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EIGENVALUES OF THE LAPLACIAN FOR THE THIRD BOUNDARY VALUE PROBLEM

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Abstract

The spectral function $\theta(t) = \sum_{n=1}^{\infty} \exp^{(-\lambda_n t)}$, where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of the two-dimensional Laplacian, is studied for a variety of domains. The dependence of $\theta(t)$ on the connectivity of a domain and the impedance boundary conditions is analysed. Particular attention is given to a doubly-connected region together with the impedance boundary conditions on its boundaries.

1. Introduction

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues λ_n for the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in the xy-plane.

Let $D \subseteq R^2$ be a bounded domain with a smooth boundary ∂D. Consider the impedance problem

$$(\Delta + \lambda)u = 0 \quad \text{in } D, \qquad \left(\frac{\partial}{\partial n} + \gamma\right)u = 0 \quad \text{on } \partial D, \tag{1.1}$$

where $\partial/\partial n$ denotes differentiation along the inward pointing normal to ∂D, γ is a positive constant and $u \in C^2(D) \cap C(\overline{D})$. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots \to \infty \quad \text{as } n \to \infty.$$

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The problem of determining the geometry of D (simply connected) and the impedance γ has been discussed recently in [6] from the asymptotic behaviour of the spectral function $\theta(t) = \operatorname{tr}(\exp^{(-\Delta t)}) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)$ for small positive t. Problem (1.1) has been investigated in [3], [5], [9] in the following special cases:

Case 1. $\gamma = 0$ (Neumann Problem)

$$\theta(t) \sim \frac{\text{area } D}{4\pi t} + \frac{\text{length } \partial D}{8(\pi t)^{1/2}} + a_0 + \frac{7t^{1/2}}{256\pi^{1/2}} \int_{\partial D} (k(\sigma))^2 \, d\sigma + O(t) \text{ as } t \to 0, \tag{1.2}$$

Case 2. $\gamma \to \infty$ (Dirichlet Problem)

$$\theta(t) \sim \frac{\text{area } D}{4\pi t} - \frac{\text{length } \partial D}{8(\pi t)^{1/2}} + a_0 + \frac{t^{1/2}}{256\pi^{1/2}} \int_{\partial D} (k(\sigma))^2 \, d\sigma + O(t) \quad \text{as } t \to 0, \tag{1.3}$$

where $k(\sigma)$ is the curvature of the boundary ∂D. The constant term a_0 has geometric significance, e.g. if D is smooth and convex, then $a_0 = 1/6$ and if D is permitted to have a finite number "h" of smooth convex holes, then $a_0 = (1 - h)/6$.

The object of this paper is to discuss the following problem: let

$$D = \{(r, \theta) \colon a \leqslant r \leqslant b, 0 \leqslant \theta \leqslant 2\pi\}$$

be a circular annulus. Suppose that the eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ are given for the impedance problem

$$(\Delta + \lambda)u = 0 \quad \text{in } D, \qquad \left(\frac{\partial u}{\partial r} + \gamma_1 u\right)_{r=a} = \left(\frac{\partial u}{\partial r} + \gamma_2 u\right)_{r=b} = 0, \tag{1.4}$$

where γ_1 and γ_2 are positive constants. The basic problem is that of determining the geometry of the circular annulus D as well as the impedances γ_1 and γ_2 from the asymptotic behaviour of $\theta(t)$ for small positive t.

Problem (1.4) has been investigated in [7] in the following special cases:

Case 1. $\gamma_1 = \gamma_2 = 0$

$$\theta(t) \sim \frac{b^2 - a^2}{4t} + \frac{\pi^{1/2}(a + b)}{4t^{1/2}} + O(t^{1/2}) \quad \text{as } t \to 0. \tag{1.5}$$

Case 2. $\gamma_1 = 0$, $\gamma_2 \to \infty$

$$\theta(t) \sim \frac{b^2 - a^2}{4t} + \frac{\pi^{1/2}(a - b)}{4t^{1/2}} + O(t^{1/2}) \quad \text{as } t \to 0. \tag{1.6}$$

Case 3. $\gamma_1 \to \infty$, $\gamma_2 = 0$

$$\theta(t) \sim \frac{b^2 - a^2}{4t} + \frac{\pi^{1/2}(b-a)}{4t^{1/2}} + O(t^{1/2}) \quad \text{as } t \to 0. \tag{1.7}$$

Case 4. $\gamma_1 = \gamma_2 \to \infty$

$$\theta(t) \sim \frac{b^2 - a^2}{4t} - \frac{\pi^{1/2}(a+b)}{4t^{1/2}} + O(t^{1/2}) \quad \text{as } t \to 0. \tag{1.8}$$

A restricted form of the results (1.8) and (1.5) has been obtained recently in [1, 2].

With reference to (1.2), (1.3), an examination of the results (1.5)–(1.8) shows that the coefficient of $(4\pi t)^{-1}$ determines the area of the annulus D and the coefficient of $(\pi t)^{-1/2}/8$ determines the total length of its boundary. We note that the constant term a_0 is zero because our domain has only one hole (i.e., $h = 1$).

2. Formulation of the mathematical problem

Following the method of Kac [3] and following closely the procedure of Section 2 in [7], it is easy to show that the spectral function $\theta(t)$ is given by

$$\theta(t) = \int\int_D G(\mathbf{x}, \mathbf{x};\ t)\, d\mathbf{x}, \tag{2.1}$$

where $G(\mathbf{x}, \mathbf{x}';\ t)$ is the Green's function for the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right)u = 0 \tag{2.2}$$

subject to the impedance boundary conditions of (1.4) and the initial condition $G(\mathbf{x}, \mathbf{x}';\ t) \to \delta(\mathbf{x} - \mathbf{x}')$ as $t \to 0$, where $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function located at the source point $\mathbf{x} = \mathbf{x}'$. Let us write

$$G(\mathbf{x}, \mathbf{x}';\ t) = G_0(\mathbf{x}, \mathbf{x}';\ t) + \chi(\mathbf{x}, \mathbf{x}';\ t), \tag{2.3}$$

where

$$G_0(\mathbf{x}, \mathbf{x}';\ t) = (4\pi t)^{-1}\exp\left\{-|\mathbf{x} - \mathbf{x}'|^2/4t\right\} \tag{2.4}$$

is the "fundamental solution" of the heat equation (2.2), while $\chi(\mathbf{x}, \mathbf{x}';\ t)$ is a "regular solution" chosen in such a way that $G(\mathbf{x}, \mathbf{x}';\ t)$ satisfies the impedance boundary conditions of (1.4).

On setting $\mathbf{x} = \mathbf{x}'$ we find that

$$\theta(t) = \frac{b^2 - a^2}{4t} + K(t), \tag{2.5}$$

where

$$K(t) = \int\int_D \chi(\mathbf{x}, \mathbf{x};\, t)\, d\mathbf{x}. \tag{2.6}$$

The problem now is to determine the asymptotic expansion of $K(t)$ for small positive t. In what follows we shall use Laplace transforms with respect to "t", and use s^2 as the Laplace transform parameter; thus

$$\overline{G}(\mathbf{x}, \mathbf{x}';\, s^2) = \int_0^\infty e^{-s^2 t} G(\mathbf{x}, \mathbf{x}';\, t)\, dt. \tag{2.7}$$

An application of the Laplace transform to the heat equation (2.2) shows that $\overline{G}(\mathbf{x}, \mathbf{x}';\, s^2)$ satisfies the membrane equation

$$(\Delta - s^2)\overline{G}(\mathbf{x}, \mathbf{x}';\, s^2) = -\delta(\mathbf{x} - \mathbf{x}') \quad \text{in } D, \tag{2.8}$$

together with the impedance boundary conditions of (1.4). The asymptotic expansion of $K(t)$ for $t \to 0$ may then be deduced directly from the asymptotic expansion of $\overline{K}(s^2)$ for $s \to \infty$, where

$$\overline{K}(s^2) = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r\overline{\chi}(r, \theta, r, \theta;\, s^2)\, dr\, d\theta. \tag{2.9}$$

3. Construction of Green's function

It is well known that the membrane equation (2.8) has the fundamental solution

$$\begin{aligned}\overline{G}_0(r, \theta, r', \theta';\, s^2) &= \frac{1}{2\pi} K_0(s|\mathbf{x} - \mathbf{x}'|)\\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} I_m(sr') K_m(sr) \cos[m(\theta - \theta')], \end{aligned} \tag{3.1}$$

where K_0 is the modified Bessel function of the second kind and of zero order, (see, for example [8]).

On solving the membrane equation (2.8) we deduce that if $r' \leqslant r \leqslant b$,

$$\begin{aligned}&\overline{G}(r, \theta, r', \theta';\, s^2)\\ &= \sum_{m=-\infty}^{\infty} \left\{\frac{1}{2\pi} K_m(sr) I_m(sr') + A_m K_m(sr) + B_m I_m(sr)\right\} \cos[m(\theta - \theta')],\end{aligned} \tag{3.2}$$

and if $r' \geqslant r \geqslant a$,

$$\bar{G}(r,\theta,r',\theta';s^2) = \sum_{m=-\infty}^{\infty}\left\{\frac{1}{2\pi}K_m(sr')I_m(sr) + A_mK_m(sr) + B_mI_m(sr)\right\}\cos[m(\theta-\theta')], \tag{3.3}$$

where A_m and B_m are constants to be determined.

Consequently, it is straightforward to show that at $r' = r$ and $\theta' = \theta$ the equation (2.8) has the regular solution

$$\begin{aligned}\bar{\chi}(r,\theta,r,\theta;s^2) = \sum_{m=-\infty}^{\infty}\frac{1}{2\pi R_m}\{&[sI'_m(sa)+\gamma_1I_m(sa)]\\ &\times[sI'_m(sb)+\gamma_2I_m(sb)]K_m^2(sr)\\ &-2[sI'_m(sa)+\gamma_1I_m(sa)]\\ &\times[sK'_m(sb)+\gamma_2K_m(sb)]I_m(sr)K_m(sr)\\ &+[sK'_m(sa)+\gamma_1K_m(sa)]\\ &\times[sK'_m(sb)+\gamma_2K_m(sb)]I_m^2(sr)\},\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}R_m = {}&[sI'_m(sa)+\gamma_1I_m(sa)][sK'_m(sb)+\gamma_2K_m(sb)]\\ &-[sK'_m(sa)+\gamma_1K_m(sa)][sI'_m(sb)+\gamma_2I_m(sb)] \neq 0.\end{aligned} \tag{3.5}$$

If we insert (3.4) into (2.9) and integrate, we find after some reduction that

$$\bar{K}(s^2) = \frac{a^2}{2}\sum_{m=-\infty}^{\infty}f_1(m;s) - \frac{b^2}{2}\sum_{m=-\infty}^{\infty}f_2(m;s), \tag{3.6}$$

where

$$\begin{aligned}f_1(m;s) = {}&\left(1+\frac{m^2}{s^2a^2}\right)\left\{I_m(sa)K_m(sa) + \frac{I_m(sa)}{a[sI'_m(sa)+\gamma_1I_m(sa)]}\right\}\\ &-I'_m(sa)K'_m(sa) + \frac{\gamma_1I'_m(sa)}{sa[sI'_m(sa)+\gamma_1I_m(sa)]}\\ &+\left[\frac{\gamma_1^2}{s^2}-\left(1+\frac{m^2}{s^2a^2}\right)\right]\frac{[sI'_m(sb)+\gamma_2I_m(sb)]}{a^2R_m[sI'_m(sa)+\gamma_1I_m(sa)]}\end{aligned} \tag{3.7}$$

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and

$$f_2(m;\, s) = \left(1 + \frac{m^2}{s^2b^2}\right)\left\{I_m(sb)K_m(sb) - \frac{I_m(sb)}{b[sI'_m(sb) + \gamma_2 I_m(sb)]}\right\}$$
$$-I'_m(sb)K'_m(sb) - \frac{\gamma_2 I'_m(sb)}{sb[sI'_m(sb) + \gamma_2 I_m(sb)]}$$
$$+\left[\frac{\gamma_2^2}{s^2} - \left(1 + \frac{m^2}{s^2b^2}\right)\right]\frac{[sI'_m(sa) + \gamma_1 I_m(sa)]}{b^2R_m[sI'_m(sb) + \gamma_2 I_m(sb)]}. \tag{3.8}$$

As γ_1, $\gamma_2 \to \infty$, we recover (2.1.3) and (2.1.4) of [7]. The series (3.6) is slowly convergent for $s \to \infty$ and it is therefore expedient to apply a Watson transformation [9] to obtain

$$\bar{K}(s^2) \sim a^2\int_0^\infty f_1(\nu;\, s)\, d\nu - b^2\int_0^\infty f_2(\nu;\, s)\, d\nu \quad \text{as } s \to \infty. \tag{3.9}$$

It now follows that the functions $f_1(\nu;\, s)$ and $f_2(\nu;\, s)$ may be expressed in terms of the asymptotic expansions of the modified Bessel functions and their derivatives due to Olver [4]; these expansions for $s \to \infty$ are uniformly valid in ν for $|\arg \nu| < \pi/2$.

4. Construction of $\theta(t)$ for our impedance problem

In this section, we look at the following cases:

Case 1. $(0 < \gamma_1, \gamma_2 \ll 1)$
In this case we deduce after some reduction that

$$f_1(\nu;\, s) \sim \frac{(\nu^2 + s^2a^2)^{1/2}}{s^2a^2}\sum_{n=0}^{\infty}\frac{A_{1,n}(\tau)}{\nu^n} \quad \text{as } s \to \infty, \tag{4.1}$$

$$f_2(\nu;\, s) \sim \frac{(\nu^2 + s^2b^2)^{1/2}}{s^2b^2}\sum_{n=0}^{\infty}\frac{A_{2,n}(\eta)}{\nu^n} \quad \text{as } s \to \infty, \tag{4.2}$$

where $\tau = \nu/(\nu^2 + s^2a^2)^{1/2}$, $\eta = \nu/(\nu^2 + s^2b^2)^{1/2}$ and for $n = 0, 1, 2, 3$

$$A_{1,0} = 0, \quad A_{1,1} = \tfrac{1}{2}(\tau - \tau^3), \quad A_{1,2} = \tau^2(\gamma_1 a - \tfrac{1}{2}) - \tau^4(\gamma_1 a - \tfrac{3}{2}) - \tau^6,$$
$$A_{1,3} = \tau^3(\tfrac{3}{8} - \gamma_1 a + \gamma_1^2a^2) + \tau^5(-\tfrac{23}{8} + 3\gamma_1 a - \gamma_1^2 a^2) + \tau^7(\tfrac{41}{8} - 2\gamma_1 a) - \tfrac{21}{8}\tau^9,$$

and

$$A_{2,0} = 0, \quad A_{2,1} = -\tfrac{1}{2}(\eta - \eta^3), \quad A_{2,2} = \eta^2(\gamma_2 b - \tfrac{1}{2}) - \eta^4(\gamma_2 b - \tfrac{3}{2}) - \eta^6,$$
$$A_{2,3} = -\eta^3(\tfrac{3}{8} - \gamma_2 b + \gamma_2^2 b^2) - \eta^5(-\tfrac{23}{8} + 3\gamma_2 b - \gamma_2^2 b^2) - \eta^7(\tfrac{41}{8} - 2\gamma_2 b) + \tfrac{21}{8}\eta^9.$$

If the asymptotic expansions (4.1) and (4.2) are now integrated, we deduce that:

$$\bar{K}(s^2) \sim \frac{\pi(a+b)}{4s} - \frac{(\gamma_2 b - \gamma_1 a)}{s^2} + \left\{7\left(\frac{1}{a}+\frac{1}{b}\right) - 32(\gamma_1+\gamma_2) + 64\left(\gamma_1^2 a + \gamma_2^2 b\right)\right\}\frac{\pi}{256 s^3} + O\left(\frac{1}{s^4}\right) \quad \text{as } s \to \infty. \tag{4.3}$$

On inverting Laplace transforms and using (2.5) we have the spectral formula:

$$\theta(t) \sim \frac{b^2-a^2}{4t} + \frac{\pi^{1/2}(a+b)}{4t^{1/2}} - (\gamma_2 b - \gamma_1 a) + \left\{7\left(\frac{1}{a}+\frac{1}{b}\right) - 32(\gamma_1+\gamma_2) + 64\left(\gamma_1^2 a + \gamma_2^2 b\right)\right\}\frac{(\pi t)^{1/2}}{128} + O(t) \quad \text{as } t \to 0. \tag{4.4}$$

Similarly the following asymptotic spectral formulae may be derived:

Case 2. $(0 < \gamma_1 \ll 1, \gamma_2 \gg 1)$

$$\theta(t) \sim \frac{b^2-a^2}{4t} + \frac{\pi^{1/2}\left[a - \left(b + \gamma_2^{-1}\right)\right]}{4t^{1/2}} - \gamma_1 a + \left\{\frac{7}{a} + \frac{1}{b} - 32\gamma_1 + 64\gamma_1^2 a - \frac{\gamma_2^{-1}}{b^2}\right\}\frac{(\pi t)^{1/2}}{128} + O(t) \quad \text{as } t \to 0. \tag{4.5}$$

Case 3. $(\gamma_1 \gg 1, 0 < \gamma_2 \ll 1)$

$$\theta(t) \sim \frac{b^2-a^2}{4t} + \frac{\pi^{1/2}\left[b - \left(a + \gamma_1^{-1}\right)\right]}{4t^{1/2}} - \gamma_2 b + \left\{\frac{1}{a} + \frac{7}{b} - 32\gamma_2 + 64\gamma_2^2 b - \frac{\gamma_1^{-1}}{a^2}\right\}\frac{(\pi t)^{1/2}}{128} + O(t) \quad \text{as } t \to 0. \tag{4.6}$$

This derives from Case 2 with the interchanges $a \leftrightarrow b$ and $\gamma_1 \leftrightarrow \gamma_2$ in the terms other than the first.

Case 4. ($\gamma_1, \gamma_2 \gg 1$)

$$\theta(t) \sim \frac{b^2 - a^2}{4t} - \frac{\pi^{1/2}\left[\left(a + \gamma_1^{-1}\right) + \left(b + \gamma_2^{-1}\right)\right]}{4t^{1/2}} + \left\{\frac{1}{a} + \frac{1}{b} - \frac{\gamma_1^{-1}}{a^2} - \frac{\gamma_2^{-1}}{b^2}\right\}\frac{(\pi t)^{1/2}}{128} + O(t) \quad \text{as } t \to 0. \qquad (4.7)$$

We remark that
(4.4) agrees with (1.5) if $\gamma_1 = \gamma_2 = 0$;
(4.5) agrees with (1.6) if $\gamma_1 = 0$ and $\gamma_2 \to \infty$;
(4.6) agrees with (1.7) if $\gamma_1 \to \infty$ and $\gamma_2 = 0$;
(4.7) agrees with (1.8) if $\gamma_1 = \gamma_2 \to \infty$.
The asymptotic expansions (4.4)–(4.7) may be interpreted as:

(i) D is a circular annulus and we have the impedance boundary conditions of (1.4) on both boundaries of D with large/small impedances γ_1, γ_2 as indicated in the specifications of the four respective cases.

(ii) For the first three terms, D is a bounded domain of area $\pi(b^2 - a^2)$.

In Case 1, it has $h = [1 + 6(\gamma_2 b - \gamma_1 a)]$ holes, a boundary of length $2\pi(a + b)$ together with Neumann conditions on the boundaries, provided h is an integer.

In Case 2, it has $h = (1 + 6\gamma_1 a)$ holes, a part of the boundary of length $2\pi a$ with Neumann conditions and the other part of length $2\pi(b + \gamma_2^{-1})$ together with Dirichlet conditions, provided h is an integer.

In Case 4, it has only one hole ($h = 1$), a boundary of length $2\pi\{(a + \gamma_1^{-1}) + (b + \gamma_2^{-1})\}$ together with Dirichlet conditions on the boundaries.

(iii) The fourth and further terms in (4.4)–(4.7), as yet undetermined, would require different interpretations.

(iv) If it is known that the domain D is a circular annulus, then both the coefficients of $t^{-1/2}$ and that of $t^{1/2}$ in (4.7) may be solved to determine γ_1 and γ_2.

(v) If, in the formula (4.4), $\gamma_1/\gamma_2 = b/a$ then the first three terms agree with the annulus with Neumann conditions. Also, if further $\gamma_1 a = \frac{1}{2} = \gamma_2 b$, then the first four terms agree with the annulus with Neumann conditions (i.e. with the case obtained by setting $\gamma_1 = \gamma_2 = 0$ in (4.4)).

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