DEAR EDITOR,

Robin McLean has shown how Dr Morris's problem can be solved straightforwardly by means of generating functions. Perhaps readers may be interested to see a second method for deriving his result: a method which brings together two apparently disparate notions, namely, "How many solutions in non-negative integers are there to the equation $x_1 + x_2 + \ldots + x_p = q$?", and secondly the inclusion-exclusion principle (i.e. the generalisation of the formula $\mathrm{n}(A \cap B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cup B)$ where $\mathrm{n}(X)$ denotes the number of elements in the set X).

First we observe that Dr Morris is in effect asking for the coefficient of x^r in the expansion of $(1 + x + \ldots + x^s)^n$, where $r = t + nI$ and $s = 2I = J - 1$. Now a term x^r arises in this expansion when we select x^{i_1} from the first bracket, x^{i_2} from the second, etc, in such a way that $i_1 + i_2 + \ldots + i_n = r$. The coefficient of x^{i_1} is then the number of solutions in non-negative integers of this equation.

First, ignoring any restrictions on the size of the i_j, we note that a typical solution $2 + 1 + 3 + \ldots + 2$ can be encoded XXOXOXXXO . . . OXX. The first two Xs indicate that $i_1 = 2$, the O that we are now going to signal the size of i_2, and so on. A particular solution is determined when we specify in which of the $n + r - 1$ code positions we shall place the r Xs (or the $n - 1$ Os). Thus the total number of solutions is $\binom{n+r-1}{r}$. (This gives us the coefficient of x^r in $(1 + x + x^2 + \ldots)^{-n}$, i.e. in $(1 - x)^{-n}$, a result quoted by Robin McLean.)

To count the number of solutions when we impose the conditions $0 \leqslant i_j \leqslant s, j = 1, 2, \ldots, n$, we must make use of the inclusion-exclusion formula. In effect we say:

number of restricted solutions
= number of unrestricted solutions
− number with at least one restriction broken
+ number with at least two restrictions broken
− . . ..

The number of solutions in which $i_1 \geqslant s + 1$, but the other i_j are unrestricted, is the same as the number of unrestricted solutions of $i_1 + i_2 + \ldots + i_n = r - (s + 1)$, and this is

$$\binom{n + r - (s+1) - 1}{r - (s+1)}.$$

Similarly, the number of solutions in which i_1 and i_2 are restricted, but the remaining i_j are unrestricted, is

$$\binom{n + r - 2(s+1) - 1}{r - 2(s+1)}.$$

A similar argument holds if i_j and i_k are restricted $(i \neq j)$ rather than i_1 and i_2.

Now applying the inclusion-exclusion formula and observing that the one variable to be restricted can be chosen in $\binom{n}{1}$ ways, the two variables in $\binom{n}{2}$ ways, etc, we obtain as the number of restricted solutions

$$\binom{n+r-1}{r} - \binom{n}{1}\binom{n+r-1-(s+1)}{r-(s+1)} + \binom{n}{2}\binom{n+r-1-2(s+1)}{r-2(s+1)} - \ldots,$$

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which is equivalent to the formula given by Robin McLean.

Yours sincerely,
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More about Pig

DEAR SIR,

In Note 63.27 (December 1979) by S. Humphrey on the game of Pig, the author seems to have missed one important point. The object of the game is, roughly, to collect as many points as possible from successive throws of two dice, with the caveat that if a six should appear then the score drops back to zero again. It is claimed that one should go for three throws and then quit.

This would indeed be the correct strategy if one had to decide at the beginning of one's turn how many throws to attempt, but the rules suggest that one can decide after each throw whether to throw again or not. If the objective is to maximise one's expected score it is possible to use the information obtained during one's turn to improve on this strategy. What is important, after all, in deciding whether to have another throw is not how many throws one has had already, but how many points one is putting at risk.

Let us suppose, then, that our current score for that turn is r points. If we have another throw then the score will be increased by 2, 3, 4, 5, 6, 7, 8, 9, 10 points with probabilities (1,2,3,4,5,4,3,2,1)/36 respectively and will be decreased by r points with probability 11/36. The expected increase for the next throw is thus

$$\frac{1 \times 2 + 2 \times 3 + \ldots + 1 \times 10 - 11 \times r}{36} = \frac{150 - 11r}{36},$$

which is positive if and only if $r < 14$. This shows that the optimal policy is to quit when one's score reaches 14 or more. This may well be after three throws but could be after 2, 4, 5, 6 or even 7 throws.

Nor is this the full story. One is after all playing against opponents and not just going for the highest expected score by oneself. There may well be times when it is worth risking all in order to reach the target before someone else, and other times when it is better to play safe. Since one's own strategy might well affect that of one's opponents, a complete analysis of the game would be quite complex, and we have not attempted to work out the details.

Yours faithfully,
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DEAR EDITOR,

In his Note S. Humphrey suggested that the best strategy when playing Pig was to aim to throw the pair of dice three times in a turn, and then stop. More generally, if the game is played with k dice, he suggested stopping after five or six throws for $k = 1$, two throws for $k = 3$ and one throw for $k > 3$.

This strategy can be improved upon by a player using his current score in a turn to decide whether or not to throw the dice again. If his present score for the turn is x, then, using k dice,

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