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ON FINITE CODING FACTORS OF A CLASS OF RANDOM MARKOV CHAINS

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ABSTRACT. For k-step Markov chains, factors generated by finite length codes split off with Bernoulli complement when maximal in entropy. Those not maximal are relatively finite in another factor which generates or splits off.

These results extend to random Markov chains with finite expected step size, implying that random Markov chains with finite expected step size can have only finitely many ergodic components, each of which is isomorphic to a finite rotation, a Bernoulli shift, or a direct product of a Bernoulli shift with a finite rotation. This result limits the type of zero entropy factors which occur in random Markov chains with finite expected step size, providing a counterpoint to the work of Kalikow, Katznelson, and Weiss, who have shown that each zero entropy process can be embedded in some random Markov chain.

Extending Rudolph and Schwarz, random Markov chains with finite expected step size are limits in $\bar{d}$ of their canonical Markov approximants. The $\bar{d}$-closure of the class is the Bernoulli cross Generalized Von Neuman processes.

Finitary isomorphism of aperiodic ergodic random Markov chains with finite expected step size is considered.

Applications are made to a class of generalized baker's transformations.

1. **Introduction.** In the following, we shall be concerned with an invertible measure preserving transformation T, acting on a probability space $(X, \mathcal{B}, \mu)$, which we take to be isomorphic to the unit interval with Lebesgue sets and Lebesgue measure. P and H are two measurable finite partitions of X, while (T, P) and (T, H) are the associated stationary processes. We shall assume that $P \vee H$ generates, *i.e.*, that $\bigvee_{-\infty}^{\infty} T^i(P \vee H) = (P \vee H)_{-\infty}^{\infty} = \mathcal{B}$.

By the the *factor* $\mathcal{H} = H_{-\infty}^{\infty}$ *generated by* H, we mean the smallest T-invariant sub-σ-algebra of $\mathcal{B}$ containing H and all subsets of measure zero. A *fiber* h in $\mathcal{H}$ is the collection of all points in X which share the same doubly-infinite H-name. For a fixed version of probability conditioned on $\mathcal{H}$ and a fixed fiber h of $\mathcal{H}$, we study the behavior of the sequence $\{T^iP\}$, $i = \ldots, -1, 0, 1, \ldots$ on h, viewed as a non-stationary process that the partition P and the transformation T induce on it.

We say that a partition P is an N-step Markov generator if $\mathcal{B} = P_{-\infty}^{\infty}$ and $\mu(P \mid P_{-\infty}^{-1}) = \mu(P \mid P_{-n}^{-1})$. For a non-negative integer l, by a *finite coding of length* l we mean a partition $H \subset P_0^{l-1}$. When l is one, we shall refer to the code as a *clumping* of P.

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DEFINITION. We say that a factor $\mathcal{H}$ generated by H *splits off* if there exists a finite partition B of X such that the sequence $\{T^iB\}$, $i = \ldots, -1, 0, 1, \ldots$, is independent, $B_{-\infty}^{\infty} \perp H_{-\infty}^{\infty}$, and $(B \vee H)_{-\infty}^{\infty} = \mathcal{B}$.

DEFINITION. Let F be a finite partition and denote the entropy of the process (T, F) by $h(T, F)$. We say that a factor $H_{-\infty}^{\infty}$ is *maximal in entropy* if whenever $H_{-\infty}^{\infty} \subset F_{-\infty}^{\infty}$ and $h(T, H) = h(T, F)$, then $H_{-\infty}^{\infty} = F_{-\infty}^{\infty}$.

DEFINITION. We say that T is K-mixing conditioned on $\mathcal{H}$ if for any measurable set A, we have $\lim_{n\to\infty} \lim_{k\to\infty} \lim_{m\to\infty} |\mu(A \mid \bigvee_{-k}^{k} T^iH) - \mu(A \mid \bigvee_{-n-m}^{-n} T^iP \bigvee_{-k}^{k} T^iH)| = 0$ a.e.

DEFINITION. For $\mathcal{H}$ and $\mathcal{G}$ two σ-algebras with $\mathcal{H} \subset \mathcal{G}$, we say that $\mathcal{H}$ is *relatively finite in* $\mathcal{G}$ if there exists a finite integer l so that each fiber h in $\mathcal{H}$ consists of exactly l fibers of $\mathcal{G}$.

It is known [8, Theorem 2] that

THEOREM 1. *If (T, P) is n-step Markov and $\mathcal{H}$ is a maximal entropy factor generated by a finite code, then $\mathcal{H}$ splits off. Moreover, if $\mathcal{H}$ does not split off, then it must be relatively finite in a larger factor $\mathcal{G}$ which either generates or itself splits off.*

In particular, [8, Theorem 3].

THEOREM 2. *Every clumping factor in a Markov chain with strictly positive transition probabilities will split off.*

The purpose of this note is to extend these known results on n-step Markov chains to a class of uniformly convergent martingales, the random Markov processes with finite expected step size. These processes were defined by Kalikow [5].

DEFINITION. Let F be a finite set. Let $\{a_i, N_i\}$ be a stationary process, where each $a_i \in F$, each $N_i \in \mathbf{N}$, N_0 is independent of $\{a_i, N_i\}_{i<0}$ and for each j

$$\mu(a_0 = k \mid a_{-1}a_{-2}\cdots a_{-j} \wedge N_0 = j) = \mu(a_0 = k \mid \{a_i\}_{i<0} \wedge N_0 = j).$$

Then $\{a_i, N_i\}_{i\in\mathbf{Z}}$ is a *complete random Markov chain.*

DEFINITION. A *random Markov chain* is the projection on the first coordinate of a complete random Markov chain, *i.e.*, if $\{a_i, N_i\}$ is a complete random Markov chain, then $\{a_i\}$ is a random Markov chain.

It is easy to see that if the values of the random variable N are bounded by some fixed integer n, then a random Markov chain is actually an n-step Markov chain.

Kalikow also defined the concept of a uniform martingale [5].

DEFINITION. Let F be a finite set, and let $\{a_i\}_{i\in\mathbf{Z}}$ be a stationary process, with all $a_i \in F$. If, for all $\epsilon > 0$, there exists $N_i \in \mathbf{N}$ such that for all $M > N_i$ and all $\{F_i\}_{i=0}^{\infty}$ with all $F_i \in F$,

$$|\mu(a_0 = F_0 \mid a_{-1} = F_1, a_{-2} = F_2, \ldots, a_{-m} = F_m) - \mu(a_0 = F_0 \mid a_{-i} = F_i \text{ for all } i)| < \epsilon,$$

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then $\{a_i\}$ is a uniform martingale.

REMARK. Note that probabilists would use the term "Markov chain" for what we have referred to as a 1-step Markov chain. Also, what they would call a martingale is not $\{a_i\}$, but rather the random sequence

$$\{\mu(a_0 = F_0 \mid a_{-1}, a_{-2}, \ldots, a_{-m})\}.$$

Kalikow then established the following fact [5, Theorem 4].

THEOREM 3. *A process is a uniform martingale iff it is a random Markov process.*

The class of all uniform martingales turns out to be somewhat large and ungainly. For instance, it is not closed under the taking of inverses [5, Example 19] and lacks other desirable properties. For this reason we restrict our discussion to those random Markov chains for which the random variable N in the associated complete random Markov chain has finite expectation. We will describe these objects as *random Markov chains with finite expected step size.*

As an application of these results, we will consider a subclass of the generalized baker's transformations. These transformations were defined by Bose [2].

Let λ be Lebesgue measure on the Lebesgue subsets B of $[0,1)$. Let $\mu = \lambda \times \lambda$ be 2-dimensional Lebesgue measure on the Lebesgue subsets G of the unit square $S = \{(x,y) \mid x \in [0,1), y \in [0,1)\}$, and let $f\colon [0,1) \longmapsto [0,1]$ be a B-measurable function. Define two mappings $\psi, \phi\colon [0,1) \longmapsto [0,1]$ by the formulæ

$$\psi_f(x) = \int_0^x f(t)\,dt$$
$$\phi_f(x) = 1 - \int_x^1 1 - f(t)\,dt.$$

For each $(x,y) \in S$, we set

$$T_f(x,y) = \begin{cases} \left(\psi_f(x), \frac{y}{f(x)}\right), & \text{if } 0 \le y \le f(x) \\ \left(\phi_f(x), 1 - \frac{1-y}{1-f(x)}\right), & \text{if } f(x) \le y < 1. \end{cases}$$

It is easy to show, that $T_f\colon S \longmapsto S$ is measurable, invertible, and preserves μ.

DEFINITION. Let P_f be the partition of S into two sets

$$P_0 = \{(x,y) \mid x \in [0,1), f(x) \le y < 1\}$$

and $P_1 = S \setminus P_0$. We may consider the process (T_f, P_f). This is said to be a *generalized baker's transformation.*

By a well known theorem of Krieger, if (T,P) is ergodic with entropy at most $\log 2$, one can construct a two-set generating partition. As shown in [2], one can then find a representation of (T,P) as a generalized baker's transformation.

Bose also defined an analogous baker's transformation with generating partition P indexed by any positive integer. In this case, instead of a single function f, there is an integer n and several functions $f_i: [0,1) \longrightarrow [0,1]$, $1 \leq i \leq n$, with $\sum_i f_i = 1$. The corresponding partition is $P = \{P_i, 1 \leq i \leq n\}$, where $P_i = \{(x,y) \in S : \sum_{j<i} f_j(x) \leq y < \sum_{j \leq i} f_j(x)\}$. The transformation T is defined in terms of the obvious sequence of functions ψ_i, $1 \leq i \leq n$.

It was shown in [9, Section 5] that when the functions f_i are uniformly continuous, then the generalized baker's transformation is a random Markov chain. Moreover, if the f_i are sufficiently well behaved, say for instance Hölder continuous and bounded away from zero, then the random Markov chain will have finite expected step size. It then follows from results in this article that under such hypotheses, clumping factors of the generating partition P will always split off.

2. **The product decomposition.** The product decomposition follows from the relativized isomorphism theorem of J.-P. Thouvenot. To apply this machinery, we need certain definitions.

DEFINITION. We say that two partitions $\mathcal{P}$ and $\mathcal{Q}$ are *ϵ-independent conditioned on* $\mathcal{H}$ if $\sum_{A \in \mathcal{P}, B \in \mathcal{Q}} |\mu_{\mathcal{H}}(A \cap B) - \mu_{\mathcal{H}}(A)\mu_{\mathcal{H}}(B)| \leq \epsilon$. (Notation: $\mathcal{P} \perp_{\mathcal{H},\epsilon} \mathcal{Q}$.)

DEFINITION. We say that (T,P) is *weakly Bernoulli conditioned on* $\mathcal{H}$ if given $\epsilon > 0$ there exists a positive integer n and a set G with $\mu(G) > 1 - \epsilon$ of $\mathcal{H}$-fibers h on which $(P_{-m}^{0} \perp_{\mathcal{H},\epsilon} P_{n}^{n+m})$ for all positive integers m.

DEFINITION. Given finite sequences of partitions $\{P_i, i = 1, 2, \ldots, n\}$ on a probability space (Z, ρ) and $\{R_i, i = 1, 2, \ldots, n\}$ on a probability space (Y, σ), we define

$$\bar{d}_n(\{P_i\}_1^n, \{R_i\}_1^n) = \inf_{\phi \in \mathcal{M}} n^{-1} \sum_{1}^{n} \rho[(P_i \setminus \phi^{-1}R_i) \cup (\phi^{-1}R_i \setminus P_i)],$$

where $\mathcal{M}$ is the class of all measure preserving transformations from Z to Y, and where we sum over corresponding sets in partitions.

DEFINITION. We say that (T,P) is *very weakly Bernoulli conditioned on* $\mathcal{H}$ if given $\epsilon > 0$ there exists a positive integer n and a set G with $\mu(G) > 1 - \epsilon$ of $\mathcal{H}$-fibers h for which $\bar{d}_n(\{T^iP \mid C\}_1^n, \{T^iP\}_1^n) < \epsilon$ on a set C_h of atoms $C \in P_{-m}^{0} \cap h$ with $\mu_{\mathcal{H}}(C_h) > 1 - \epsilon$. (The $\bar{d}_n$ is measured using $\mu_{\mathcal{H}}$.)

It is easy to show [8, Lemma 2] that

LEMMA 1. *If (T,P) is weakly Bernoulli conditioned on $\mathcal{H}$, then (T,P) is very weakly Bernoulli conditioned on $\mathcal{H}$.*

The proof of the following lemma may be found in [7, Section 3] and [12]

LEMMA 2. *The process (T,P) is very weakly Bernoulli conditioned on $\mathcal{H}$ iff (T,P) is finitely determined conditioned on $\mathcal{H}$.*

From Thouvenot's relativized isomorphism theorem [12], if (T,P) is finitely determined relative to $\mathcal{H}$, then $\mathcal{H}$ splits off with an orthogonal Bernoulli complement.

It is well known that if a finite state Markov chain is mixing, then its generator is weakly Bernoulli. In the next section we show that if $\mathcal{H}$ is a finite coding factor of a random Markov chain (T,P), then (T,P) is nearly Markovian along fibers of $\mathcal{H}$. If $\mathcal{H}$ is maximal in entropy, then (T,P) is K-mixing along fibers, hence one can show that (T,P) is weakly Bernoulli along fibers.

3. **The main result.**

THEOREM 4. *Let (T,P) be a random Markov chain with finite expected step size. Let $H \in P_0^{l-1}$ generate a finite coding factor $\mathcal{H}$ which is maximal in entropy. Then (T,P) is weakly Bernoulli conditioned on $\mathcal{H}$.*

PROOF. Given ϵ, we can find a positive integer R such that for all except a set of measure ϵ of fibers h of $\mathcal{H}$, for all except a set of conditional measure ϵ of atoms P_{-m}^0, the $\mathcal{H}$-conditional distributions of P_R^∞ on any two atoms A, B of P_{-m}^0 are ϵ-close in $\bar{d}$. To see this, consider the above distributions as lying in the complete random Markov chain. Let $N(\omega)$ denote the step size, which is a random variable. Use the finite expectation of $N(\omega)$ to find a set C with $\mu(C) > 1-\epsilon$ and a positive integer $\hat{N}$ sufficiently large that we have $N(\omega) \le \hat{N}$, $N(T\omega) \le \hat{N}+1, \ldots$, *i.e.*, the random Markov process never looks back farther than a certain fixed past of length $\hat{N}$. Note that C is independent of $P_{-\infty}^0$. Then form a joining of the (nonstationary) distributions in h, conditioned on A and B, as follows. The fact that $\mathcal{H}$ is maximal in entropy, implies K-mixing on fibers. Hence, given $\hat{N}$, there exists a positive integer L such that $P_{-\infty}^0 \perp_{\mathcal{H},\epsilon} P_L^{L+\hat{N}}$. Join the two distributions arbitrarily in the first L places, so that the $P_L^{L+\hat{N}+l}$ overlap within ϵ. Now apply the independence of the past beyond $\hat{N}$ which holds on C to get the $\bar{d}$-matching for arbitrarily large intervals. Thus, the $\bar{d}$ distance between future distributions in a fiber conditioned on past atoms can be made arbitrarily small, as desired.

It is shown in [12] that if the process (T,P) is $\mathcal{H}$-conditionally very weakly Bernoulli, then $\mathcal{H}$ must be maximal in entropy. Hence, splitting occurs in the Markov chain context iff a finite coding factor $\mathcal{H}$ is maximal in entropy.

On the other hand, if $\mathcal{H}$ is not maximal in entropy, then by Zorn's Lemma it is contained in another factor which has the same entropy and which is either maximal in entropy or itself generates. (An example of such a situation is given in [8].)

The following lemma shows that such a factor, while *not* necessarily a finite coding factor, must still split off. (Note, however, that not *all* factors will split off.)

LEMMA 3. *Let (T,P) be a random Markov chain with finite expected step size, and $\mathcal{H}$ a finite coding factor generated by $H \in P_0^{l-1}$. Suppose $\mathcal{H}$ is not maximal in entropy, and denote by $\mathcal{G}$ the (unique) maximal factor which contains $\mathcal{H}$ and has the same entropy. Then $\mathcal{G}$ either splits off with Bernoulli complement or generates.*

PROOF. The essence of the proof is the observation that (T,P) is nearly Markovian on fibers of $\mathcal{G}$, *i.e.*, for $A \in P_1^\infty$ and $B \in P_{-\infty}^0$, then almost everywhere

$$\lim_{n\to\infty} |\mu_{\mathcal{G}}(A \mid P_{-n+1}^0 B) - \mu_{\mathcal{G}}(A \mid P_{-n+1}^0)| = 0.$$

Thus the argument in the proof of the preceding theorem goes through as before.

Indeed, denote by $\pi(T \mid \mathcal{H}) = \bigcap_{n>0}(\bigvee_{-\infty}^{-n} T^iP \vee \mathcal{H}) = \bigcap_{n>0}(\bigvee_{n}^{\infty} T^iP \vee \mathcal{H})$ the Pinsker algebra of T relative to $\mathcal{H}$. Then given $\epsilon > 0$ there exists a positive integer $\hat{N}$ such that $\bigvee_{-\infty}^{-\hat{N}} T^iP \perp_{\epsilon,\mathcal{H}} \bigvee_{0}^{\infty} T^iP$, given the partition $P_0^{\hat{N}}$. Hence $\pi(T \mid \mathcal{H}) \perp_{\epsilon,\mathcal{H}} \pi(T \mid \mathcal{H})$ given $P_{-\hat{N}}^0$. Thus, the partition $\pi(T \mid \mathcal{H})$ is ϵ-contained in $P_{-\hat{N}}^0$. Since $\mathcal{H} \in \mathcal{G}$ and the entropies are equal, we see that $\mathcal{G}$ is ϵ-contained in $\mathcal{H} \vee P_{-\hat{N}}^0$.

The next theorem indicates the manner in which a finite coding factor $\mathcal{H}$ which is not maximal in entropy must sit in its extension $\mathcal{G}$.

THEOREM 5. *For P a finite partition, let (T, P) be a random Markov chain with finite expected step size, and $\mathcal{H}$ a finite coding factor generated by $H \in P_0^{l-1}$. Suppose $\mathcal{H}$ is not maximal in entropy, and denote by $\mathcal{G}$ the (unique) maximal factor which contains $\mathcal{H}$ and has the same entropy. Then $\mathcal{H}$ is relatively finite in $\mathcal{G}$.*

PROOF. The (T, P) process is embedded in a complete random Markov chain. Hence $\mathcal{G}$ also sits in the complete random Markov chain. Denote by $\mathcal{Q} = \{Q_{n,\bigvee_{-n}^{0} T^iP}\}$ the countable partition in the complete chain, where $n = n(\omega) = \sup\{N(T^i\omega) - i, i = 1, 2, \ldots\}$. Since $\mathcal{G} \perp_{\mathcal{H}\vee\mathcal{Q}} \mathcal{G}$, it follows that $\mathcal{G}$ is relatively countable for each fiber h in $\mathcal{H}$. Indeed, on the fibers of $\mathcal{H}$, the atoms of $\mathcal{G}$ are countable unions $\{\bigcup D_i : D_i \in \bigvee_{-N_i}^{0} T^iP\}$. The Markov property and the fact that P is finite guarantee that there can be only finitely many ergodic components. For a given ergodic component, on each fiber h, the action of T on $\mathcal{G}$ is ergodic, hence we have a finite rotation. Indeed, one can further conclude that there is a positive integer n_0 and a finite partition $Q_0 \subset \bigvee_{-n_0}^{0} T^iP$ so that $\mathcal{G} = Q_0 \vee \mathcal{H}$.

4. **A bouquet of corollaries.** First, we consider the general form of a random Markov chain with finite expected step size.

For (T, P) any random Markov chain with finite expected step size, there is a natural clumping in which all states of P coalesce to one single set. This clumping factor has entropy zero. The Pinsker algebra contains this trivial factor and has entropy zero. It follows that the Pinsker algebra must split off. It also follows that the trivial clumping factor is relatively finite in the Pinsker algebra. Hence the action of T on the Pinsker algebra is a finite rotation.

We summarize this in the following:

COROLLARY 1. *Each of the finitely many ergodic component of a random Markov chain with finite expected step size is isomorphic to a finite rotation, or a Bernoulli shift, or the direct product of a Bernoulli shift and a finite rotation.*

Moreover,

COROLLARY 2. *If an ergodic component of a random Markov chain with finite expected step size has period p, then each finite step Markov chain in the skew product that constitutes the associated complete random Markov chain with finite expected step size will also have a rotation factor with period p.*

Moreover, we can extend the results of Rudolph and Schwartz [11], who showed that the closure in the $\bar{d}$ metric of k-step Markov chains is the set of BGVN (Bernoulli cross generalized von Neumann) processes.

COROLLARY 3. *A random Markov process with finite expected step size is the limit in $\bar{d}$ of its canonical Markov approximants.*

PROOF. By the above corollary, a positive entropy random Markov chain with finite expected step size is the product of a Bernoulli with a finite rotation, hence a BGVN process. Moreover, the rotation factor is measurable with respect to a finite block of variables. Hence, by [11, Theorem II], it must be the limit in $\bar{d}$ of its canonical Markov approximants.

Furthermore,

COROLLARY 4. *The closure in the $\bar{d}$-metric of the random Markov chains with finite expected step size is the* BGVN *processes.*

PROOF. Let (T, P) be the $\bar{d}$-limit of a sequence (T_k, P_k) of random Markov chains with finite expected step size. By the previous corollary, we can select canonical n_k-step Markov approximations to the (T_k, P_k) which are within 2^{-k} in $\bar{d}$ of (T_k, P_k). The result follows from [10, Theorem I], since the sequence (T_{n_k}, P_{n_k}) also converges to (T, P) in $\bar{d}$.

One can also comment on the isomorphism class of ergodic random Markov chains with finite expected step size.

COROLLARY 5. *Two ergodic random Markov chains with finite expected step size are isomorphic iff they have the same entropy and period.*

Indeed, using the criterion of Rudolph [10], we can even show the following result.

THEOREM 6. *Two aperiodic ergodic random Markov chains with finite expected step size are finitarily isomorphic iff they have the same entropy.*

PROOF. We utilize Rudolph's finitarily Bernoulli criterion [10, p. 3]. One can choose $A_i(\epsilon)$, $\bar{A}(\epsilon)$, $\hat{A}(\epsilon)$, $C(\epsilon, n)$, and markers $S(\epsilon)$ as in the proof [10, p. 6] that a finite state mixing Markov chain is finitarily Bernoulli.

One does not have the strict independence of the Markov property; but one does have ϵ-independence conditioned on long strings, and the ϵ-independence is uniform over strings of the same length, since the random step size has the property that N_1^∞ is independent of $N_{-\infty}^0$ and $P_{-\infty}^0$. Thus, CBI, properties 4 and 5, and USM all follow.

Next we apply these results to generalized baker's transformations. It follows from the above that if (T, P) is a random Markov chain with finite expected step size, and $\mathcal{H}$ is a clumping factor in which each atom leads to every other atom, then $\mathcal{H}$ is maximal in entropy and hence splits off. Thus, COROLLARY 6. *Let (T, P) be a generalized baker's transformation with generator $P = \{P_i\}$ determined by functions $\{f_i, i = 1, \ldots, n\}$. Suppose that the functions $\{f_i\}$ are such that the Lebesgue measure $\lambda\{x : f_i = 0\} = 0$ for $i = 1, \ldots, n$; and suppose that (T, P) is a random Markov chain with finite expected step size. Then a clumping factor $\mathcal{H}$ is maximal in entropy, hence on each ergodic component either splits off or generates.*

Moreover,

COROLLARY 7. *Suppose (T, P) is a generalized baker's transformation with generator $P = \{P_i\}$ determined by Hölder continuous functions $\{f_i, i = 1, 2, \ldots, n\}$ uniformly bounded away from zero by some constant c. If $\mathcal{H}$ is a finite coding factor, then on each ergodic component either $\mathcal{H}$ splits off, or it is relatively finite in a factor $\mathcal{G}$ which generates or splits off.*

PROOF. By the results in [9, Section 6], (T, P) is random Markov with finite expected step size.

Finally, we note that these results allow us to weaken the requirement in [9, Section 7] on the Lebesgue measure of the sets where the f_i are zero in a generalized baker's transformation, in return for a slightly weaker result.

COROLLARY 8. *Let (T, P) be a generalized baker's transformation with generator $P = \{P_i\}$ determined by functions $\{f_i, i = 1, \ldots, n\}$. Suppose that (T, P) is a random Markov chain with finite expected step size. Then there are only finitely many ergodic components; and restricted to any component, (T, P) is either Bernoulli or the direct product of a Bernoulli with a finite rotation.*

Moreover,

COROLLARY 9. *Suppose (T, P) and $(\hat{T}, \hat{P})$ are generalized baker's transformations with generators $P = \{P_i\}$ and $\hat{P} = \{\hat{P}_i\}$ determined by Hölder continuous functions $\{f_i, i = 1, 2, \ldots, n\}$ and $\{\hat{f}_i, i = 1, 2, \ldots, \hat{n}\}$, respectively, which are uniformly bounded away from zero by some constant c. Then (T, P) and $(\hat{T}, \hat{P})$ are finitarily isomorphic iff they have the same entropy.*

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