, and let $\rho$ be the loser permutation of $D$'s white tournament. Define a *weight* for $D$, denoted wt($D$), by

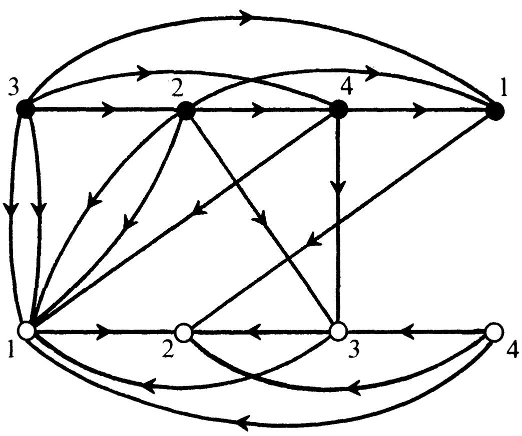
$$\mathrm{wt}(D) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\rho)\, x_1^{a_1} \dots x_n^{a_n},$$

and for any subset $\mathscr{S} \subseteq \mathscr{D}$ let the *generating function* for $\mathscr{S}$ with respect to this weight be denoted by $\Phi(\mathscr{S})$, so

$$\Phi(\mathscr{S}) = \sum_{D \in \mathscr{S}} \mathrm{wt}(D).$$

Finally, let $N(i, j)$ denote the number of edges directed from black vertex $i$ to white vertex $j$ in $D$, where $i, j = 1, \dots, n$.

For example, for $\alpha = (4, 2, 2, 0)$ the graph $E_1$ in Figure 1 is in $\mathscr{D}$, with $\sigma = 3241$, $\rho = 2134$, $\mathrm{sgn}(\sigma) = 1$, $\mathrm{sgn}(\rho) = -1$, $N(2, 1) = N(3, 1) = 2$, $N(1, 2) = N(2, 3) = N(4, 1) = N(4, 3) = 1$, and $\mathrm{wt}(E_1) = -x_1^1 x_2^5 x_3^5 x_4^3$.




$E_1$

Figure 1: An element of $\mathscr{D}$ for $\alpha = (4, 2, 2, 0)$.

Note that $E_1$ is drawn with the vertices arranged in two rows. The top row contains the black vertices, in order $\sigma(1), \dots, \sigma(n)$ from left to right and the bottom row contains the white vertices, in order $1, \dots, n$ from left to right. Moreover, white vertex $i$ is directly below black vertex $\sigma(i)$, for $i = 1, \dots, n$. We shall follow this convention when drawing elements of $\mathscr{D}$, since it allows a convenient geometrical description of the bijections in Section 3.

Condition (iv) on $\mathscr{D}$ implies that if $\alpha_j = 0$, then $\rho(j) = j$ and $N(\sigma(i), j) = 0$ for $i = 1, \dots, n$. This is illustrated by $E_1$, where $\alpha_4 = 0$.

The significance of the set $\mathscr{D}$ is revealed by the next result, which shows that the LHS of Theorem 1.1 is the generating function for $\mathscr{D}$.

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PROPOSITION 2.1. *For* $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n \geqq 0$, *we have*

$$\Phi(\mathscr{D}) = |x_i^{n-j}|_{n\times n} \cdot |h_{\alpha_i - i + j}|_{n\times n}.$$

*Proof.* Let $\mathscr{D}_{\sigma,\rho}$ consist of all elements of $\mathscr{D}$ whose black tournament has winner permutation $\sigma$ and whose white tournament has loser permutation $\rho$. Then clearly

$$\Phi(\mathscr{D}) = \sum_{\alpha\in S_n} \sum_{\rho \in S_n} \Phi(\mathscr{D}_{\sigma,\rho}).$$

Now from the definition of winner and loser permutation, for elements of $\mathscr{D}_{\sigma,\rho}$ we have

$$a_{\sigma(k)} = n - k + \sum_{l=1}^{n} N(\sigma(k), \rho(l)), \quad k = 1, \ldots, n,$$

and restriction (iv) for $\mathscr{D}$ yields

$$n - l + \sum_{k=1}^{n} N(\sigma(k), \rho(l)) = \alpha_{\rho(l)} + n - \rho(l),$$

or equivalently,

$$(*) \quad \sum_{k=1}^{n} N(\sigma(k), \rho(l)) = \alpha_{\rho(l)} - \rho(l) + l, \quad l = 1, \ldots, n.$$

The definition of $\Phi$ now gives

$$\Phi(\mathscr{D}_{\sigma,\rho}) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\rho) \sum \prod_{k=1}^{n} x_{\sigma(k)}{}^{n-k+\sum_{l=1}^{n} N(\sigma(k),\rho(l))},$$

where the summation is over $N(\sigma(k), \rho(l)) \geqq 0$ for $k, l = 1, \ldots, n$, subject to restriction (*). Thus

$$\Phi(\mathscr{D}_{\sigma,\rho}) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\rho) \prod_{k=1}^{n} x_{\sigma(k)}^{n-k} \prod_{l=1}^{n} \Lambda_l$$

where

$$\Lambda_l = \sum \prod_{k=1}^{n} x_{\sigma(k)}^{N(\sigma(k),\rho(l))},$$

in which the summation is over all $N(\sigma(k), \rho(l)) \geqq 0$ for $k = 1, \ldots, n$, with the restriction that

$$\sum_{k=1}^{n} N(\sigma(k), \rho(l)) = \alpha_{\rho(l)} - \rho(l) + l.$$

From the definition of the homogeneous symmetric functions of $n$ variables, it follows that

$$\Lambda_l = h_{\alpha_{\rho(l)} - \rho(l) + l},$$

so

$$\Phi(\mathscr{D}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} x_{\sigma(k)}^{n-k} \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_{l=1}^{n} h_{\alpha_{\rho(l)} - \rho(l) + l},$$

and the result follows immediately.

Now let $\mathscr{A}$ be the subset of $\mathscr{D}$ consisting of all graphs $A$ in $\mathscr{D}$ in which:

(i) the white tournament of $A$ has loser permutation $\rho = 12 \dots n$.

(ii) $N(\sigma(i), j) = 0$ for $i \neq j$, $i, j = 1, \dots, n$ where $\sigma$ is the winner permutation of $A$'s black tournament.

For example, if $\alpha = (4, 2, 2, 0)$ then the graph $E_2$ in Figure 2 is in $\mathscr{A}$, with $\sigma = 3241$, and

$$\operatorname{wt}(E_2) = x_1^0 x_2^4 x_3^7 x_4^3.$$

Note that

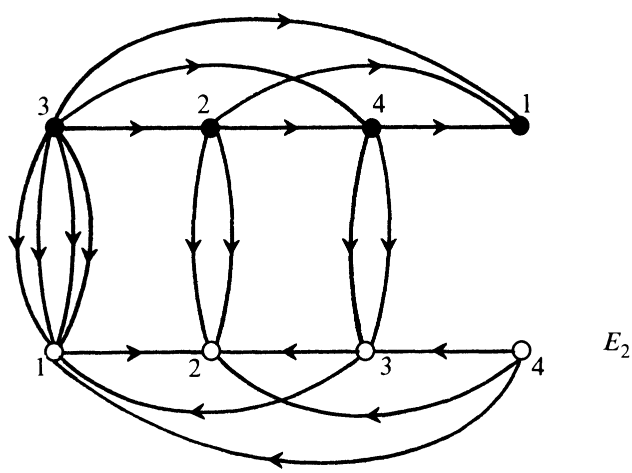


Figure 2: An element of $\mathscr{A}$ for $\alpha = (4, 2, 2, 0)$.

condition (i) forces all edges between white vertices to be directed from right to left, and condition (ii) forces any edge from a black vertex to a white vertex to lie in a single column.

The significance of the subset $\mathscr{A}$ of $\mathscr{D}$ is revealed by the next result, which shows that the RHS of Theorem 1.1 is the generating function for $\mathscr{A}$.

PROPOSITION 2.2. *For* $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n \geqq 0$, *we have*

$$\Phi(\mathscr{A}) = |x_i^{\alpha_j+n-j}|_{n\times n}.$$

*Proof.* Let $\mathscr{A}_\sigma$ consist of all elements of $\mathscr{A}$ whose black tournament has winner permutation $\sigma$. Then clearly

$$\Phi(\mathscr{A}) = \sum_{\sigma\in S_n} \Phi(\mathscr{A}_\sigma).$$

For elements of $\mathscr{A}_\sigma$ we have

$$a_{\sigma(k)} = n - k + N(\sigma(k), k)$$

for $k = 1, \ldots, n$ by conditions (i) and (ii) for $\mathscr{A}$. Also, restriction (iv) for $\mathscr{D}$ applied to elements of $\mathscr{A}_\sigma$ yields

$$\alpha_k + n - k = n - k + N(\sigma(k), k) \quad \text{for } k = 1, \ldots, n,$$

so

$$N(\sigma(k), k) = \alpha_k \quad \text{for } k = 1, \ldots, n.$$

Combining these results, we find that $\mathscr{A}_\sigma$ consists of a single graph, for which $a_{\sigma(k)} = \alpha_k + n - k$, $k = 1, \ldots, n$, so

$$\Phi(\mathscr{A}) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} x_{\sigma(k)}^{\alpha_k+n-k},$$

and the result follows immediately.

Let $\mathscr{B}$ be the subset of $\mathscr{D}$ consisting of those graphs in which $N(\sigma(i), j) > 0$ for some $n \geqq i > j \geqq 1$. Clearly $\mathscr{A}$ and $\mathscr{B}$ are disjoint, since the existence of such $i > j$ violates condition (ii) for $\mathscr{A}$.

Finally, we let $\mathscr{C} = \mathscr{D} - \mathscr{A} - \mathscr{B}$, so $\mathscr{B}$ and $\mathscr{C}$ partition $\mathscr{D} - \mathscr{A}$. In Section 3 we complete an involutionary proof of Theorem 1.1 by finding weight-reversing involutions for $\mathscr{B}$ and $\mathscr{C}$.

**3. Involutions and the Jacobi-Trudi identity.** First we consider a mapping $\psi$ for $\mathscr{B}$. For $D \in \mathscr{D}$ let

$$\mathscr{R}(D) = \{ (i, j) \,|\, N(\sigma(i), j) > 0, n \geqq i > j \geqq 1\}.$$

Now $\mathscr{B}$ consists precisely of those graphs $B$ for which $\mathscr{R}(B) \neq \emptyset$. For example, $E_1$ in Figure 1 is in $\mathscr{B}$, since

$$\mathscr{R}(E_1) = \{ (2, 1), (3, 1), (4, 2) \} \neq \emptyset.$$

Thus, for $B \in \mathscr{B}$, if

$$(s, t) = \operatorname*{Max}_i(\operatorname*{Min}_j\{\mathscr{R}(B)\})$$

then $(s, t)$ is well-defined. We obtain $\psi(B) = B'$ from $B$ as follows: reverse

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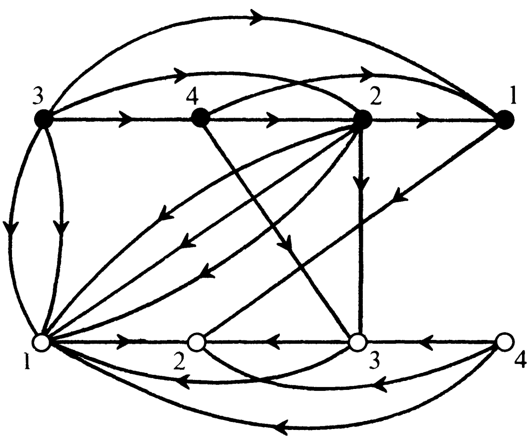
the edge $(\sigma(s - 1), \sigma(s))$ in the black tournament, and replace one of the $N(\sigma(s), t) > 0$ edges from black vertex $\sigma(s)$ to white vertex $t$ by an additional edge from black vertex $\sigma(s - 1)$ to white vertex $t$.

For example $\psi(E_1) = E_3$ and $\psi(E_3) = E_1$, where $E_1, E_3 \in \mathscr{B}$ are given in Figures 1 and 3. Note that

$$\mathrm{wt}(E_3) = x_1^1 x_2^5 x_3^5 x_4^3 = -\mathrm{wt}(E_1).$$




$E_3$

Figure 3: An image under $\psi$ for $\alpha = (4, 2, 2, 0)$.

We now prove that $\psi$ is a weight-reversing involution in general.

THEOREM 3.1. *The mapping* $\psi:\mathscr{B} \rightarrow \mathscr{B}:B \mapsto B'$ *is an involution with* $\mathrm{wt}(B') = -\mathrm{wt}(B)$, *so*

$$\Phi(\mathscr{B}) = 0.$$

*Proof.* Let the parameters of the graph $B'$ be denoted by a prime (e.g., the loser permutation of the white transitive tournament is denoted by $\rho'$ and the out-degree of black vertex $i$ by $a_i'$).

Since no edges in the white tournament are affected, we have $\rho' = \rho$. Since $\sigma(s - 1)$ and $\sigma(s)$ are consecutive in the winner permutation of the white transitive tournament, reversing the edge between them yields a transitive tournament, with $\sigma(s - 1)$ and $\sigma(s)$ trading places in the winner permutation, so

$$\sigma' = \sigma(1) \ldots \sigma(s - 2)\sigma(s)\sigma(s - 1)\sigma(s + 1) \ldots \sigma(n).$$

The out-degrees of black vertices $\sigma(1), \ldots, \sigma(s - 2), \sigma(s + 1), \ldots, \sigma(n)$ are unaffected by $\psi$, and the black vertices $\sigma(s - 1)$ and $\sigma(s)$ each has one out-directed edge replaced by another out-directed edge, so $a_i' = a_i$ for $i = 1, \ldots, n$. Similarly, the in-degrees of the white vertices are unchanged by $\psi$.

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Since the only edges between black and white vertices that are changed are those between black vertices $\sigma(s-1) = \sigma'(s)$, $\sigma(s) = \sigma'(s-1)$ and white vertex $t$, we obtain

$$(**)\quad \begin{array}{lll} N'(\sigma'(i), j) & = N(\sigma(i), j) & , i, j = 1, \ldots, n;\ i \neq s-1, s, \\ N'(\sigma'(s-1), j) & = N(\sigma(s), j) & , j = 1, \ldots, n;\ j \neq t, \\ N'(\sigma'(s), j) & = N(\sigma(s-1), j) & , j = 1, \ldots, n;\ j \neq t, \\ N'(\sigma'(s-1), t) & = N(\sigma(s), t), & \\ N'(\sigma'(s), t) & = N(\sigma(s-1), t) + 1. & \end{array}$$

Thus we have shown that $B' \in \mathscr{B}$, since $(s, t) \in \mathscr{R}(B')$ and conditions (i)-(iv) for $\mathscr{D}$ are satisfied. Also $\mathrm{wt}(B') = -\mathrm{wt}(B)$, since $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$. We complete the proof of this result by showing that $B'' = B$, where $B'' = \psi(B')$.

An inspection of (**) shows that

$$\{j \mid (i, j) \in \mathscr{R}(B) \text{ for some } i = 1, \ldots, n\}$$
$$= \{j \mid (i, j) \in \mathscr{R}(B') \text{ for some } i = 1, \ldots, n\},$$

so $t = t'$, since $t$ and $t'$ are the smallest elements of these two sets, respectively. Similarly

$$\{i \mid (i, t) \in \mathscr{R}(B)\} \cup \{s-1\} = \{i \mid (i, t) \in \mathscr{R}(B')\} \cup \{s-1\},$$

and $s' = s$.

Thus $(s', t') = (s, t)$, so $\sigma'' = \sigma$, $\rho'' = \rho$, and two applications of (**) yields

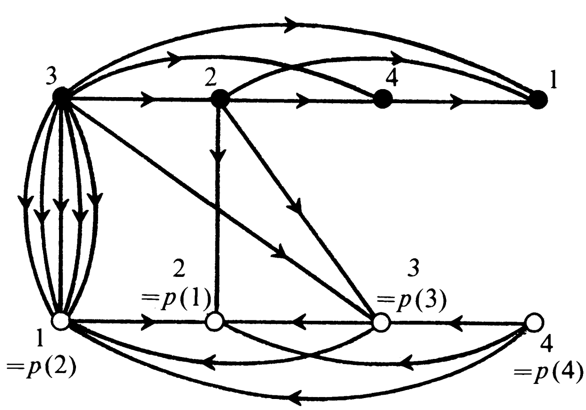
$$N''(\sigma''(i), j) = N(\sigma(i), j), \quad \text{for } i, j = 1, \ldots, n.$$

Thus $B'' = B$ and the result follows.

Now we consider a mapping $\xi$ for $C$. If $D \in \mathscr{D} - \mathscr{B}$, let

$$\mathscr{F}(D) = \{(i, j) \mid N(\sigma(i), \rho(j)) > 0,\ 1 \leqq i < j \leqq n\}.$$

For example, for $E_4$, $E_5 (\in \mathscr{C})$ in Figure 4, $\mathscr{F}(E_4) = \{(1, 2), (1, 3), (2, 3)\}$, $\mathscr{F}(E_5) = \{(1, 3)\}$, and for $E_2 (\in \mathscr{A})$ in Figure 2, $\mathscr{F}(E_2) = \emptyset$.




$E_4$

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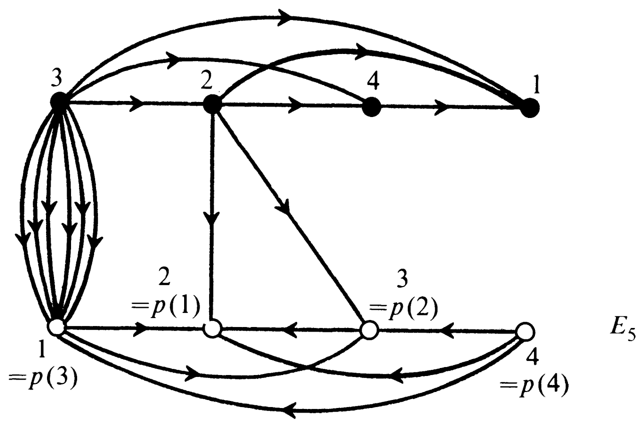


Figure 4: Two elements of $\mathscr{C}$ for $\alpha = (4, 2, 2, 0)$.

In fact, it is true in general that for $D \in \mathscr{D} - \mathscr{B}, \mathscr{F}(D) = \emptyset$ if and only if $D \in \mathscr{A}$. The "if" part of this statement is true since $D \in \mathscr{A}$ implies that $\rho(j) = j$ for all $j$, so $N(\sigma(i), \rho(j)) = 0$ for all $i \neq j$, and thus $i < j$.

This, and the fact that $D \notin \mathscr{B}$ also tells us that the "only if" part is true for $\rho = 12 \dots n$. To prove the "only if" part for other $\rho$, assume $\mathscr{F}(D) = \emptyset$ for $D \notin \mathscr{A}$, and let $k$ be the maximum $j$ such that $\rho(j) \neq j$. Then $k \geqq 2$, $\rho(k) < k$ and $N(\sigma(i), j) = 0$ for $i \neq j$ when $i > k$ or $j > k$, since $\mathscr{F}(D) = \emptyset$ implies

$$N(\sigma(i), \rho(j)) = 0 \quad \text{for } i < j,$$

and $D \notin \mathscr{B}$ implies

$$N(\sigma(i), j) = 0 \quad \text{for } i > j.$$

These combine to give

$$N(\sigma(i), \rho(k)) = 0 \quad \text{for all } i = 1, \dots, n.$$

But the in-degree restriction gives

$$\sum_{i=1}^{n} N(\sigma(i), \rho(k)) = \alpha_{\rho(k)} - \rho(k) + k > 0,$$

since $\alpha_{\rho(k)} \geqq 0$ and $k > \rho(k)$, so we have a contradiction, finishing the proof.

Thus for $C \in \mathscr{C}$, if

$$(u, v) = \operatorname*{Max}_j(\operatorname*{Min}_i\{\mathscr{F}(C)\},$$

then $(u, v)$ is well-defined. We obtain $\xi(C) = C'$ from $C$ as follows: reverse the edge $(\rho(v), \rho(v - 1))$ in the white tournament, and replace one of the $N(\sigma(u), \rho(v))$ edges from black vertex $\sigma(u)$ to white vertex $\rho(v)$ by an additional edge from black vertex $\sigma(u)$ to white vertex $\rho(v-1)$.

For example $\xi(E_4) = E_5$ and $\xi(E_5) = E_4$. Note that

$$\mathrm{wt}(E_4) = -x_1^0 x_2^4 x_3^9 x_4^1 = -\mathrm{wt}(E_5)$$

and $(u, v) = (1, 3)$ for both graphs, which holds in general by the next result, completing the involutionary proof of Theorem 1.1.

THEOREM 3.2. *The mapping* $\xi:\mathscr{C} \rightarrow \mathscr{C}:C \mapsto C'$ *is an involution with* $\mathrm{wt}(C') = -\mathrm{wt}(C)$, *so*

$$\Phi(\mathscr{C}) = 0.$$

*Proof.* We follow closely the proof of Theorem 3.1, since $\psi$ and $\xi$ are very similar, and omit details that are common.

The out-degrees of black vertices, in-degrees of white vertices, and the permutation $\sigma$, are all unaffected by $\xi$. The positions of $\rho(v-1)$ and $\rho(v)$ are interchanged in $\rho$ to create $\rho'$ with $\mathrm{sgn}(\rho') = -\mathrm{sgn}(\rho)$, so for all $C \in \mathscr{C}$, we have created $C' \in \mathscr{D}$ with $\mathrm{wt}(C') = -\mathrm{wt}(C)$. To prove that $\xi$ is an involution (so $C' \in \mathscr{C}$) it is sufficient to prove that $(u', v') = (u, v)$.

Now the minimality of $u$ implies

$$N(\sigma(i), \rho(j)) = 0 \quad \text{for } i < j, i < u.$$

We obtain $\rho(i) = i$ for $i = 1, \ldots, u-1$ and $N(\sigma(i), j) = 0$ for $i < u$ or $j < u$ by the same argument (reversed) that was used above to prove that $\mathscr{F}(D) = \emptyset$ implies $D \in \mathscr{A}$ for $D \in \mathscr{D} - \mathscr{B}$. Thus $\rho(v-1) \geqq u$, so $C' \notin \mathscr{B}$. Furthermore, $(u, v) \in \mathscr{F}(D')$ while $(i, j) \notin \mathscr{F}(D')$ for $i < u$, so $u' = u$ and $C' \notin \mathscr{A}$, giving $C' \in \mathscr{C}$. Finally, the maximality of $v$ implies that there are no edges between black vertex $\sigma(u)$ and white vertex $\rho(j) = \rho'(j)$ for $j > v$ in both $C$ and $C'$, so $v' = v$.

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