A GENERALIZATION OF THE PAPPUS CONFIGURATION

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1. Introduction. A configuration is a system of m points and n lines such that each point lies on μ of the lines and each line contains ν of the points. It is usually denoted by the symbol (m_μ, n_ν), with $m\mu = n\nu$. Two configurations corresponding to the same symbol are said to be equivalent if there exist 1–1 mappings of the points and lines of one onto the points and lines of the other which preserve the incidence relations. It is a combinatorial problem to determine whether a given set of integers m, n, μ, ν with $m\mu = n\nu$ corresponds to an abstract configuration, and a geometric problem to determine whether the configuration exists in a given geometry. For example, there are two inequivalent configurations corresponding to the symbol $(12_4, 16_3)$, both of which exist in the real projective plane. A configuration is said to be inscriptible in a plane cubic if there exists an equivalent configuration whose points lie on the cubic. For such a configuration $\nu = 3$.

A family of configurations K_n corresponding to the symbol $(3n_n, n^2{}_3)$ $(n = 1, 2, \ldots)$ will be studied in this paper. K_1 is a line containing three distinct points, K_2 is the complete quadrilateral, K_3 is the Pappus configuration, and K_4 is a configuration studied by Zacharias [5]. In section **2** it will be shown that K_n contains configurations of the type K_q if n is a multiple of q. In section **3** it will be shown that K_n is inscriptible in the plane cubic curve as a real configuration with two degrees of freedom, and consequently exists in the real projective plane. This generalizes a result proved by Feld [2] for the Pappus configuration.

2. The family of configurations K_n. Let A_i, B_i, and C_i $(i = 0, 1, \ldots, n-1)$ be called points, and let (ij) $(i, j = 0, 1, \ldots, n-1)$ be called lines, where (ij) represents the triple of points A_i, B_j, C_k subject to the condition

$$2.1 \qquad i + j + k \equiv 0 \pmod{n}.$$

K_n is defined abstractly as the system of $3n$ points A_i, B_i, C_i $(i = 0, 1, \ldots, n-1)$ and n^2 lines (ij) $(i, j = 0, 1, \ldots, n-1)$. It can easily be verified that each of the $3n$ points lie on n of the lines, and each of the n^2 lines contains 3 of the points, so that the configuration has the symbol $(3n_n, n^2{}_3)$. The $3n$ points of K_n are the vertices of 3 n-gons in perspective in pairs from the vertices of the third, the n^2 lines of K_n being the lines of perspectivity. K_n can also be visualized as a $2n$-gon $A_0B_0A_1B_1 \ldots A_{n-1}B_{n-1}$ with the lines A_iB_j passing through the point C_k $(i + j \equiv -k \bmod n;\ k = 0, 1, \ldots, n-1)$.

If n is not a prime number, the configuration K_n has non-trivial components

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which are configurations belonging to the same family. In the proof of the following theorem the matrix (a_{ij}) $(i = 1, 2; j = 1, 2, \ldots, f)$ will represent the f^2 lines $(a_{1r}a_{2s})$ $(r, s = 1, 2, \ldots, f)$.

THEOREM 2. 1. *If n is a multiple of q, K_n contains $(n/q)^2$ distinct configurations K_q no two of which have a line in common. Each line of K_n is a line of one of the K_q, and each point of K_n is a point of n/q of the K_q.*

Consider the r^2 matrices

$$K(ij) \equiv \begin{pmatrix} i & r+i & 2r+i & \ldots & (q-1)r+i \\ j & r+j & 2r+j & \ldots & (q-1)r+j \end{pmatrix} \quad (i, j = 0, 1, \ldots, r-1)$$

where $r = n/q$. The lines represented by $K(ij)$ are the lines of a K_q for all $i, j = 0, 1, \ldots, r-1$. To see this define

2.2 $\quad A_{kr+i} \equiv A^*_k, \quad B_{kr+j} \equiv B^*_k, \quad C_{kr-i-j} \equiv C^*_k \qquad (k = 0, 1, \ldots, q-1).$

The $3q$ points 2.2 are the only points on the lines represented by $K(ij)$. From the condition 2.1 for collinearity it follows that the points A^*_k, B^*_l, C^*_m will be collinear if and only if

2.3 $$r(k + l + m) \equiv 0 \pmod{n}.$$

Since $rq = n$, 2.3 holds if and only if

2.4 $$k + l + m \equiv 0 \pmod{q}.$$

The points 2.2 and the lines A^*_k, B^*_l, C^*_m subject to the condition 2.4 form a K_q by definition.

The r^2 configurations K_q represented by $K(ij)$ $(i, j = 0, 1, \ldots, n-1)$ are all distinct. By a consideration of the matrices $K(ij)$ it is seen that no two have a line in common. Furthermore, any point of K_n occurs in exactly r of the K_q. The $q^2r^2 = n^2$ lines of the r^2 K_q make up all the lines of K_n.

COROLLARY 1. *The 16 lines of K_4 can be divided into four sets of four lines which form complete quadrilaterals.*

This result was obtained by Zacharias [5].

COROLLARY 2. *K_{3q} contains q^2 distinct Pappus configurations.*

3. The inscription of K_n in the non-singular plane cubic curve. Any real non-singular cubic $\mathfrak{C}$ may be transformed into the Weierstrass canonical form by a suitable choice of the triangle of reference. Then the co-ordinates of any point on $\mathfrak{C}$ can be expressed parametrically in the form $(\wp u, \wp' u, 1)$ where $\wp u$ is the Weierstrass elliptic function. The point having the parameter u will be denoted by u. The necessary and sufficient condition that the points u, v, w be collinear is that

3.1 $$u + v + w \equiv 0 \pmod{2\omega, 2\omega'}$$

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where 2ω and $2\omega'$ are the periods of $\wp u$. The real plane cubics fall into two classes, unipartite and bipartite, depending upon whether they have one or two real circuits. For the bipartite cubic 2ω and $2\omega'/i$ are real and positive, while for the unipartite cubics 2ω and $2\omega'$ are conjugate complex. The points on the even branch of the bipartite cubic are given by values of the parameter of the form $u + \omega'$ where u is real. Points on the odd branch of either type are given by real values of the parameter.

The conditions that the $3n$ points

3.2 $$A_i,\ B_i,\ C_i \qquad (i = 0, 1, \ldots, n-1)$$

of $\mathfrak{C}$ should be points of a K_n are

3.3 $$A_i + B_j + C_k \equiv 0 \qquad (\text{mod } 2\omega, 2\omega')$$

with

3.4 $$i + j + k \equiv 0 \qquad (\text{mod } n).$$

Sum those equations of 3.3 having A_i in common:

$$\sum_{j,\,k=1}^{n-1} (A_i + B_j + C_k) \equiv 0 \qquad (\text{mod } 2\omega, 2\omega')$$

so that

$$n\,A_i \equiv -\sum_{j=0}^{n-1} (B_j + C_j) \qquad (\text{mod } 2\omega, 2\omega')$$

for $i = 0, 1, \ldots, n-1$. Thus

3.5 $$n\,A_0 \equiv n\,A_1 \equiv \ldots \equiv n\,A_{n-1} \qquad (\text{mod } 2\omega, 2\omega').$$

Similarly

3.6 $$nB_0 \equiv nB_1 \equiv \ldots \equiv nB_{n-1} \qquad (\text{mod } 2\omega, 2\omega'),$$

3.7 $$nC_0 \equiv nC_1 \equiv \ldots \equiv nC_{n-1} \qquad (\text{mod } 2\omega, 2\omega').$$

The equation $nu \equiv v$ (mod 2ω, $2\omega'$) has n^2 distinct solutions

3.8 $$u \equiv v/n + 2(r\omega + s\omega')/n \quad (\text{mod } 2\omega, 2\omega') \quad (r, s = 0, 1, \ldots, n-1).$$

If $\mathfrak{C}$ is unipartite and v real, u will be real if and only if $r = s$. This leaves n distinct real solutions

$$u \equiv v/n + 2r(\omega + \omega')/n \quad (\text{mod } 2\omega, 2\omega_0) \qquad (r = 0, 1, \ldots, n-1).$$

Thus, since the points A_i $(i = 0, 1, \ldots, n-1)$ are all distinct and since 3.5 holds we may take

3.9 $$A_i \equiv A + 2i(\omega + \omega')/n \quad (\text{mod } 2\omega, 2\omega') \qquad (i \equiv 0, 1, \ldots, n-1),$$

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with A real. Similarly we may take

3.10 $B_i \equiv B + 2i(\omega + \omega')/n \pmod{2\omega, 2\omega'} \qquad (i = 0, 1, \ldots, n-1),$

3.11 $C_i \equiv C + 2i(\omega + \omega')/n \pmod{2\omega, 2\omega'} \qquad (i = 0, 1, \ldots, n-1),$

with B, C real. The condition 3.3 will be satisfied by the points 3.2 if and only if

3.12 $$A + B + C \equiv 0 \pmod{2\omega, 2\omega'}.$$

The configuration will degenerate if any two of the sets of points 3.9, 3.10. 3.11 are the same. Thus nA, nB and nC must be different modulo 2ω, $2\omega'$.

If $\mathfrak{C}$ is bipartite and v real, u will be real if and only if $s = 0$. This leaves n distinct real solutions

3.13 $u \equiv v/n + 2r\omega/n \pmod{2\omega, 2\omega'} \qquad (r = 0, 1, \ldots, n-1).$

Thus we may take

3.14 $A_i \equiv A + 2i\omega/n \pmod{2\omega, 2\omega'} \qquad (i = 0, 1, \ldots, n-1)$

3.15 $B_i \equiv B + 2i\omega/n \pmod{2\omega, 2\omega'} \qquad (i = 0, 1, \ldots, n-1)$

3.16 $C_i \equiv C + 2i\omega/n \pmod{2\omega, 2\omega'} \qquad (i = 0, 1, \ldots, n-1)$

with A, B, and C real satisfying condition 3.12. Thus nA, nB, and nC must be different, as before, so that the configuration will not degenerate.

If $\mathfrak{C}$ is bipartite and $u - \omega'$ real, then the points 3.13 will all be real and on the even branch. Thus if any one of the points 3.14 lies on the even branch, i.e. if $A - \omega'$ is real, all the points 3.14 lie on the even branch. A similar statement holds for the points 3.15 and 3.16. By condition 3.12 which must be satisfied by the points of K_n, either none or exactly two of the sets of points 3.14, 3.15, 3.16 lie on the even branch.

We have proved

THEOREM 3.1. *K_n may be inscribed in a non-singular plane cubic $\mathfrak{C}$ with two degrees of freedom. Any two real points u, v such that nu, nv, and $-n(u + v)$ are different* (mod 2ω, $2\omega'$) *may be selected as a pair of points of the configuration, and the remaining points are uniquely determined. If $\mathfrak{C}$ is bipartite the $3n$ points of K_n fall into three sets of n points such that either two or none of the sets lie on the even branch.*

We have also proved

THEOREM 3.2. *K_n exists in the real projective plane for all n.*

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