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First Variations of the Best Sobolev Trace Constant with Respect to the Domain

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Abstract. In this paper we study the best constant of the Sobolev trace embedding $H^1(\Omega) \to L^2(\partial\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^N$. We find a formula for the first variation of the best constant with respect to the domain. As a consequence, we prove that the ball is a critical domain when we consider deformations that preserve volume.

1 Introduction

Sobolev inequalities have been studied by many authors and is by now a classical subject. It at least goes back to Aubin [2]; for more references see [4]. The Sobolev trace inequality is relevant for the study of boundary value problems for differential operators. The Sobolev trace embedding has been intensively studied; see for example, [3, 5–9, 14–16].

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain (C^2 is enough for our arguments); we have a compact inclusion $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$. Hence there exists a constant S such that $S\|u\|^2_{L^2(\partial\Omega)} \leq \|u\|^2_{H^1(\Omega)}$. The best Sobolev trace constant is the largest S such that the inequality holds, *i.e.,*

$$S(\Omega) = \inf_{v \in H^1(\Omega) \setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 + |v|^2 \, dx}{\int_{\partial\Omega} |v|^2 \, d\sigma}. \tag{1.1}$$

The embedding is compact, so we have existence of extremals, *i.e.,* functions where the infimum is attained. The extremal is strictly positive in Ω and smooth up to the boundary; see [5]. These extremals, normalized with

$$\int_{\partial\Omega} u^2 = 1, \tag{1.2}$$

are weak solutions of the following Steklov eigenvalue problem:

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = S(\Omega) u & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

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where $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative. In the rest of this article we will assume that the extremals are normalized according to (1.2).

Our main goal in this paper is to look at the dependence of the best Sobolev trace constant on the domain. We study the first variation (also known as the shape derivative) of the best Sobolev trace constant with respect to the domain.

Let us summarize what is known for (1.1). The constant $S(\Omega)$ is not homogeneous under dilatations of the domain. The asymptotic behavior of $S(\mu\Omega)$ and the extremals in expanding ($\mu \to \infty$), and contracting domains ($\mu \to 0$), were studied in [5, 7, 8, 14]. In [8], it was shown that

$$\lim_{\mu \to 0+} \frac{S(\mu\Omega)}{\mu} = \frac{|\Omega|}{|\partial\Omega|}. \tag{1.4}$$

It was proved in [16] that if Ω is a ball, then the extremals are radial functions. Moreover, in a ball the best constant $S(r) = S(B_r)$ is not explicit, but it can be obtained from the fact that it verifies the ODE $S' = 1 - S^2 - (N-1)S/r$ with $S(0) = 0$; see [14].

To study the variations of the best Sobolev trace constant with respect to variations of the domain we consider a smooth map (C^2 is enough) $V \colon \mathbb{R}^N \to \mathbb{R}^N$, the deformation field, and for small $t \in \mathbb{R}$, the perturbed domains

$$\Omega_t = (\mathrm{Id} + tV)(\Omega) = \{x + tV(x), x \in \Omega\}.$$

Let us denote by $S(\Omega_t)$ the best Sobolev trace constant of Ω_t. Our main result is the following.

Theorem 1.1 *The function $S(\Omega_t)$ is differentiable with respect to t at $t = 0$, and one has*

$$S'(\Omega) = -\int_{\partial\Omega} \left(|\nabla_T u|^2 + u^2 - S(\Omega)Hu^2 - S^2(\Omega)u^2\right) \langle V, \nu \rangle. \tag{1.5}$$

Here H is the mean curvature of $\partial\Omega$, and ∇_T is the gradient with respect to the tangential variables.

Maximization or minimization of eigenvalues is an active subject of research; see the survey [11].

Concerning (1.1), if we take $u \equiv 1$ in (1.1), we obtain $S(\Omega) \le |\Omega|/|\partial\Omega|$. Therefore if we consider the class of domains with fixed volume, $\Lambda = \{\Omega : |\Omega| = \lambda\}$, there is no minimum of $S(\Omega)$ in this class.

From the limit (1.4), we get that for fixed Ω and $B(0, R)$ belonging to Λ, there exists μ_0 such that for every $\mu < \mu_0$ we have $S(\mu\Omega) \le S(\Omega)(\mu B(0, R))$. This suggests that the ball maximizes $S(\Omega)$ among the sets of a given area. To support this conjecture, we obtain as a direct corollary of our main result that the ball is a critical domain when we restrict ourselves to deformations in the class Λ.

Corollary 1.2 *One has $S'(B(0, R)) = 0$ for every deformation V such that* $\operatorname{div} V = 0$, *that is, we consider deformations that preserve volume at the first variation.*

2 Proofs of the Results

To prove Theorem 1.1, we will need to differentiate integrals of the form

$$J_1(\Omega) = \int_\Omega f(u) \quad \text{and} \quad J_2(\Omega) = \int_{\partial\Omega} g(u).$$

If u, f and g are smooth, these functionals are differentiable with respect to t, that is, there exist $dJ_i = \lim_{t\to 0}(J_i(t) - J_i(0))/t$. (See [12] for the details and precise assumptions on u, f and g.) In the following theorem explicit formulas for the derivatives are computed; we refer again to [12] for the proof.

Proposition 2.1 *The shape derivatives of J_1 and J_2 are given by*

$$dJ_1 = \int_\Omega f'(u)u' + \int_{\partial\Omega} f(u)\langle V,\nu\rangle,$$

$$dJ_2 = \int_{\partial\Omega} g'(u)u' + \int_{\partial\Omega} Hg(u)\langle V,\nu\rangle + \int_{\partial\Omega} \frac{\partial g(u)}{\partial\eta}\langle V,\nu\rangle.$$

Here H stands for the mean curvature of $\partial\Omega$.

Proof of Theorem 1.1 Remark that we are facing an eigenvalue problem of Stecklov type, (1.3). The first eigenvalue λ_1 coincides with $S_2(\Omega)$, [8]. This first eigenvalue turns out to be simple (see [16]) and therefore it will be differentiable. The proof of this fact uses the general theory for families of operators of type A developed in [13, Ch. VII].

Now, to prove our main result, the expression for $S'(\Omega)$ in Theorem 1.1, we first find a problem that is satisfied by the derivative u', namely

$$\begin{cases} \Delta u' = u' & \text{in } \Omega, \\ \frac{\partial u'}{\partial\nu} = \nabla_T u \nabla_T \langle V,\nu\rangle + S'(\Omega)u + S(\Omega)u' & \\ \qquad - \frac{\partial^2 u}{\partial\nu^2}\langle V,\nu\rangle + S(\Omega)\frac{\partial u}{\partial\nu}\langle V,\nu\rangle, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Next we use this expression together with the weak form of the equation (1.3) verified by u to compute the formula for the derivative of $S(\Omega)$ (1.5). All these computations can be made rigorous using the results of [10, 12].

To obtain (2.1), we observe that the extremal u is a weak solution of (1.3), *i.e.,*

$$\int_\Omega \nabla u \nabla v + \int_\Omega uv = S(\Omega)\int_{\partial\Omega} uv,$$

for every $v \in H^1(\Omega)$. Using Theorem 2.1, we have

$$\begin{aligned} \int_\Omega \nabla u' \nabla v + \int_\Omega u'v + \int_{\partial\Omega} \nabla u \nabla v \langle V,\nu\rangle + \int_{\partial\Omega} uv\langle V,\nu\rangle \\ = S'(\Omega)\int_{\partial\Omega} uv + S(\Omega)\int_{\partial\Omega} u'v + S(\Omega)\int_{\partial\Omega} Huv\langle V,\nu\rangle \\ + S(\Omega)\int_{\partial\Omega} \frac{\partial(uv)}{\partial\nu}\langle V,\nu\rangle. \end{aligned} \tag{2.2}$$

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Taking v with compact support in Ω we obtain that u' verifies

$$\Delta u' = u', \quad \text{in } \Omega. \tag{2.3}$$

Going back to (2.2) and integrating by parts we get

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u'}{\partial \nu} v + \int_{\partial\Omega} \nabla u \nabla v \langle V, \nu\rangle + \int_{\partial\Omega} uv\langle V, \nu\rangle \\ = S'(\Omega)\int_{\partial\Omega} uv + S(\Omega)\int_{\partial\Omega} u'v + S(\Omega)\int_{\partial\Omega} Huv\langle V, \nu\rangle \\ + S(\Omega)\int_{\partial\Omega} \frac{\partial(uv)}{\partial \nu}\langle V, \nu\rangle.\end{aligned}$$

As $\nabla u \nabla v = \nabla_T u \nabla_T v + u_\nu v_\nu$, we obtain,

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u'}{\partial \nu} v + \int_{\partial\Omega} \nabla_T u \nabla_T v \langle V, \nu\rangle + \int_{\partial\Omega} uv\langle V, \nu\rangle \\ = S'(\Omega)\int_{\partial\Omega} uv + S(\Omega)\int_{\partial\Omega} u'v + S(\Omega)\int_{\partial\Omega} Hu^{q-1}v\langle V, \nu\rangle \\ + S(\Omega)\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\langle V, \nu\rangle.\end{aligned} \tag{2.4}$$

Using that

$$\int_{\partial\Omega} \nabla_T u \nabla_T v \langle V, \nu\rangle = -\int_{\partial\Omega} \Delta_T u v\langle V, \nu\rangle - \int_{\partial\Omega} \nabla_T u \nabla_T \langle V, \nu\rangle v,$$

and that

$$\Delta u = \Delta_T u + H\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2}, \tag{2.5}$$

equation (2.4) becomes

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u'}{\partial \nu} v = \int_{\partial\Omega} \nabla_T u \nabla_T \langle V, \nu\rangle v + S'(\Omega)\int_{\partial\Omega} uv + S(\Omega)\int_{\partial\Omega} u'v \\ - \int_{\partial\Omega} \frac{\partial^2 u}{\partial \nu^2} v\langle V, \nu\rangle + S(\Omega)\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\langle V, \nu\rangle.\end{aligned} \tag{2.6}$$

Therefore,

$$\begin{aligned}\frac{\partial u'}{\partial \nu} = \nabla_T u \nabla_T \langle V, \nu\rangle + S'(\Omega)u + S(\Omega)u' \\ - \frac{\partial^2 u}{\partial \nu^2}\langle V, \nu\rangle + S(\Omega)\frac{\partial u}{\partial \nu}\langle V, \nu\rangle, \quad \text{on } \partial\Omega.\end{aligned} \tag{2.7}$$

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Equations (2.3) and (2.7) give (2.1).

Now we find the expression for $S'(\Omega)$. From $\Delta u' = u'$ and $\Delta u = u$ we obtain

$$\int_{\partial\Omega} \frac{\partial u'}{\partial \nu} u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u'.$$

Hence,

$$\int_{\partial\Omega} \nabla_T u \nabla_T \langle V, \nu\rangle u + S'(\Omega) \int_{\partial\Omega} u^2 + S(\Omega) \int_{\partial\Omega} uu' \\ - \int_{\partial\Omega} \frac{\partial^2 u}{\partial \nu^2} u\langle V, \nu\rangle + S(\Omega) \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \langle V, \nu\rangle = S(\Omega) \int_{\partial\Omega} uu'.$$

Using that we have normalized u by (1.2), that is $\int_{\partial\Omega} u^2 = 1$, we get

$$S'(\Omega) = -\int_{\partial\Omega} \nabla_T u \nabla_T \langle V, \nu\rangle u + \int_{\partial\Omega} \frac{\partial^2 u}{\partial \nu^2} u\langle V, \nu\rangle - S(\Omega) \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \langle V, \nu\rangle. \tag{2.8}$$

Now we just differentiate our normalization constraint $\int_{\partial\Omega} u^2 = 1$ to obtain

$$0 = 2\int_{\partial\Omega} uu' + \int_{\partial\Omega} Hu^2 \langle V, \nu\rangle + \int_{\partial\Omega} \frac{\partial u^2}{\partial \nu} \langle V, \nu\rangle.$$

With this in mind, (2.8) can be written as

$$S'(\Omega) = -\int_{\partial\Omega} \nabla_T u \nabla_T \langle V, \nu\rangle u - S^2(\Omega) u^2 + \frac{\partial^2 u}{\partial \nu^2} u\langle V, \nu\rangle.$$

Finally, using again (2.5) and integrating by parts on $\partial\Omega$ we obtain

$$S'(\Omega) = -\int_{\partial\Omega} \left(|\nabla_T u|^2 + u^2 - S(\Omega) Hu^2 - S^2(\Omega) u^2\right) \langle V, \nu\rangle,$$

as we wanted to prove. ■

Proof of Corollary 1.2 Now we take $\Omega = B(0, R)$. Our first step is to recall that from the results of [16] the extremals are radial. Since u is constant on the boundary, we have that there exist constants c_1, c_2 and c_3 such that

$$\nabla_T u = 0, \quad u = c_1, \quad \frac{\partial u}{\partial \nu} = c_2 \quad \text{and} \quad \frac{\partial^2 u}{\partial \nu^2} = c_3, \tag{2.9}$$

on the boundary $\partial B(0, R)$. Now we observe that since the deformation preserves volume at the first variation, V verifies $\operatorname{div}(V) = 0$ in Ω. Then

$$0 = \int_\Omega \operatorname{div}(V) = \int_{\partial\Omega} \langle V, \nu\rangle. \tag{2.10}$$

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Hence, from (2.9) and (2.10), we obtain

$$\int_{\partial B(0,R)} \nabla_T u \nabla_T \langle V, \nu\rangle u = 0, \quad \int_{\partial B(0,R)} \frac{\partial^2 u}{\partial \nu^2} u \langle V, \nu\rangle = 0,$$

$$S(B(0,R)) \int_{\partial B(0,R)} u \frac{\partial u}{\partial \nu} \langle V, \nu\rangle = 0.$$

So, from (1.5), we have $S'(B(0,R)) = 0$. The proof is complete. ∎

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