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Jordan ∗-Derivations of Finite-Dimensional Semiprime Algebras

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Abstract. In this paper, we characterize Jordan $*$-derivations of a 2-torsion free, finite-dimensional semiprime algebra R with involution $*$. To be precise, we prove the following. Let $\delta\colon R \to R$ be a Jordan $*$-derivation. Then there exists a $*$-algebra decomposition $R = U \oplus V$ such that both U and V are invariant under δ. Moreover, $*$ is the identity map of U and $\delta\mid_U$ is a derivation, and the Jordan $*$-derivation $\delta\mid_V$ is inner. We also prove the following. Let R be a noncommutative, centrally closed prime algebra with involution $*$, char $R \neq 2$, and let δ be a nonzero Jordan $*$-derivation of R. If δ is an elementary operator of R, then $\dim_C R < \infty$ and δ is inner.

1 Results

Throughout the paper, R always denotes an associative ring. An additive map $d\colon R \to R$ is called a *derivation* if $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. Let $*$ be an involution of R; that is, $*$ is an anti-automorphism of R satisfying $(x^*)^* = x$ for all $x \in R$. When R is an algebra over a field F, the involution $*$ is not necessarily F-linear in general. An additive mapping $\delta\colon R \to R$ is called a Jordan $*$-derivation if $\delta(x^2) = \delta(x)x^* + x\delta(x)$ for all $x \in R$. A Jordan $*$-derivation of R is said to be *inner* if it is of the form $x \mapsto xa - ax^*$ for some $a \in R$. For the motivation to study Jordan $*$-derivations, we refer the reader to the references in [3, 10].

In [2] Brešar and Vukman proved that if a unital $*$-ring R contains $\frac{1}{2}$ and a central invertible skew-hermitian element μ (*i.e.*, $\mu^* = -\mu$), then every Jordan $*$-derivation of R is inner. In particular, every Jordan $*$-derivation of a unital complex $*$-algebra is inner. In [10] Šemrl showed that every Jordan $*$-derivation of $\mathcal{B}(H)$, the algebra of all bounded linear operators on a real Hilbert space H, with $\dim_{\mathbb{R}} H > 1$ is inner (see also [3]). Clearly, the algebra $\mathcal{B}(H)$ is a prime ring with nonzero socle and is not a division ring if $\dim_{\mathbb{R}} H > 1$. The following $*$-version of [4, Theorem 1.2] gives a generalization of Šemrl's theorem.

Theorem 1.1 *Let R be a prime ring with involution $*$, char $R \neq 2$, and let $\delta\colon R \to R$ be a Jordan $*$-derivation. Suppose that R has nonzero socle but is not a division ring. Then there exists $a \in Q_s(R)$ such that $\delta(x) = xa - ax^*$ for all $x \in R$, where $Q_s(R)$ is the symmetric Martindale ring of quotients of R.*

In the theorem above, the case that R is a division ring is not yet solved. This paper is a continuation of the recent paper [4] concerning Jordan $*$-derivations. An ideal I

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of a ring (resp. algebra) R with involution $*$ is called a $*$-ideal if $I^* = I$. By a $*$-ring (resp. $*$-algebra) decomposition of R, we mean a ring (resp. algebra) decomposition $R = U \oplus V$, where U and V are $*$-ideals of R. The first purpose of this paper is to prove the following theorem.

Theorem 1.2 *Let R be a 2-torsion free, finite-dimensional, semiprime algebra with involution $*$ and let $\delta \colon R \to R$ be a Jordan $*$-derivation. Then there exists a $*$-algebra decomposition $R = U \oplus V$ such that U and V are invariant under δ. Moreover, $*$ is the identity map of U, $\delta\mid_U$ is a derivation, and the Jordan $*$-derivation $\delta\mid_V$ is inner.*

As an application of Theorem 1.2, we characterize Jordan $*$-derivations of a prime ring R when these $*$-derivations are "elementary operators". For simplicity of notation, we assume that the prime ring R is centrally closed; that is, $R = RC + C$, where C is the extended centroid of R. In this case, R is a prime algebra over C. By an elementary operator of R we mean an additive map $\phi \colon R \to R$, which is of the form $x \mapsto \sum_i a_i x b_i$ for $x \in R$, where a_i, b_i are finitely many elements in R. When $\dim_C R < \infty$, R is a finite-dimensional central simple C-algebra (see [6, Theorem 2 (p. 57)]). Let R^{op} denote the C-algebra opposite to the C-algebra R. It is known that there exists an isomorphism $\Phi \colon R \otimes_C R^{\mathrm{op}} \to \mathrm{End}_C(R)$ defined by

$$\Phi\left(\sum_i a_i \otimes b_i\right)(x) = \sum_i a_i x b_i$$

for $\sum_i a_i \otimes b_i \in R \otimes_C R^{\mathrm{op}}$ and $x \in R$. This implies that every C-linear map of R into itself is an elementary operator. In the next theorem, we prove that a centrally closed prime ring R must be finite-dimensional over C if it admits a nonzero Jordan $*$-derivation that is also an elementary operator. Although the theorem below has an analog in the case of semiprime rings, to avoid a lengthy argument we only prove the case of prime rings.

Theorem 1.3 *Let R be a noncommutative, centrally closed, prime algebra with involution $*$, $\operatorname{char} R \neq 2$, and let $\delta \colon R \to R$ be a nonzero Jordan $*$-derivation that is also an elementary operator. Then $\dim_C R < \infty$ and δ is inner.*

For $a, b \in R$, $[a, b]$ denotes the element $ab - ba$. Given two additive subgroups A and B of R, $[A, B]$ (resp. AB) will denote the additive subgroup of R generated by all elements $[a, b]$ (resp. ab) for $a \in A$ and $b \in B$.

2 Proof of Theorem 1.2

We first prove Theorem 1.2 with R a division algebra.

Theorem 2.1 *Let D be a noncommutative, finite-dimensional, central division C-algebra with involution $*$ and $\operatorname{char} D \neq 2$. Then every Jordan $*$-derivation of D is inner.*

We first recall a result due to Herstein. Suppose that L is a Lie ideal of a ring R; that is, L is an additive subgroup of R satisfying $[L, R] \subseteq L$. It follows from the proof of [5, Lemma 1.3] that $R[L,L]R \subseteq L + L^2$. Let D be as in Theorem 2.1. Since $[D,D]$ is a Lie ideal of D and $\big[[D,D],[D,D]\big] \neq 0$ by the fact that $\operatorname{char} D \neq 2$, we have $D = [D,D] + [D,D]^2$. We will use the result in the proof below. The involution $*$ of D is said to be of the first kind if $\beta^* = \beta$ for all $\beta \in C$. Otherwise, $*$ is said to be of the second kind.

Proof of Theorem 2.1 Let $\delta\colon D \to D$ be a nonzero Jordan $*$-derivation. Let $x, y \in D$. Expanding $\delta((x+y)^2) = \delta(x+y)(x+y)^* + (x+y)\delta(x+y)$, we see that

$$\delta(xy + yx) = \delta(x)y^* + \delta(y)x^* + x\delta(y) + y\delta(x). \tag{2.1}$$

Case 1. Suppose that $*$ is of the second kind. Choose a nonzero $\beta^* = -\beta \in C$. For $x \in D$, by (2.1) we have

$$2\delta(\beta x) = \delta(\beta x + x\beta) = \delta(\beta)x^* + \delta(x)(-\beta) + x\delta(\beta) + \beta\delta(x).$$

That is, $2\delta(\beta x) = \delta(\beta)x^* + x\delta(\beta)$. Replacing x by $\beta^{-1}x$, we see that $\delta(x) = xa - ax^*$ for all $x \in D$, where $a := \delta(\beta)/2\beta$.

Case 2. Suppose that $*$ is of the first kind. We claim that δ is C-linear. Fix a $\beta \in C$ and set $f(w) := \delta(\beta w) - \beta\delta(w)$ for $w \in D$. Let $x, y \in D$. By (2.1) we have

$$\delta((\beta x)y + y(\beta x)) = \delta(\beta x)y^* + \delta(y)\beta x^* + \beta x\delta(y) + y\delta(\beta x). \tag{2.2}$$

On the other hand,

$$\delta(x(\beta y) + (\beta y)x) = \delta(x)\beta y^* + \delta(\beta y)x^* + x\delta(\beta y) + \beta y\delta(x). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$f(x)y^* - f(y)x^* - xf(y) + yf(x) = 0. \tag{2.4}$$

Replacing y by 1 in (2.4) and using $\delta(1) = 0$, we see that

$$f(x) = bx^* + xb \tag{2.5}$$

for all $x \in D$, where $b := \delta(\beta)/2$. It follows from (2.4) that

$$(bx^* + xb)y^* - (by^* + yb)x^* - x(by^* + yb) + y(bx^* + xb) = 0.$$

That is, $b[x^*, y^*] = [x,y]b$, *i.e.*, $b[x,y]^* + [x,y]b = 0$ for all $x, y \in D$. By (2.5), we see that $f([D,D]) = 0$. That is,

$$\delta(\beta x) = \beta\delta(x) \quad \text{for all } x \in [D,D] \text{ and } \beta \in C. \tag{2.6}$$

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Let $u, v \in [D, D]$ and $\gamma \in C$. Then $u^*, v^* \in [D, D]$. By (2.1) and (2.6), we have

$$\begin{aligned}\delta\big(\gamma(uv + vu)\big) &= \delta\big((\gamma u)v + v(\gamma u)\big)\\ &= \delta(\gamma u)v^* + \delta(v)\gamma u^* + \gamma u\delta(v) + v\delta(\gamma u)\\ &= \gamma\big(\delta(u)v^* + \delta(v)u^* + u\delta(v) + v\delta(u)\big)\\ &= \gamma\delta(uv + vu).\end{aligned} \tag{2.7}$$

Since $uv - vu = [u, v] \in [D, D]$, it follows from (2.6) that

$$\delta(\gamma(uv - vu)) = \gamma\delta(uv - vu). \tag{2.8}$$

By (2.7) and (2.8), we have $\delta(\gamma uv) = \gamma\delta(uv)$ for all $u, v \in [D, D]$ and $\gamma \in C$. In view of the fact that $D = [D, D] + [D, D]^2$, every element of D is of the form $u + \sum_i u_i v_i$ for some $u \in [D, D]$ and finitely many $u_i, v_i \in [D, D]$; this implies that $\delta\colon D \to D$ is C-linear, as asserted.

Let $\overline{C}$ be the algebraic closure of C and let $\widehat{D} := D \otimes_C \overline{C}$. Then $\widehat{D} \cong \mathrm{M}_n(\overline{C})$ for some n. Moreover, $n > 1$, since D is not a field. Since $*$ is of the first kind, the involution $*$ of D can be extended to a first kind involution on $\widehat{D}$, also denoted by $*$, by the following rule:

$$\Big(\sum_i x_i \otimes \beta_i\Big)^* = \sum_i x_i^* \otimes \beta_i \quad \text{for } x_i \in D \text{ and } \beta_i \in \overline{C}.$$

Moreover, $\delta\colon D \to D$ can be extended to a well-defined map on $\widehat{D}$, also denoted by δ, by

$$\delta\Big(\sum_i x_i \otimes \beta_i\Big) = \sum_i \delta(x_i) \otimes \beta_i \quad \text{for } x_i \in D \text{ and } \beta_i \in \overline{C}.$$

Note that C is an infinite field. By the C-linearity of δ and the fact that $*$ is of the first kind, we claim that $\delta(y^2) = \delta(y)y^* + y\delta(y)$ for all $y \in \widehat{D}$.

Let $C[\lambda_1, \ldots, \lambda_m]$ denote the polynomial ring over C in commutative indeterminates $\lambda_1, \ldots, \lambda_m$, where $m := \dim_C D$. Choose a basis $\{e_1, \ldots, e_m\}$ for D over C. Write

$$e_i e_j = \sum_{k=1}^m \alpha_{ijk} e_k, \quad e_i^* = \sum_{k=1}^m \beta_{ik} e_k, \quad \text{and} \quad \delta(e_i) = \sum_{k=1}^m \gamma_{ik} e_k \tag{2.9}$$

for $1 \le i, j \le m$, where all $\alpha_{ijk}, \beta_{ik}, \gamma_{ik} \in C$. For $x \in D$, write

$$x = \sum_{i=1}^m \mu_i e_i \in D, \quad \text{where } \mu_1, \ldots, \mu_m \in C.$$

Using expansion formulas (2.9) to expand $\delta(x^2) - \delta(x)x^* - x\delta(x)$, we see that

$$\begin{aligned}0 &= \delta(x^2) - \delta(x)x^* - x\delta(x)\\ &= \sum_{s=1}^m p_s(\mu_1, \ldots, \mu_m)e_s = \sum_{s=1}^m e_s \otimes p_s(\mu_1, \ldots, \mu_m),\end{aligned} \tag{2.10}$$

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where

$$p_s(\lambda_1,\dots,\lambda_m) = \sum_{1\le i,j\le m}\sum_{k=1}^{m}\Big(\alpha_{ijk}\gamma_{ks} - \sum_{t=1}^{m}\alpha_{tks}\beta_{jk}\gamma_{it} - \alpha_{iks}\gamma_{jk}\Big)\lambda_i\lambda_j \in C[\lambda_1,\dots,\lambda_m]$$

for $s = 1,\dots,m$. Note that $p_s(\lambda_1,\dots,\lambda_m)$'s depend only on $\alpha_{ijk},\beta_{ik},\gamma_{ik}$. Since $p_s(\mu_1,\dots,\mu_m)\in C$ for all $\mu_i\in C$, it follows from (2.10) that $p_s(\mu_1,\dots,\mu_m)=0$ for $1\le i\le m$. Thus, $p_s(\lambda_1,\dots,\lambda_m)=0$ in the polynomial ring $C[\lambda_1,\dots,\lambda_m]$, since C is an infinite field.

In particular, we have $p_s(\nu_1,\dots,\nu_m)=0$ for all $\nu_i\in\overline{C}$. Thus,

$$\sum_{s=1}^{m} e_i\otimes p_s(\nu_1,\dots,\nu_m)=0$$

in $D\otimes_C\overline{C}$. Reversing the expansion of (2.10), we see that $\delta(y^2)=\delta(y)y^*+y\delta(y)$, where $y=\sum_{i=1}^{m}e_i\otimes\nu_i\in D\otimes_C\overline{C}$. This proves our claim.

Clearly, $\widehat{D}$ is a prime locally matrix ring (see [4, 7]). In view of [4, Theorem 1.1] or Theorem 1.1, there exists an element $\overline{c}\in\widehat{D}$ such that

$$\delta(x\otimes 1)=(x\otimes 1)\overline{c}-\overline{c}(x^*\otimes 1)$$

for all $x\in D$. Write $\overline{c}=a\otimes 1+c_1\otimes\gamma_1+\cdots$, where $a,c_i\in D$ and $1,\gamma_1,\dots$ are C-independent. This implies that

$$\big(\delta(x)-xa+ax^*\big)\otimes 1+(\,\cdot\,)\otimes\gamma_1+\cdots=0$$

for all $x\in D$. Thus, $\delta(x)=xa-ax^*$ for all $x\in D$. ■

By applying the same arguments as in the proof of Theorem 2.1, we have the following theorem.

Theorem 2.2 *Let D be a 2-torsion free, noncommutative, central division C-algebra with involution $*$. Suppose that there exists an extension field F of C such that $D\otimes_C F\cong M_s(\Delta)$ for some division F-algebra Δ and some $s>1$. Then every Jordan $*$-derivation of D is inner.*

We next deal with the case that R is a ring with exchange involution τ; that is, R has a ring decomposition $R=T\oplus T^{\mathrm{op}}$ with involution $(x,y)^\tau=(y,x)$ for $x,y\in T$, where T^{op} is the ring opposite to T.

Theorem 2.3 *Let R be a unital ring with exchange involution τ. Then every Jordan τ-derivation of R is inner.*

Proof Write $R = T \oplus T^{\text{op}}$ with involution $(x, y)^\tau = (y, x)$ for $x, y \in T$. Let $\delta\colon R \to R$ be a Jordan τ-derivation. Since R is a unital ring, $1_R = e_1 + e_2$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$, and 1 is the identity of the ring T. Thus, $e_1^\tau = e_2$. For $x = (x_1, x_2) \in R$, define $\widetilde{x} = (x_1, -x_2)$. Then

$$\delta\big((x_1 + 1, 0)^2\big) = \delta\big((x_1 + 1, 0)\big)(0, x_1 + 1) + (x_1 + 1, 0)\delta\big((x_1 + 1, 0)\big).$$

On the other hand,

$$\begin{aligned}\delta\big((x_1 + 1, 0)^2\big) &= \delta\big((x_1^2 + 2x_1 + 1, 0)\big)\\ &= \delta\big((x_1, 0)\big)(0, x_1) + (x_1, 0)\delta\big((x_1, 0)\big) + 2\delta\big((x_1, 0)\big) + \delta\big((1, 0)\big).\end{aligned}$$

Comparing the two equalities above, we see that

$$\delta\big((x_1, 0)\big) = (x_1, 0)\delta(e_1) + \delta(e_1)(0, x_1). \tag{2.11}$$

Similarly, we have

$$\delta\big((0, x_2)\big) = (0, x_2)\delta(e_2) + \delta(e_2)(x_2, 0) = -(0, x_2)\delta(e_1) - \delta(e_1)(x_2, 0), \tag{2.12}$$

where we have used the identity $\delta(e_2) = -\delta(e_1)$ at the second equality above. By (2.11) and (2.12), we have $\delta(x) = \widetilde{x}\delta(e_1) + \delta(e_1)(\widetilde{x})^\tau$ for all $x \in R$. A direct computation shows that $\delta(x) = xa - ax^\tau$ for all $x \in R$, where $a := \widetilde{\delta(e_1)}$. ■

Lemma 2.4 *Let N be a field with involution $*$, $\operatorname{char} N \neq 2$, and let $K = \{x \in N \mid x^* = -x\}$. Then the following hold:*

(i) *Every Jordan $*$-derivation of N is inner if $K \neq \{0\}$.*
(ii) *Every Jordan $*$-derivation of N is a derivation if $K = \{0\}$.*

Proof Let $\delta\colon N \to N$ be a Jordan $*$-derivation. Since N is a field, $\delta(x^2) = (x + x^*)\delta(x)$ for all $x \in N$.

Case 1. Suppose that $K \neq \{0\}$. Choose a nonzero $k \in K$. Let $x \in N$. Note that $\delta(k^2) = 0$. Thus,

$$\delta\big((x + k)^2\big) = (x + x^*)\delta(x + k),$$

implying that $2\delta(kx) = (x+x^*)\delta(k)$. Replacing x by $k^{-1}x$, we see that $\delta(x) = xa - ax^*$, where $a := \delta(k)/2k \neq 0$. This proves (i).

Case 2. Suppose that $K = \{0\}$; that is, $*$ is the identity map of N. Thus $\delta(x^2) = 2x\delta(x)$ for all $x \in N$. By the linearization on x, we get $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in N$. That is, δ is a derivation of N. ■

Proof of Theorem 1.2 Let R be a 2-torsion free, finite-dimensional, semiprime F-algebra with involution $*$, where F is a field and let $\delta\colon R \to R$ be a Jordan $*$-derivation. By the Wedderburn–Artin Theorem,

$$R = W_1 \oplus W_2 \oplus \cdots \oplus W_t,$$

where all W_i are finite-dimensional, simple, Artinian F-algebras. Note that W_i's are the only minimal ideals of R. Thus, for each W_i, either $W_i^* = W_i$ (that is, W_i is a $*$-ideal of R) or $W_i^* = W_j$ for some $j \neq i$.

Case 1. Suppose that $W_i^* = W_i$. Then W_i is a 2-torsion free, finite-dimensional, simple F-algebra with involution $*$. Note that, as an additive group, W_i is generated by elements x^2 for $x \in W_i$, since $2W_i = W_i$ and $2x = (x + e_i)^2 - x^2 - e_i$, where e_i denotes the identity of W_i. This implies that $\delta(W_i) \subseteq W_i$. Thus $\delta\colon W_i \to W_i$ is a Jordan $*$-derivation. By the Wedderburn–Artin Theorem, $W_i \cong \mathrm{M}_s(\Delta)$ for some division algebra Δ and some integer $s \geq 1$. If $s > 1$, then δ is inner according to [4, Theorem 1.2]. If $s = 1$, then δ is inner on W_i unless W_i is a field and $*$ is the identity map on W_i (see Theorem 2.1 and Lemma 2.4). By Lemma 2.4, $\delta\colon W_i \to W_i$ is a derivation when $*$ is the identity map on W_i.

Case 2. Suppose that $W_i^* = W_j$ for some $j \neq i$. Let $T := W_i \oplus W_j = W_i \oplus W_i^*$. Since T is generated by elements x^2 for $x \in T$ as an additive group, T is invariant under δ. In fact, $T \cong W_i \oplus W_i^{\mathrm{op}}$ via the map $\phi\colon T \to W_i \oplus W_i^{\mathrm{op}}$ defined by

$$\phi(x + y^*) = (x, y) \text{ for } x, y \in W_i.$$

Let τ denote the exchange involution on $W_i \oplus W_i^{\mathrm{op}}$; that is, $(x, y)^\tau = (y, x)$ for $x, y \in W_i$. Then $\phi(z^*) = \phi(z)^\tau$ for $z \in T$. Thus, T is isomorphic to $W_i \oplus W_i^{\mathrm{op}}$ as rings with involution. In view of Theorem 2.3, δ is inner on T.

Let

$$\Gamma = \{i \mid W_i \text{ is a field and } * \text{ is the identity map on } W_i\}.$$

Set $U = \bigoplus_{i\in\Gamma} W_i$ and $V = \bigoplus_{j\notin\Gamma} W_j$. Clearly, U and V are $*$-ideals of R and are invariant under δ. Moreover, δ is a derivation on U by Lemma 2.4 and the Jordan $*$-derivation δ on V is inner by Theorems 2.1 and 2.3 and Lemma 2.4. This proves the theorem. ■

Using the proofs above, we can establish an analog of Theorem 1.2 in the context of semiprime Artinian rings. Recall that a semiprime Artinian ring R is the direct sum of finitely many simple Artinian rings. These simple Artinian rings are the only minimal ideals of R, which are called the components of R. A division component I of R means that the component I is itself a division ring.

Theorem 2.5 *Let R be a 2-torsion free, semiprime, Artinian ring with involution $*$ such that every $*$-invariant division component of R is a finite-dimensional central division algebra. Suppose that $\delta\colon R \to R$ is a Jordan $*$-derivation. Then there exists a $*$-ring decomposition $R = U \oplus V$ such that U and V are invariant under δ. Moreover, $*$ is the identity map of U, $\delta\mid_U$ is a derivation, and the Jordan $*$-derivation $\delta\mid_V$ is inner.*

3 Proof of Theorem 1.3

Throughout this section, R always denotes a noncommutative, centrally closed, prime ring with involution $*$. Thus, R is a prime C-algebra, where C is the extended centroid of R. In order to prove Theorem 1.3, we need the well-known result of Martindale [9, Theorem 2(a)], stated below in a form convenient for our purpose.

Lemma 3.1 *Let $a_i, b_i, c_j, d_j \in R$ be such that $\sum_{i=1}^{\ell} a_i x b_i + \sum_{j=1}^{m} c_j x d_j = 0$ for all x in a nonzero ideal of R. If $a_1, \ldots, a_\ell$ are linearly independent over C, then each b_i is a C-linear combination of the d_j's. Analogously, if $b_1, \ldots, b_\ell$ are linearly independent over C, then each a_i is a C-linear combination of the c_j's.*

Applying the same argument as given in the proof of [8, Lemma 2.6], after replacing Q by R, we have the following lemma.

Lemma 3.2 *Let $a_1, \ldots, a_n \in R$ be C-independent. If $\dim_C R \geq \frac{n^2(n+5)^2}{4}$, then there exists $y \in R$ such that $a_1, \ldots, a_n, a_1 y, \ldots, a_n y$ are C-independent.*

Lemma 3.3 *Suppose that the map $x \mapsto cx^*$ for $x \in R$ is an elementary operator of R, where c is a fixed nonzero element of R. Then $\dim_C R < \infty$.*

Proof Suppose on the contrary that $\dim_C R = \infty$. Write $cx^* = \sum_{i=1}^{n} a_i x b_i$ for all $x \in R$, where a_i, b_i are fixed elements in R and n is a positive integer. Choose n to be minimal. Then $a_1, \ldots, a_n$ are C-independent.

Let $x, y \in R$. Then $c(xy)^* = \sum_{i=1}^{n} a_i x y b_i$. On the other hand, $c(xy)^* = cy^* x^* = \sum_{i=1}^{n} a_i y b_i x^*$. Thus,

$$\sum_{i=1}^{n} (a_i x) y b_i = \sum_{i=1}^{n} a_i y (b_i x^*). \tag{3.1}$$

In view of Lemma 3.2, there exists $x_0 \in R$ such that $a_1, \ldots, a_n, a_1 x_0, \ldots, a_n x_0$ are C-independent. By (3.1) we have $\sum_{i=1}^{n} (a_i x_0) y b_i = \sum_{i=1}^{n} a_i y (b_i x_0^*)$ for all $y \in R$. It follows from Lemma 3.1 that $b_i = 0$ for all i, a contradiction. ■

Proof of Theorem 1.3 Suppose on the contrary that $\dim_C R = \infty$. Since $\delta \colon R \to R$ is an elementary operator, there exist finitely many $a_i, b_i \in R$, $1 \leq i \leq m$, such that $\delta(x) = \sum_{i=1}^{m} a_i x b_i$ for all $x \in R$. Let $x, y \in R$. Since δ is Jordan $*$-derivation of R, it follows from (2.1) that

$$\sum_{i=1}^{m} a_i (xy + yx) b_i = \delta(x) y^* + \sum_{i=1}^{m} a_i y b_i x^* + x \sum_{i=1}^{m} a_i y b_i + y \delta(x),$$

implying that

$$\sum_{i=1}^{m} [a_i, x] y b_i + \sum_{i=1}^{m} a_i y (x b_i - b_i x^*) - y \delta(x) = \delta(x) y^*. \tag{3.2}$$

Choose a basis $1 = c_0, c_1, \ldots, c_t$ for the C-space $C + \sum_{i=1}^{m} C a_i$ and a basis $e_1, \ldots, e_s$ for the C-space $\sum_{i=1}^{m} C b_i$. We rewrite (3.2) as

$$\sum_{i=1}^{s} [d_i, x] y e_i + c_0 y \big(x h_0 - h_0 x^* - \delta(x) \big) + \sum_{j=1}^{t} c_j y (x h_j - h_j x^*) = \delta(x) y^*$$

for all $x, y \in R$, where all $d_i, h_j \in R$ are fixed. Write

$$f_0(x) := xh_0 - h_0x^* - \delta(x) \quad \text{and} \quad f_j(x) := xh_j - h_jx^* \text{ for } j = 1, \dots, t.$$

Then we have

$$\sum_{i=1}^{s}[d_i, x]ye_i + \sum_{j=0}^{t} c_j y f_j(x) = \delta(x)y^* \tag{3.3}$$

for all $x, y \in R$. Let $x, y, z \in R$. By (3.3), we see that

$$\delta(x)(yz)^* = \delta(x)z^*y^* = \sum_{i=1}^{s}[d_i, x]ze_iy^* + \sum_{j=0}^{t} c_jzf_j(x)y^*.$$

On the other hand, $\delta(x)(yz)^* = \sum_{i=1}^{s}[d_i, x]yze_i + \sum_{j=0}^{t} c_jyzf_j(x)$. Thus,

$$\sum_{j=0}^{t}(c_jy)zf_j(x) - \sum_{j=0}^{t} c_jz(f_j(x)y^*) = \sum_{i=1}^{s}[d_i, x]z(e_iy^*) - \sum_{i=1}^{s}\big([d_i, x]y\big)ze_i. \tag{3.4}$$

Since $\dim_C R = \infty$, it follows from Lemma 3.2 that $e_0, \dots, e_s, e_0y_0^*, \dots, e_sy_0^*$ are C-independent for some $y_0 \in R$. By (3.4), we have

$$\sum_{j=0}^{t}(c_jy_0)zf_j(x) - \sum_{j=0}^{t} c_jz(f_j(x)y_0^*) = \sum_{i=1}^{s}[d_i, x]z(e_iy_0^*) - \sum_{i=1}^{s}([d_i, x]y_0)ze_i.$$

By Lemma 3.1, $[d_i, R] \subseteq \sum_{j=0}^{t} C(c_jy_0) + \sum_{j=0}^{t} Cc_j$ for $i = 1, \dots, s$. Since $\dim_C R = \infty$, it follows from [1, Theorem 2] that $d_i \in C$ for $i = 1, \dots, s$. Thus, (3.4) is reduced to

$$\sum_{j=0}^{t}(c_jy)zf_j(x) - \sum_{j=0}^{t} c_jz(f_j(x)y^*) = 0. \tag{3.5}$$

By Lemma 3.2 again, there exists $y_0 \in R$ such that $c_0, \dots, c_t, c_0y_0, \dots, c_ty_0$ are C-independent. By (3.5), we have

$$\sum_{j=0}^{t}(c_jy_0)zf_j(x) - \sum_{j=0}^{t} c_jz\big(f_j(x)y_0^*\big) = 0$$

for all $x, z \in R$. In view of Lemma 3.1, $f_j(x) = 0$ for all $x \in R$ and all j. In particular, $f_0 = 0$; that is, $\delta(x) = xh_0 - h_0x^*$ for all $x \in R$. Since δ is a nonzero elementary operator, $h_0 \neq 0$ and the map $x \mapsto h_0x^*$ for $x \in R$ is also an elementary operator of R. In view of Lemma 3.3, $\dim_C R < \infty$, a contradiction.

Up to now, we have proved that $\dim_C R < \infty$. In view of Posner's Theorem [6, Theorem 2 (p. 57)] R is a finite-dimensional central simple C-algebra. Thus, by Theorem 1.2, the Jordan ∗-derivation δ of R is inner, as asserted. ■

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