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ON THE GROWTH OF LINEAR RECURRENCES IN FUNCTION FIELDS

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Abstract

Let $(G_n)_{n=0}^{\infty}$ be a nondegenerate linear recurrence sequence whose power sum representation is given by $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$. We prove a function field analogue of the well-known result in the number field case that, under some nonrestrictive conditions, $|G_n| \geq (\max_{j=1,\ldots,t} |\alpha_j|)^{n(1-\varepsilon)}$ for n large enough.

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1. Introduction

Let $(G_n)_{n=0}^{\infty}$ be a nondegenerate linear recurrence sequence with power sum representation $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$. This expression makes sense for a sequence $(G_n)_{n=0}^{\infty}$ taking values in any field K; the characteristic roots α_i as well as the coefficients of the polynomials a_i then lie in a finite extension L of K. In this paper K is either a number field or a function field in one variable of characteristic zero. The nondegenerate condition means in the number field case that no ratio α_i/α_j for $i \neq j$ is a root of unity and in the function field case that no ratio α_i/α_j for $i \neq j$ is contained in the field of constants. In the number field case it is well known that, if $\max_{j=1,\ldots,t} |\alpha_j| > 1$, then for any $\varepsilon > 0$ the inequality

$$|G_n| \geq (\max_{j=1,\ldots,t} |\alpha_j|)^{n(1-\varepsilon)} \tag{1.1}$$

is satisfied for every sufficiently large n.

The purpose of this paper is to prove an analogous result in the case of a function field in one variable of characteristic zero. Thus we answer, in the setting we are

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working in, Open Question 3 in [11]; we are grateful to Shparlinski for bringing this paper to our attention. Firstly, we will prove a theorem which states an inequality for an arbitrary valuation in the splitting field L of the characteristic polynomial belonging to the linear recurrence sequence. Secondly, we will derive a corollary for the special case of polynomial power sums. In this special case the inequality takes a form very similar to (1.1). At this point we make the following observation: Theorem 1 in [6] already implies that under some nondegeneracy conditions the degree of polynomials in a linear recurrence sequence with polynomial roots cannot be bounded and therefore must grow to infinity as n does. But this theorem does not say how fast it must grow. In the present paper we will give a bound depending on n for the minimal possible degree of G_n.

The number field case is often mentioned (see [3, 8] or more recently [1, 2]), but it is not that easy to access a proof of it (see [10] or the formulation in [9]). So we give a complete proof based on results of Evertse and Schmidt in the Appendix. By doing so we contribute to the goal that well-known facts should be fully accessible with proof following van der Poorten's wise statement in [9] that 'all too frequently, the well known is [often] not generally known, let alone known well'.

2. Results and notations

Throughout the paper we denote by K a function field in one variable over $\mathbb{C}$. By L we usually denote a finite algebraic extension of K. For the convenience of the reader we give a short summary of the notion of valuations that can also be found in [4]. For $c \in \mathbb{C}$ and $f(x) \in \mathbb{C}(x)$, where $\mathbb{C}(x)$ is the rational function field over $\mathbb{C}$, we denote by $\nu_c(f)$ the unique integer such that $f(x) = (x-c)^{\nu_c(f)} p(x)/q(x)$ with $p(x), q(x) \in \mathbb{C}[x]$ such that $p(c)q(c) \neq 0$. Further, we write $\nu_\infty(f) = \deg q - \deg p$ if $f(x) = p(x)/q(x)$. These functions $\nu : \mathbb{C}(x) \to \mathbb{Z}$ are up to equivalence all valuations in $\mathbb{C}(x)$. If $\nu_c(f) > 0$, then c is called a zero of f, and if $\nu_c(f) < 0$, then c is called a pole of f, where $c \in \mathbb{C} \cup \{\infty\}$. For a finite extension K of $\mathbb{C}(x)$ each valuation in $\mathbb{C}(x)$ can be extended to no more than $[K : \mathbb{C}(x)]$ valuations in K. This again gives up to equivalence all valuations in K. Both in $\mathbb{C}(x)$ as well as in K the sum formula

$$\sum_\nu \nu(f) = 0$$

holds, where the sum is taken over all valuations in the relevant function field. Valuations have the properties $\nu(fg) = \nu(f) + \nu(g)$ and $\nu(f+g) \geq \min(\nu(f), \nu(g))$ for all $f, g \in K$. Each valuation in a function field corresponds to a place and vice versa. The places can be thought of as the equivalence classes of valuations. For more information about valuations and places we refer to [13].

For any power sum $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$ with $a_j(n) = \sum_{k=0}^{m_j} a_{jk} n^k$ and any valuation μ (in a function field L/K containing the α_j and the coefficients of the a_j) we have the trivial bound

$$\begin{aligned}\mu(G_n) &= \mu(a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n) \geq \min_{j=1,\dots,t} \mu(a_j(n)\alpha_j^n)\\ &\geq \min_{j=1,\dots,t} \mu(a_j(n)) + \min_{j=1,\dots,t} \mu(\alpha_j^n)\\ &\geq \min_{j=1,\dots,t} \min_{k=0,\dots,m_j} \mu(a_{jk}n^k) + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j)\\ &= \min_{\substack{j=1,\dots,t\\ k=0,\dots,m_j}} \mu(a_{jk}) + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j) = \widetilde{C} + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j).\end{aligned}$$

Our main result is now the following theorem which gives a bound in the other direction.

THEOREM 2.1. *Let $(G_n)_{n=0}^{\infty}$ be a nondegenerate linear recurrence sequence taking values in K with power sum representation $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$. Let $L = K(\alpha_1, \dots, \alpha_t)$ be the splitting field of the characteristic polynomial of that sequence and let μ be a valuation on L. Then there is an effectively computable constant C, independent of n, such that, for every sufficiently large n, the inequality*

$$\mu(G_n) \leq C + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j)$$

holds.

For the special case of a linear recurrence sequence of complex polynomials having complex polynomials as characteristic roots we get the following lower bound for the degree of the nth member of the sequence.

COROLLARY 2.2. *Let $(G_n)_{n=0}^{\infty}$ be a nondegenerate linear recurrence sequence of polynomials in $\mathbb{C}[x]$ with power sum representation $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$ such that $\alpha_1, \dots, \alpha_t \in \mathbb{C}[x]$. Then there is an effectively computable constant C, independent of n, such that, for every sufficiently large n, the inequality*

$$\deg G_n \geq n \cdot \max_{j=1,\dots,t} \deg \alpha_j - C$$

holds.

In the case of a binary recurrence sequence of polynomials, that is, $t = 2$ in Corollary 2.2, one can use Mason's function field *abc* theorem (see [7]) to show that the number of distinct zeros of G_n must go to infinity as n does. By considering this in slightly more detail, the number of distinct zeros of G_n can be bounded above (trivially) and below (by means of the *abc* theorem) both by linear polynomials in n.

It would be interesting to prove a function field variant of Corollary 3.1 in [1]. However, because of Lemma 3.1, which is based on Dirichlet's classical approximation theorem, we are not (yet) able to prove such a statement.

In the proof given in the next section we will apply the following special case of Theorem 1 in [6].

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LEMMA 2.3. *Let K be as above and L be a finite extension of K of genus $\mathfrak{g}$. Furthermore, let $\alpha_1, \ldots, \alpha_d \in L^*$ with $d \geq 2$ be such that $\alpha_i/\alpha_j \notin \mathbb{C}^*$ for each pair of subscripts i, j with $1 \leq i < j \leq d$. Moreover, for every $i = 1, \ldots, d$, let $\pi_{i1}, \ldots, \pi_{ir_i} \in L$ be r_i linearly independent elements over $\mathbb{C}$. Put*

$$q = \sum_{i=1}^{d} r_i.$$

Then, for every $n \in \mathbb{N}$ such that

$$\{\pi_{il}\alpha_i^n : l = 1, \ldots, r_i, i = 1, \ldots, d\}$$

is linearly dependent over $\mathbb{C}$, but no proper subset of this set is linearly dependent over $\mathbb{C}$, we have

$$n \leq C = C(q, \mathfrak{g}, \pi_{il}, \alpha_i : l = 1, \ldots, r_i, i = 1, \ldots, d).$$

The proof will also make use of height functions in function fields. Let us therefore define the height of an element $f \in L^*$ by

$$\mathcal{H}(f) := -\sum_{\nu} \min(0, \nu(f)) = \sum_{\nu} \max(0, \nu(f)),$$

where the sum is taken over all valuations in the function field $L/\mathbb{C}$. For every $z \in L \setminus \mathbb{C}$,

$$\begin{aligned}\mathcal{H}(z) &= \sum_{\nu} \max(0, \nu(z)) = \sum_{P} \max(0, \nu_P(z)) \\ &= \deg \sum_{P} \max(0, \nu_P(z))P = \deg(z)_0 = [L : \mathbb{C}(z)] = \deg_{\mathbb{C}}(z),\end{aligned}$$

by Theorem I.4.11 in [13], where we have used the fact that all places have degree one since we are working over $\mathbb{C}$ (instead of the height, one can use $\deg_{\mathbb{C}}(z) = [L : \mathbb{C}(z)]$ as in [14]). Additionally, we define $\mathcal{H}(0) = \infty$. This height function satisfies some basic properties that are listed in the next lemma which is proven in [5].

LEMMA 2.4. *Let $\mathcal{H}$ denote the height on $L/\mathbb{C}$ as above. Then, for $f, g \in L^*$:*

(a) $\mathcal{H}(f) \geq 0$ *and* $\mathcal{H}(f) = \mathcal{H}(1/f)$;
(b) $\mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(f + g) \leq \mathcal{H}(f) + \mathcal{H}(g)$;
(c) $\mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(fg) \leq \mathcal{H}(f) + \mathcal{H}(g)$;
(d) $\mathcal{H}(f^n) = |n| \cdot \mathcal{H}(f)$;
(e) $\mathcal{H}(f) = 0 \iff f \in \mathbb{C}^*$;
(f) $\mathcal{H}(A(f)) = \deg A \cdot \mathcal{H}(f)$ *for any* $A \in \mathbb{C}[T] \setminus \{0\}$.

We will also use the following function field analogue of the Schmidt subspace theorem.

PROPOSITION 2.5 (Zannier [14]). *Let $F/\mathbb{C}$ be a function field in one variable and of genus $\mathfrak{g}$. Let $\varphi_1, \ldots, \varphi_n \in F$ be linearly independent over $\mathbb{C}$ and let $r \in \{0, 1, \ldots, n\}$.*

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Let S be a finite set of places of F containing all the poles of $\varphi_1, \ldots, \varphi_n$ and all the zeros of $\varphi_1, \ldots, \varphi_r$. Put $\sigma = \sum_{i=1}^{n} \varphi_i$. Then

$$\sum_{\nu \in S} (\nu(\sigma) - \min_{i=1,\ldots,n} \nu(\varphi_i)) \leq \binom{n}{2} (|S| + 2\mathfrak{g} - 2) + \sum_{i=r+1}^{n} \mathcal{H}(\varphi_i).$$

3. Proofs

Proof of Theorem 2.1. Denote the coefficients of the polynomial $a_j(n) \in L[n]$ by $a_{j0}, a_{j1}, \ldots, a_{jm_j}$ where m_j is the degree of $a_j(n)$. So

$$a_j(n) = \sum_{k=0}^{m_j} a_{jk} n^k.$$

First assume that the recurrence sequence is of the shape $G_n = a_1(n)\alpha_1^n$. Using Lemma 2.4,

$$\begin{aligned}\mu(G_n) = \mu(a_1(n)) + n\mu(\alpha_1) &\leq \mathcal{H}(a_1(n)) + n\mu(\alpha_1)\\ &\leq \sum_{k=0}^{m_1} \mathcal{H}(a_{1k}n^k) + n\mu(\alpha_1) = \sum_{k=0}^{m_1} \mathcal{H}(a_{1k}) + n\mu(\alpha_1).\end{aligned}$$

Thus from now on we can assume that $t \geq 2$. Let $\pi_{j1}, \ldots, \pi_{jk_j}$ be a maximal $\mathbb{C}$-linear independent subset of $a_{j0}, a_{j1}, \ldots, a_{jm_j}$. Then we can write the sequence as

$$G_n = \sum_{j=1}^{t} \bigg(\sum_{i=1}^{k_j} b_{ji}(n)\pi_{ji} \bigg) \alpha_j^n$$

with polynomials $b_{ji}(n) \in \mathbb{C}[n]$. Since $a_j(n)$ is not the zero polynomial, there is for each j at least one index i such that $b_{ji}(n)$ is not the zero polynomial. Without loss of generality we can assume that no $b_{ji}(n)$ is the zero polynomial since otherwise we can throw out all zero polynomials and renumber the remaining terms. It does not matter whether all π_{ji} occur in the sum or not. Moreover, we assume that n is large enough such that $b_{ji}(n) \neq 0$ for all j, i.

Consider as a next step the set

$$M := \{\pi_{ji}\alpha_j^n : i = 1, \ldots, k_j, j = 1, \ldots, t\}.$$

We intend to apply Lemma 2.3. If M is linearly dependent over $\mathbb{C}$, then we choose a minimal linearly dependent subset $\widetilde{M}$ of M, that is, a linearly dependent subset $\widetilde{M}$ with the property that no proper subset of $\widetilde{M}$ is linearly dependent. Let $\widetilde{G_n}$ be the linear recurrence sequence associated with this subset $\widetilde{M}$, that is,

$$\widetilde{G_n} = \sum_{j=1}^{s} \bigg(\sum_{i=1}^{\widetilde{k_j}} b_{ji}(n)\pi_{ji} \bigg) \alpha_j^n$$

for $s \leq t$ and after a suitable renumbering of the summands. Since $\pi_{j1}, \ldots, \pi_{jk_j}$ are $\mathbb{C}$-linearly independent we have $s \geq 2$. Applying Lemma 2.3 to

$$\widetilde{M} := \{\pi_{ji}\alpha_j^n : i = 1, \ldots, \widetilde{k_j}, j = 1, \ldots, s\}$$

gives an upper bound for n. Thus for n large enough this subset $\widetilde{M}$ of M cannot be linearly dependent. Because of the fact that there are only finitely many subsets of M, for n large enough the set M must be linearly independent.

We assume from here on that n is large enough such that M is linearly independent. For each fixed n we have $b_{ji}(n) \in \mathbb{C}^*$. Thus the set

$$M' := \{b_{ji}(n)\pi_{ji}\alpha_j^n : i = 1, \ldots, k_j, j = 1, \ldots, t\}.$$

is linearly independent over $\mathbb{C}$ and contains for each $j = 1, \ldots, t$ at least one element. Let S be a finite set of places of L containing all zeros and poles of α_j for $j = 1, \ldots, t$ and of the nonzero a_{ji} for $j = 1, \ldots, t$ and $i = 1, \ldots, m_j$ as well as μ and the places lying over ∞. Now applying Proposition 2.5 yields

$$\sum_{\nu \in S} \Big(\nu(G_n) - \min_{\substack{j=1,\ldots,t \\ i=1,\ldots,k_j}} \nu(b_{ji}(n)\pi_{ji}\alpha_j^n) \Big) \leq \binom{\sum_{j=1}^t k_j}{2} (|S| + 2\mathfrak{g} - 2) =: C_1$$

and, since each summand in the sum on the left-hand side is nonnegative,

$$\mu(G_n) - \min_{\substack{j=1,\ldots,t \\ i=1,\ldots,k_j}} \mu(b_{ji}(n)\pi_{ji}\alpha_j^n) \leq C_1.$$

Therefore for all $j_0 = 1, \ldots, t$ and $i_0 = 1, \ldots, k_{j_0}$,

$$\begin{aligned}
\mu(G_n) &\leq C_1 + \min_{\substack{j=1,\ldots,t \\ i=1,\ldots,k_j}} \mu(b_{ji}(n)\pi_{ji}\alpha_j^n) \\
&\leq C_1 + \mu(b_{j_0 i_0}(n)\pi_{j_0 i_0}\alpha_{j_0}^n) \\
&= C_1 + \mu(\pi_{j_0 i_0}) + n\mu(\alpha_{j_0}) \\
&\leq C_1 + \max_{\substack{j=1,\ldots,t \\ i=0,\ldots,m_j,\ a_{ji}\neq 0}} \mu(a_{ji}) + n\mu(\alpha_{j_0}) \\
&\leq C_1 + \max_{\substack{j=1,\ldots,t \\ i=0,\ldots,m_j,\ a_{ji}\neq 0}} \mathcal{H}(a_{ji}) + n\mu(\alpha_{j_0}) \\
&= C_2 + n\mu(\alpha_{j_0}).
\end{aligned}$$

Since this holds for all $j_0 = 1, \ldots, t$,

$$\mu(G_n) \leq C_2 + n \cdot \min_{j=1,\ldots,t} \mu(\alpha_j). \qquad \square$$

Proof of Corollary 2.2. We can apply Theorem 2.1 with $L = K = \mathbb{C}(x)$ and $\mu = \nu_\infty$. This yields

$$-\deg G_n = \nu_\infty(G_n) \leq C + n \cdot \min_{j=1,\ldots,t} \nu_\infty(\alpha_j) = C - n \cdot \max_{j=1,\ldots,t} \deg \alpha_j$$

which immediately implies the inequality in question. $\square$

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Appendix A. The number field case

In this appendix we will give a proof of the following theorem.

THEOREM A.1. *Let $(G_n)_{n=0}^{\infty}$ be a nondegenerate linear recurrence sequence taking values in a number field K and let $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$ with algebraic integers $\alpha_1, \ldots, \alpha_t$ be its power sum representation satisfying $\max_{j=1,\ldots,t} |\alpha_j| > 1$. Denote by $|\cdot|$ the usual absolute value on $\mathbb{C}$. Then, for any $\varepsilon > 0$, the inequality*

$$|G_n| \geq (\max_{j=1,\ldots,t} |\alpha_j|)^{n(1-\varepsilon)}$$

is satisfied for every sufficiently large n.

Note that this result is not effective in the sense that we do not give a bound n_0 such that the inequality is satisfied for all n greater than n_0. If we were to look more precisely at the limitations placed on n in the proof given below, it would be possible to give an (admittedly) rather complicated upper bound on the number of exceptions. This bound would have the following form: if $n/\log n > B_1$, then there are at most B_2 values of n for which the inequality is not valid. Since the explicit constants are not so enlightening we will not calculate them in detail.

From here on K will denote a number field. In the proof we will need three auxiliary results which are listed below. The first one is a result of Schmidt.

LEMMA A.2 (Schmidt [12]). *Suppose that $(G_n)_{n\in\mathbb{Z}}$ is a nondegenerate linear recurrence sequence of complex numbers, whose characteristic polynomial has k distinct roots of multiplicity at most a. Then the number of solutions $n \in \mathbb{Z}$ of the equation*

$$G_n = 0$$

can be bounded above by

$$c(k, a) = e^{(7k^a)^{8k^a}}.$$

The second is a result of Evertse. We use the notation

$$\|\mathbf{x}\| = \max_{\substack{k=0,\ldots,t \\ i=1,\ldots,D}} |\sigma_i(x_k)|$$

with $\{\sigma_1, \ldots, \sigma_D\}$ the set of all embeddings of K in $\mathbb{C}$ and $\mathbf{x} = (x_0, x_1, \ldots, x_t)$. Moreover, we denote by $\mathcal{O}_K$ the ring of integers in K.

LEMMA A.3 (Evertse [3]). *Let t be a nonnegative integer and S a finite set of places in K, containing all infinite places. Then for every $\varepsilon > 0$ a constant C exists, depending only on ε, S, K, t such that for each nonempty subset T of S and every vector $\mathbf{x} = (x_0, x_1, \ldots, x_t) \in \mathcal{O}_K^{t+1}$ with*

$$x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0$$

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for each nonempty subset $\{i_0, i_1, \ldots, i_s\}$ *of* $\{0, 1, \ldots, t\}$,

$$\Big(\prod_{k=0}^{t}\prod_{v\in S}\|x_k\|_v\Big)\prod_{v\in T}\|x_0 + x_1 + \cdots + x_t\|_v \ge C\Big(\prod_{v\in T}\max_{k=0,\ldots,t}\|x_k\|_v\Big)\|\mathbf{x}\|^{-\varepsilon}.$$

Furthermore, we will need the following lemma which also can be found in [3].

LEMMA A.4. *Suppose* K *is a number field of degree* D*, let* $f(X) \in K[X]$ *be a polynomial of degree* m *and* T *a nonempty set of primes on* K*. Then there exists a positive constant* c*, depending only on* K, f *such that for all* $r \in \mathbb{Z}$ *with* $r \neq 0$ *and* $f(r) \neq 0$*,*

$$\begin{aligned} c^{-1}|r|^{-Dm} \le \Big(\prod_{v}\max(1, \|f(r)\|_v)\Big)^{-1} &\le \prod_{v\in T}\|f(r)\|_v \\ &\le \prod_{v}\max(1, \|f(r)\|_v) \le c|r|^{Dm}. \end{aligned}$$

PROOF OF THEOREM A.1. Since the characteristic roots α_j of G_n are algebraic integers we can find a nonzero integer z such that $za_j(n)\alpha_j^n$ are algebraic integers for all $j = 1, \ldots, t$ and all $n \in \mathbb{N}$. Set $L = K(\alpha_1, \ldots, \alpha_t)$, the splitting field of the characteristic polynomial of the sequence G_n. Choose S as a finite set of places in L containing all infinite places as well as all places such that $\alpha_1, \ldots, \alpha_t$ are S-units. Let μ be such that $\|\cdot\|_\mu = |\cdot|$ is the usual absolute value on $\mathbb{C}$. In particular, $\mu \in S$. Further, define $T = \{\mu\}$.

As G_n is nondegenerate, the sequence $\widetilde{G_n} = zG_n$ is also nondegenerate. Therefore by Lemma A.2, for n large enough,

$$za_{j_1}(n)\alpha_{j_1}^n + \cdots + za_{j_s}(n)\alpha_{j_s}^n \neq 0$$

for each non-empty subset $\{j_1, \ldots, j_s\}$ of $\{1, \ldots, t\}$. Thus we can apply Lemma A.3 and get

$$\Big(\prod_{j=1}^{t}\prod_{v\in S}\|za_j(n)\alpha_j^n\|_v\Big)|zG_n| \ge C\max_{j=1,\ldots,t}|za_j(n)\alpha_j^n|\,\|z\mathbf{x}\|^{-\varepsilon}$$

for $\mathbf{x} = (a_1(n)\alpha_1^n, \ldots, a_t(n)\alpha_t^n)$. Without loss of generality, we can assume that $|\alpha_1| = \max_{j=1,\ldots,t}|\alpha_j|$. Since z is a fixed integer and the α_j are S-units, we can rewrite this as

$$\begin{aligned} \Big(\prod_{j=1}^{t}\prod_{v\in S}\|a_j(n)\|_v\Big)|G_n| &\ge C_1\max_{j=1,\ldots,t}|a_j(n)\alpha_j^n|\,\|\mathbf{x}\|^{-\varepsilon} \\ &\ge C_1|a_1(n)\alpha_1^n|\,\|\mathbf{x}\|^{-\varepsilon} = C_1|a_1(n)|\,|\alpha_1|^n\|\mathbf{x}\|^{-\varepsilon}. \qquad \text{(A.1)} \end{aligned}$$

In preparation for the next step, note that there exists a positive constant A such that

$$\max_{\substack{j=1,\ldots,t\\ i=1,\ldots,D}}|\sigma_i(\alpha_j)| \le A\cdot|\alpha_1|.$$

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We decompose $\varepsilon = \gamma \cdot \delta$ with small δ and $A^\gamma \leq |\alpha_1|$. This gives the estimates

$$\begin{aligned}
\|\mathbf{x}\| &= \max_{\substack{j=1,\dots,t\\ i=1,\dots,D}} |\sigma_i(a_j(n)\alpha_j^n)| = \max_{\substack{j=1,\dots,t\\ i=1,\dots,D}} |\sigma_i(a_j(n))\sigma_i(\alpha_j)^n| \\
&\leq \max_{\substack{j=1,\dots,t\\ i=1,\dots,D}} |\sigma_i(a_j(n))| \cdot \max_{\substack{j=1,\dots,t\\ i=1,\dots,D}} |\sigma_i(\alpha_j)|^n \\
&\leq C_2 n^m \cdot \max_{\substack{j=1,\dots,t\\ i=1,\dots,D}} |\sigma_i(\alpha_j)|^n \leq C_2 n^m A^n |\alpha_1|^n,
\end{aligned}$$

with $m = \max_{j=1,\dots,t} \deg a_j$, and

$$\|\mathbf{x}\|^\varepsilon \leq C_3 n^{m\varepsilon} A^{\gamma n \delta} |\alpha_1|^{n\varepsilon} \leq C_3 n^{m\varepsilon} |\alpha_1|^{n(\varepsilon+\delta)}.$$

Now we insert this into inequality (A.1), giving

$$\Big(\prod_{j=1}^{t} \prod_{v\in S} \|a_j(n)\|_v\Big) |G_n| \geq C_4 |a_1(n)|\,|\alpha_1|^n n^{-m\varepsilon} |\alpha_1|^{-n(\varepsilon+\delta)} \geq C_5 n^{-m\varepsilon} |\alpha_1|^{n(1-\varepsilon-\delta)}.$$

Applying Lemma A.4 to the product in the brackets on the left hand side gives the bound

$$\prod_{j=1}^{t} \prod_{v\in S} \|a_j(n)\|_v \leq \prod_{j=1}^{t} C_6^{(j)} n^{Dm} \leq C_7 n^{tDm}.$$

Altogether, for n large enough,

$$|G_n| \geq C_8 n^{-tDm-m\varepsilon} |\alpha_1|^{n(1-\varepsilon-\delta)}.$$

Hence, recalling that $|\alpha_1| = \max_{j=1,\dots,t} |\alpha_j|$, for n large enough,

$$|G_n| \geq (\max_{j=1,\dots,t} |\alpha_j|)^{n(1-\widetilde{\varepsilon})}.$$

This proves the theorem. □

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