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ORDINARY SINGULARITIES WITH DECREASING HILBERT FUNCTION

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1. Introduction. Let A be the co-ordinate ring of a reduced curve over a field k. This means that A is an algebra of finite type over k, A has no nilpotent elements, and that if P is a minimal prime ideal of A, then A/P is an integral domain of Krull dimension one. Let M be a maximal ideal of A. Then $G(A)$ (the graded ring of A relative to M) is defined to be $\bigoplus_{k=0}^{\infty} M^i/M^{i+1}$. We get the same graded ring if we first localize at M, and then form the graded ring of A_M relative to the maximal ideal MA_M. That is

$$G(A) \cong \bigoplus_{i=0}^{\infty} (MA_M)^i/(MA_M)^{i+1}.$$

Let $\bar{A}$ be the integral closure of A. If $P_1, P_2, \ldots, P_s$ are the minimal primes of A then

$$\bar{A} = \prod_{i=1}^{s} \overline{(A/P_i)},$$

where A/P_i is a domain and $\overline{A/P}_i$ is the integral closure of A/P_i in its quotient field. Let $M_1, \ldots, M_n$ be those maximal ideals of $\bar{A}$ that lie over M. That is, $A \subset \bar{A}$ and $M_i \cap A = M$. Let

$$J = M_1 \cap \ldots \cap M_n \quad \text{and} \quad G(\bar{A}) = \bigoplus_{i=0}^{\infty} J^i/J^{i+1}.$$

The maximal ideal M is called an ordinary singular point if Proj $G(A)$ is reduced. Equivalently M is ordinary if $M\bar{A} = M_1 \ldots M_n$ (each M_i with exponent one) and the tangent directions to the branches M_i are distinct. This equivalence is discussed in [**2**]. If $G(A)$ is reduced, then Proj $G(A)$ is reduced. Furthermore $G(A)$ is reduced if and only if the induced map $G(A) \to G(\bar{A})$ is an inclusion [**3**]. However $G(A)$ need not be reduced at an ordinary singular point, as the examples show.

The M^i/M^{i+1} are finite dimensional vector spaces over k. The hilbert function of M is defined by

$$f(i) = \dim_k(M^i/M^{i+1}).$$

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The embedding dimension is

$$f(1) = \dim_k(M/M^2).$$

The function $f(i)$ becomes constant if i is sufficiently large, this constant value being the multiplicity of M. We always have $f(i) \leqq$ multiplicity, and in the ordinary case described above the multiplicity is n. In this note we show how to construct examples of curves with ordinary singular points, and examine the behaviour of the hilbert function for these examples. In particular the hilbert function can decrease before ultimately increasing to the multiplicity. Examples of singularities with a hilbert function that temporarily decreases are mentioned in [**4**], although these examples are for non-ordinary singularities.

During the preparation of this paper I had a number of helpful conversations with B. Dayton, A. V. Geramita, and F. Orecchia. In particular Geramita pointed out in a seminar that little is known about the behaviour of the hilbert function before it stabilizes, and Orecchia suggested that I might try to construct ordinary singularities by an identification construction similar to that used for seminormal rings.

2. Lines through a point. First we examine the case of n straight lines in k^r, which pass through one (rational) point. Without loss of generality we can assume that the lines all pass through the origin, and that there are r linearly independent directions (otherwise choose a smaller r). We can choose co-ordinates so that the directions of lines $1, \ldots, r$ are the co-ordinate axes. Let the directions of the lines be the columns of the $r \times n$ matrix $C = (c_{i,j})$, $1 \leqq i \leqq r$, $1 \leqq j \leqq n$ where $c_{i,j} = \delta_{i,j}$ if $1 \leqq j \leqq r$. Let $B = k[X_1, \ldots, X_r]/I$ be the reduced co-ordinate ring of these n straight lines in k^r. Then

$$\bar{B} = \prod_{i=1}^{n} k[t_i],$$

and the surjection $\pi_i : B \rightarrow k[t_i]$ is given by

$$\pi_i(X_j) = c_{j,i}t_i$$

(i.e., by the i^{th} column of C). The image of X_j in $\bar{B}$ is then $C(j)t$ where $C(j)$ is the j^{th} row of C, $t = (t_1, \ldots, t_n)$ and the vectors are multiplied co-ordinatewise. Finally B is the subring of $\bar{B}$ generated by the $C(j)t$.

Let N be the ideal of all functions in B vanishing at the origin. B is graded, with N being the ideal of elements of degree $\geqq 1$. Elements of degree 1 generate B, so $G(B) = B$. Furthermore $J = (t_1, \ldots, t_n)\bar{B}$ so $G(\bar{B}) = \bar{B}$. Thus N is an ordinary singular point and N^i/N^{i+1} is isomorphic to the subspace of k^n ($= i^{\text{th}}$ graded piece of $\bar{B}$) generated by the images of all monomials of degree i in the r variables $X_1, \ldots, X_r$.

There are $\binom{i+r-1}{r-1}$ such monomials.

Thus

$$b_i = \dim_k (N^i/N^{i+1}) \leqq \text{minimum} \left(\binom{i+r-1}{r-1}, n \right).$$

Let C^i be the $\binom{i+r-1}{r-1}$ by n matrix whose rows are all possible i-fold products of rows of C. Then $b_i = \text{rank } C^i$.

The columns of C can be thought of as n points Q_j in $\mathbf{P}_k^{r-1}$, and $b_i < \binom{i+r-1}{r-1}$ if and only if the Q_j lie on a hypersurface of degree i. The condition $b_i = \min\left(\binom{i+r-1}{r-1}, n\right)$ is precisely the definition of generic position of the points Q_j that is given in [**3**]. In [**1**] Theorem 4 it is shown that "most of the time" $Q_1, \ldots, Q_n$ are in generic position (at least for k infinite). (These values for b_i are also given by Theorem 3.4 of [**2**].)

Even if the Q_j are not in generic position, we know that $\dim_k(\bar{B}/B) < \infty$, so $b_j = n$ for j large enough. The sequence $\{b_j\}$ can never decrease because B contains non-zero divisors of degree one (k infinite). In fact $\{b_j\}$ must be strictly increasing until it reaches n.

3. The definition of A**.** In this section we define a one dimensional domain A with ordinary singular point M. We will preserve the notation of Section 2. Let $x_1, \ldots, x_n$ be distinct elements of k, and let

$$A = \left\{ f \in k[t] \,|\, f(x_1) = \ldots = f(x_n) \quad \text{and} \quad f'(x_j) = \sum_{i=1}^{r} c_{i,j} f'(x_i), \quad r+1 \leqq j \leqq n \right\}.$$

Clearly A is a k-subspace of $k[t]$, with

$$\dim_k (k[t]/A) \leqq 2n - r - 1,$$

because we have put $2n - r - 1$ linear conditions on A. If $j \geqq r + 1$ and $f, g \in A$ then

$$\begin{aligned}(fg)'(x_j) &= g(x_j)f'(x_j) + f(x_j)g'(x_j) = g(x_j)\left[\sum_{i=1}^{r} c_{i,j} f'(x_i)\right] \\ &\quad + f(x_j)\left[\sum_{i=1}^{r} c_{i,j} g'(x_i)\right] = \sum_{i=1}^{r} c_{i,j}[g(x_i)f'(x_i) + f(x_i)g'(x_i)] \\ &= \sum_{i=1}^{r} c_{i,j}(fg)'(x_i),\end{aligned}$$

so $fg \in A$. Thus A is a sub-k-algebra of $k[t]$. Since

$$\dim_k (k[t]/A) < \infty$$

we have that $k[t]$ is the integral closure of A, and A is a k-algebra of finite type.

Let $M = \{f \in A \mid f(x_i) = 0\}$. Then $A/M = k$ so M is a maximal ideal of A. Let

$$M_i = (t - x_i)k[t] \ (1 \leqq i \leqq n).$$

Then clearly $M_i \cap A = M$. Let

$$F = \prod_{i=1}^{n} (t - x_i).$$

Then $F^2 \in M$, so the M_i are precisely the maximal ideals of $\bar{A} = k[t]$ that contain M, and $J = Fk[t]$. Furthermore $F^2k[t]$ is a $k[t]$ ideal of A. The conductor I of A in $k[t]$ therefore contains $F^2k[t]$. Thus the only primes of $k[t]$ containing I are $\{M_1, \ldots, M_n\}$, and

$$(\operatorname{Spec} A) - M \cong (\operatorname{Spec} k[t]) - \{M_1, \ldots, M_n\}$$

(as schemes). We can thus think of Spec A as having been obtained by identifying the M_i in a certain way to form M. If $A = k[X_1, \ldots, X_s]$ (so that Spec $A \subset k^s$) and we define the tangent vector at x_i in the naive way from calculus to be

$$\mathbf{t}_i = (X_1'(x_i), X_2'(x_i), \ldots, X_s'(x_i))$$

then the equations defining A yield

$$\mathbf{t}_j = \sum_{i=1}^{r} c_{ij}\mathbf{t}_i \ (j \geqq r + 1),$$

so intuitively at least, the tangents of the branches $M_1, \ldots, M_r$ are linearly independent, and the tangents of the remaining branches are specified linear combinations of them. A is the largest subring of $k[t]$ in which $M_1, \ldots, M_n$ are identified to one point, with the specified tangent vectors. Example 2 in [**2**] shows that not all ordinary singularities are of the type described here.

4. The powers of M. In this section we describe the powers M^i, and show that $G(A)_{\text{reduced}} \cong B$, B as described in Section 2. Again let

$$F = \prod_{i=1}^{n} (t - x_i),$$

and let

$$f_i = c_iF^2/(t - x_i),$$

with $c_i = F'(x_i)^{-2}$, so that $f_i'(x_j) = \delta_{ij}$. Let

$$h_i = f_i + \sum_{j=r+1}^{n} c_{ij} f_j \ (1 \leqq i \leqq r).$$

Then $h_i'(x_j) = \delta_{ij}$ $(1 \leqq i, j \leqq r)$ and $h_i'(x_l) = c_{i,l}$ $(1 \leqq i \leqq r,\ r + 1 \leqq l \leqq n)$. Thus $h_i \in A$. Note that

$$\begin{aligned} F^2k[t] &= \{ f \in k[t] \mid f(x_1) = \ldots = f(x_n) = 0, f'(x_1) = \ldots \\ &= f'(x_n) = 0\} = \{ f \in A \mid f(x_1) = \ldots = f(x_n) = 0, \\ &\qquad f'(x_1) = \ldots = f'(x_n) = 0\}. \end{aligned}$$

Clearly $M^2 \subset F^2k[t] \subset M \subset A$. We claim that the h_i $(1 \leqq i \leqq r)$ form a k-basis of $M/F^2k[t]$. Suppose $f \in M$ with $f'(x_i) = a_i$ $(1 \leqq i \leqq r)$. Then $f - \sum_{i=1}^{r} a_i h_i = h$ satisfies $h(x_i) = 0$ and $h'(x_i) = 0$ $(1 \leqq i \leqq r)$. Since $h \in A$ this also holds for $r + 1 \leqq i \leqq n$. Thus $h \in F^2k[t]$. If $\sum_{i=1}^{r} a_i h_i \in F^2k[t]$ evaluate the derivative at x_i $(1 \leqq i \leqq r)$. This yields $a_i = 0$, so the h_i $(1 \leqq i \leqq r)$ are linearly independent in $M/F^2k[t]$, and thus form a basis of $M/F^2k[t]$.

Now we claim that

$$(h_1, \ldots, h_r, F^2)k[t] = Fk[t].$$

All the h_i are divisible by F, so it suffices to show that $t - x_i$ has exponent one in at least one of the h_i. First suppose $1 \leqq i \leqq r$. Then h_i has one term (f_i) in which $(t - x_i)$ has exponent one, and all the rest divisible by $(t - x_i)^2$. Now recall that the tangent directions are assumed to be non-zero. For each $i \geqq r + 1$, there exists j such that $c_{j,i} \neq 0$. Then h_j has one term (i.e., $c_{j,i} f_i$) in which $(t - x_i)$ has exponent one and all other terms divisible by $(t - x_i)^2$. This proves

LEMMA 1. *If the columns of C are all non-zero then*

$$(h_1, \ldots, h_r, F^2)k[t] = Fk[t].$$

Now we can describe the powers M^i. First of all

$$M = \{h_1, \ldots, h_r, F^2k[t]\}.$$

By this notation we mean linear combinations of $h_1, \ldots, h_r, F^2$ in which the h_i have coefficient in k, and F^2 has coefficient in $k[t]$. Using the same notation

$$M^2 = \{h_ih_j, h_jF^2k[t], F^4k[t]\}$$

and by Lemma 1, this equals $\{h_ih_j, F^3k[t]\}$ $(1 \leqq i, j \leqq r)$. Next

$$M^3 = M^2M = \{h_ih_jh_k, h_ih_jF^2k[t], h_iF^3k[t], F^5k[t]\}.$$

The contribution from the last three types of generator is $(h_ih_jF^2, h_iF^3,$

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$F^5)k[t]$. All generators are divisible by F^4, and

$$(h_1F^3, \ldots, h_rF^3, F^5)k[t] = F^3(h_1, \ldots, h_r, F^2)k[t] = F^4k[t],$$

using Lemma 1. Thus

$$(h_ih_jF^2, h_iF^3, F^5)k[t] = F^4k[t] \quad \text{and}$$
$$M^3 = \{h_ih_jh_k, F^4k[t]\}.$$

A simple induction then yields

LEMMA 2. *If the columns of C are all non-zero then $F^{i+1}k[t] \subset M^i$. Also M^i is the k-span of all monomials of degree i in the h_j, and $F^{i+1}k[t]$.*

We should perhaps check that the discussion of tangent directions in the preceding section is compatible with the more usual approach. Since $Mk[t] = M_1 \ldots M_n$ we have $M/M^2 \to M_i/M_i^2$ onto, and $\mathrm{Hom}_k(M_i/M_1^2, k)$ is a subspace of $\mathrm{Hom}\,(M/M^2, k)$ (the Zariski tangent space at M) ([**2**] Lemma 1.6). $\mathrm{Hom}_k\,(M_i/M_i^2, k)$ is one dimensional, generated by $f \to f'(x_i)$ $(f \in M_i)$. Let $\mathbf{t}_i$ $(1 \leqq i \leqq n)$ denote the image of this form in $\mathrm{Hom}_k\,(M/M^2, k)$. Then

$$\mathbf{t}_i(h_j) = h_j'(x_i) = \delta_{ij} \; (1 \leqq i \leqq r)$$

so the $\mathbf{t}_i$ are linearly independent $(1 \leqq i \leqq r)$. The defining relations for A then yield

$$\mathbf{t}_j(f) = \sum_{i=1}^{r} c_{i,j}\mathbf{t}_i(f) \quad \text{for all} \quad f \in M,$$

i.e.,

$$\mathbf{t}_j = \sum_{i=1}^{r} c_{i,j}\mathbf{t}_i \quad \text{in } \mathrm{Hom}_k\,(M/M^2, k) \quad (r + 1 \leqq j \leqq n).$$

This is the relation among the tangent directions that was claimed earlier. As we will see later, the $\mathbf{t}_i$ need not span $\mathrm{Hom}_k\,(M/M^2, k)$. If the columns of C are pairwise linearly independent, the $\mathbf{t}_i$ are distinct, so M is an ordinary singularity.

Now we consider the homomorphism

$$G(A) \to G(\bar{A}) = \bigoplus_{i=0}^{\infty} J^i/J^{i+1}.$$

We have $J = Fk[t]$ so

$$G(\bar{A}) = \bigoplus_{i=0}^{\infty} F^ik[t]/F^{i+1}k[t] \cong \prod_{i=1}^{n} k[t_i],$$

where t_i is the image of $t - x_i$ in the i^{th} component of

$$Fk[t]/F^2k[t] = \prod_{i=1}^{n} ((t - x_i)k[t]/(t - x_i)^2).$$

We wish to find the image of M^i/M^{i+1} in $F^ik[t]/F^{i+1}k[t]$. To do this it suffices to find the image of h_i in $Fk[t]/F^2k[t]$, because $G(A) \to G(\bar{A})$ is a ring homomorphism and $F^{i+1}k[t] \subset M^i$ gets sent to zero in $F^ik[t]/F^{i+1}k[t]$. In the first co-ordinate every term of h_1 vanishes except f_1, which maps to

$$c_1(t - x_1)F'(x_1)^2 = t - x_1 = t_1.$$

If $j \geqq r + 1$, all terms of h_1 map to zero except $c_{1j}f_j$ which maps to

$$c_{1j}c_j(t - x_j)F'(x_j)^2 = c_{1j}t_j.$$

If $2 \leqq j \leqq r$, f_1 maps to zero. Thus in $Fk[t]/F^2k[t] = (k^n)t$, h_1 maps to the first row of the matrix C. Similarly h_i maps to the i^{th} row of C. Now we have $G(\bar{A}) = \bar{B}$ (B, $\bar{B}$ as in Section 2) and the image of $G(A)$ in $\bar{B} = \prod_{i=1}^n k[t_i]$ is the same as the image of B. For i large enough we observed in Section 2 that (so long as the columns of C are pairwise linearly independent)

$$N^i/N^{i+1} \to F^ik[t]/F^{i+1}k[t]$$

is onto (in fact this was an isomorphism). For such i,

$$M^i/M^{i+1} \to F^ik[t]/F^{i+1}k(t)$$

is onto. Since M^i contains $F^{i+1}k[t]$ this implies that $M^i = F^ik[t]$, i large enough. The homomorphisms

$$M^i/M^{i+1} \to F^ik[t]/F^{i+1}k[t]$$

eventually become isomorphisms. Now suppose that

$$u \in \ker\,[G(A) \to G(\bar{A})]$$

(u homogeneous). If

$$M^i/M^{i+1} \to F^ik[t]/F^{i+1}k[t]$$

is an isomorphism for $i \geqq i_0$ then $u^{i_0} = 0$, so $\ker\,[G(A) \to G(\bar{A})]$ consists of nilpotent elements. Since $G(\bar{A})$ is reduced, the kernel is precisely the nilradical of $G(A)$, and

$$G(A)_{\text{reduced}} \cong B.$$

The nilpotents in low degree disappear in Proj $G(A)$, so we see that Proj $G(A)$ is reduced, again verifying that M is an ordinary singularity of A. These results can be summarized as follows:

THEOREM 3. *Let A be as defined in Section* 3, *with maximal ideal M. If the tangents of the branches at M are pairwise linearly independent, then M is an ordinary singular point, and $G(A)_{\text{reduced}} \cong B$ = co-ordinate ring of the reducible variety consisting of the tangents to the branches.*

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The following somewhat more general result appears to be true. Let M be a singular point with nonsingular branches on any algebraic curve Spec A (that is $M\bar{A} = M_1 \ldots M_s\bar{A}$, i.e., exponents one). As usual we assume $\bar{A}/M_i = k$. Then the kernel of $G(A) \to G(\bar{A})$ is nilpotent, and $G(A)_{\text{reduced}}$ = co-ordinate ring of the tangents to the branches. (Some of the tangent directions may co-incide, so $G(A) \to G(\bar{A})$ need not be onto in high degrees.)

5. The calculation of the hilbert function. We are now ready to calculate the hilbert function of M. We have the following commutative diagram, where all the arrows are inclusions. The number denotes the codimension over k. In particular $c_i = \dim_k(M^i/M^{i+1})$ is the hilbert function of M.

$$\begin{tikzcd}
M \arrow[r, "d_1"] & Fk[t] \\
M^2 \arrow[u, "c_1"'] \arrow[r, "d_2"] & F^2k[t] \arrow[u, "n"'] \\
M^3 \arrow[u, "c_2"'] \arrow[r, "d_3"] & F^3k[t] \arrow[u, "n"'] \\
{} \arrow[u, "c_3"'] & {} \arrow[u]
\end{tikzcd}$$

Since M^i contains $F^{i+1}k[t]$, d_i is equal to the codimension of the image of M^i/M^{i+1} in $F^ik[t]/F^{i+1}k[t]$. Thus $d_i = n - b_i$ (b_i as in Section 2. Recall that the image of $G(A)$ in $G(\bar{A})$ is the same as that of B). This yields the formulas

$$c_i + d_i = d_{i+1} + n \quad \text{or} \quad c_i = d_{i+1} - d_i + n.$$

Then

$$c_{i+1} - c_i = d_{i+2} - 2d_{i+1} + d_i = -b_{i+2} + 2b_{i+1} - b_i.$$

This is the negative second difference of the b_i. Thus as long as the sequence $\{b_i\}$ is "concave upwards" we have a decreasing hilbert function.

If $r = 2$, then $b_i = i + 1$ $(i \leqq n - 1)$, and $b_i = n$ $(i \geqq n - 1)$. (Points in $\mathbf{P}^1$ are always in generic position.) Thus $c_i = b_i - b_{i+1} + n = n - 1$ for $i \leqq n - 2$, and $c_i = n$ for $i \geqq n - 1$. Thus c_i does not decrease, but if four or more points are identified, the embedding dimension c_1 is $\geqq 3$, even though the tangent directions lie in a plane. This at first seemed surprising to me.

The simplest case of a hilbert function with a decrease appears to be $r = 3$, $n = 10$. Assume that the ten tangent directions are in generic position. Then using the above equations we can produce the following table:

i	b_i	d_i	c_i
1	3	7	7
2	6	4	6
3	10	0	10

The values for $i \geqq 4$ are the same as for $i = 3$. Here we have embedding dimension 7.

If $r = 3$ and there are more lines the hilbert function decreases by 1 for awhile since the second difference of the sequence $b_i = \binom{i+2}{2}$ is $\{1\}$. Then it increases to the multiplicity in one or two steps depending on whether or not $n + 1$ is a binomial coefficient $\binom{i+2}{2}$. Here are two examples illustrating this (with tangents in generic position):

$n = 21$				$n = 23$			
i	b_i	d_i	c_i	i	b_i	d_i	c_i
1	3	18	18	1	3	20	20
2	6	15	17	2	6	17	19
3	10	11	16	3	10	13	18
4	15	6	15	4	15	8	17
5	21	0	21	5	21	2	21
6	21	0	21	6	23	0	23

For $i \geqq 7$ the values are the same as for $i = 6$.

If $r \geqq 4$ the hilbert function decreases even more rapidly because the second difference of $b_i = \binom{i+r-1}{r-1}$ is a polynomial of degree $r - 3 \geqq 1$. The final increase to the multiplicity $n + 1$ will be in one or two steps. Here are three examples with $r = 4$. (Tangents are in generic position.)

$n = 35$			$n = 37$			$n = 53$		
i	b_i	c_i	i	b_i	c_i	i	b_i	c_i
1	4	29	1	4	31	1	4	47
2	10	25	2	10	27	2	10	43
3	20	20	3	20	22	3	20	38
4	35	35	4	35	35	4	35	35
5	35	35	5	37	37	5	53	53

For $i \geqq 6$ the values are the same as for $i = 5$. The $n = 53$ example shows that the final increase can come in one step, even though 53 is not of the form $\binom{i+3}{3}$.

The fact that $b_i = c_i$ one step before the final stabilization in all the above examples is not an accident, but rather is a consequence of the formula $c_i = b_i - b_{i+1} + n$ and $b_{i+1} = n$.

At first glance these examples seem to contradict Theorem 3.3 of [**2**]. However here generic position means generic in the subspace of $\operatorname{Hom}_k(M/M^2, k)$ spanned by the tangent directions whereas in [**2**] generic position means generic in the (possibly larger) space $\operatorname{Hom}_k(M/M^2, k)$. See Theorem 4 below.

6. Further remarks. Let

$$N_i = \ker (M^i/M^{i+1} \to F^i k[t]/F^{i+1}).$$

The image of M^i/M^{i+1} has dimension b_i, so

$$n_i = \dim_k N_i = c_i - b_i.$$

From the above examples we see that n_i can be quite large. Furthermore $c_i - b_i = n - b_{i+1}$ so the sequence $\{n_i\}$ is strictly decreasing (until 0 is reached). Thus $G(A)$ is reduced if and only if $n_1 = 0$. This is equivalent to

$$F^2k[t] \cap M = M^2.$$

But $F^2k[t] \subset M$ so $n_1 = 0$ is equivalent to $F^2k[t] = M^2$. This proves

THEOREM 4. *The following are equivalent* (*for A as defined in Section* 3):
(a) *$G(A)$ is reduced.*
(b) $n_1 = 0$.
(c) $M^2 = F^2k[t]$.
(d) $b_2 = n$.
(e) $G(A) \cong B$.
(f) *The embedding dimension equals the dimension of the space spanned by the tangent directions.*
(g) *$r(r + 1)/2 \geqq n$ and the tangent directions are in generic position in $\mathbf{P}^{r-1}$.*

Here are a couple of examples to show that the isomorphism class of A depends on the tangent vectors themselves, and not just on their directions. Let

$$A = \{ f \in k[t] \mid f(0) = f(1) = f(-1), 4f'(0) + f'(1) + f'(-1) = 0\}.$$

One can show that $A = k[t^3 - t, t^4 - t^2]$, so A is a plane curve.

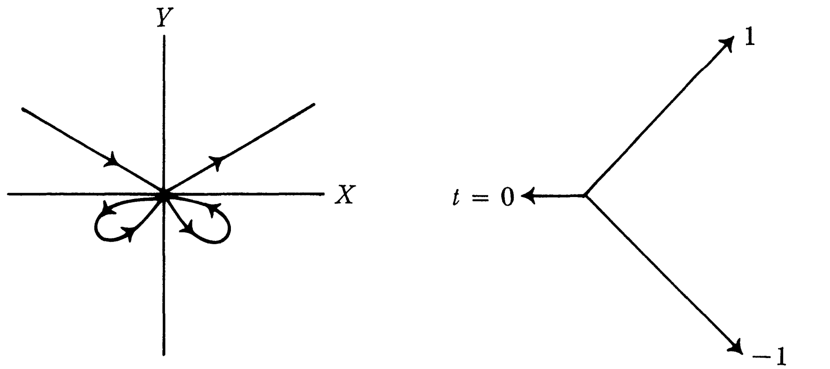

FIGURE 1. The curve $X = t^3 - t$, $Y = t^4 - t^2$ and its tangent vectors at $X = 0$, $Y = 0$.

Now let

$$A' = \{f \in k[t] \mid f(1) = f(-1) = f(0),\ \ f'(0) = f'(1) + f'(-1)\}.$$

One can show that

$$A' = k[t^4 - t^2, (t^2 - \tfrac{4}{5})(t^3 - t), t(t^3 - t)^2].$$

However topological considerations show that A' cannot be a plane curve (i.e., A' is not generated as a k-algebra by two elements), at least if $k = \mathbf{R}$. The tangent vectors of A' at $t = -1, 0, 1$ are as in figure 2.

It is clear that one cannot draw a real plane curve with tangents at $t = -1, 0, 1$ in the indicated directions, and no singular points except the origin. If $k = C$ this topological argument does not seem to work, but A' still appears not to be a plane curve.

Our basic construction can be used if some tangent directions are equal. For example

$$A = \{f \in k[t] \mid f(1) = f(-1), f'(1) + f'(-1) = 0\}$$

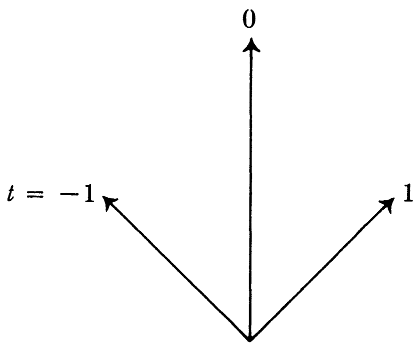

FIGURE 2.

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yields the tacnode $A = k[t^2, t^5 - 2t^3 + t]$. If instead we try

$$A' = \{ f \in k[t] \mid f(1) = f(-1), f'(1) = f'(-1)\}$$

then A' cannot be a plane curve (at least if $k = \mathbf{R}$), by a topological argument similar to that given above. Finally if we repeat one of the columns of C then b_i stays the same and the calculation in Section 5 shows that c_i is increased by one. Thus our method also gives non-ordinary singularities with nonsingular branches and decreasing hilbert function.

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