19

Behavior at very large and very small distances

For the classical Abraham model, and its relativistic generalization, we had to accept a phenomenological charge distribution. The physically appealing idea to let this charge distribution shrink to a point charge failed because the charged particle acquires a mass which grows beyond any limit. There is simply no bare parameter in the model which would balance the divergence in a meaningful way. Nevertheless the situation is much less dramatic than it sounds. When probed over distances that are large compared to the size of the charge distribution and correspondingly long times, only global properties of the charge distribution, like total charge and total electrostatic energy, are needed, thereby greatly reducing the dependence on the choice of the form factor. In the quantized version one has to investigate the problem anew, which requires the study of the properties of the Pauli–Fierz Hamiltonian at very small distances. The form factor $\widehat{\varphi}$ cuts off the interaction with the Maxwell field at large wave numbers. The point-charge limit thus means removing this ultraviolet cutoff. If it could be done, we would be in the very satisfactory position of having the empirical masses and empirical charges of the quantum particles as the only model parameters. Of course, the validity of the theory would not extend beyond what we have discussed already. In particular, relativistic corrections are not properly accounted for.

As we will see, the ultraviolet behavior of the Pauli–Fierz model is not so well understood. If the Maxwell field is replaced by a scalar Bose field, the ultraviolet divergencies simplify considerably and have been studied by E. Nelson in detail. To have a sort of blueprint we therefore include a section on the scalar field model.

Since the photons have zero mass, the Coulomb potential decreases as $-e^2/4\pi|x|$. In a quantized field theory one has to check whether states which have such a slow decay for the average fields still lie in Fock space, the Hilbert space which we used throughout to develop our theory. This issue leads to a study of the infrared behavior of the Pauli–Fierz Hamiltonian. Note that for this purpose the dispersion relation $\omega(k) = |k|$ is crucial, whereas an ultraviolet cutoff in the

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interaction can be accommodated without harm. On the other hand, for the point-charge limit we may assign the photons a small mass. The infrared and ultraviolet behavior appear as disjoint properties.

19.1 Infrared photons

A classical charge traveling at constant velocity v carries with it the electric field $E_v^{\rm cl}$ and the magnetic field $B_v^{\rm cl}$, see Eq. (4.5), where we omitted the boldface and added the superscript "cl" to distinguish from the quantized sister. One would expect that the quantized theory reproduces these fields on the average, at least very far away from the charge. Thus we are led to consider states ψ in Fock space such that

$$\langle\psi, E_\varphi(x)\psi\rangle_{\mathcal{F}} = E_{v\perp}^{\rm cl}(x), \quad \langle\psi, B_\varphi(x)\psi\rangle_{\mathcal{F}} = B_v^{\rm cl}(x). \tag{19.1}$$

Under these constraints the average number of photons is minimal for the coherent state $\psi_v^{\rm coh}$ having averages (19.1) and the minimum is given by

$$\langle\psi_v^{\rm coh}, N_{\rm f}\psi_v^{\rm coh}\rangle_{\mathcal{F}} = \frac{e^2}{2}\int {\rm d}^3k|\widehat{\varphi}(k)|^2(k^2-(v\cdot k)^2)^{-2} \times\omega(1+\omega^{-2}(v\cdot k)^2)(v\cdot Q^\perp v). \tag{19.2}$$

If $\widehat{\varphi}(0) = (2\pi)^{-3/2}$ and $\omega(k) = |k|$, then the integrand diverges as $|k|^{-3}$ for small k which makes the integral in (19.2) logarithmically infrared divergent. There is no vector in Fock space which satisfies (19.1), unless $v = 0$.

A natural consequence is to take $\psi_v^{\rm coh}$ as the basic object and to build the Fock space $\mathcal{F}_v$ out of finite photon excitations away from it. If in $\mathcal{F}_v$ one searches for a vector reproducing the classical fields at velocity u on the average, then the constraint (19.1) becomes

$$\langle\psi, E_\varphi(x)\psi\rangle_{\mathcal{F}_v} = E_{u\perp}^{\rm cl}(x) - E_{v\perp}^{\rm cl}(x), \quad \langle\psi, B_\varphi(x)\psi\rangle_{\mathcal{F}_v} = B_u^{\rm cl}(x) - B_v^{\rm cl}(x). \tag{19.3}$$

The minimal photon number consistent with (19.3) is

$$\frac{1}{2}\int {\rm d}^3k|\widehat{\varphi}(k)|^2\omega\Big((v\widehat{\phi}_v - u\widehat{\phi}_u)\cdot Q^\perp(v\widehat{\phi}_v - u\widehat{\phi}_u) + \big(v\widehat{\phi}_v\frac{1}{\omega}(v\cdot k) - u\widehat{\phi}_u\frac{1}{\omega}(u\cdot k)\big)\cdot Q^\perp\big(v\widehat{\phi}_v\frac{1}{\omega}(v\cdot k) - u\widehat{\phi}_u\frac{1}{\omega}(u\cdot k)\big)\Big), \tag{19.4}$$

$\widehat{\phi}_v(k)$ from (4.6), which again diverges logarithmically for small k, unless $u = v$. The family of coherent states $\{\psi_v^{\rm coh}\,|\,|v|<1\}$ leads to mutually inequivalent

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representations of the canonical commutation relations. Mathematically it is bad news, since there is no single Hilbert space which can accommodate states corresponding to the electron freely traveling at arbitrary uniform velocity.

To probe the subject further let us consider the scattering of photons where, to simplify matters, it is assumed that the motion of the quantized particle is replaced by a classical current. To figure out the Hamiltonian we return to (13.47) and regard $j(x,t)$ as a given current. In the Coulomb gauge Eq. (13.47) reads

$$\partial_t A = -E, \quad \partial_t E = -\Delta A - j, \tag{19.5}$$

where it is understood that (19.5) refers to the transverse components only. The longitudinal piece of E is determined through the Poisson equation. Equations (19.5) are the Heisenberg equations of motion for the time-dependent Hamiltonian

$$H(t) = H_{\mathrm{f}} - \int \mathrm{d}^3x\, j(x,t)A(x) \tag{19.6}$$

acting on $\mathcal{F}$. Since $H(t)$ is quadratic in a, a^*, its unitary propagator can be computed explicitly. For $t \geq 0$ one obtains, with time ordering denoted by $\mathcal{T}$,

$$\begin{aligned} U(t,0) &= \mathcal{T}\exp\Big[-\mathrm{i}\int_0^t \mathrm{d}s\; H(s)\Big] \\ &= \mathrm{e}^{-\mathrm{i}H_{\mathrm{f}}t}\exp\Big[\mathrm{i}\int_0^t \mathrm{d}s \sum_{\lambda=1,2}\int \mathrm{d}^3k(2\omega)^{-1/2}\big(e_\lambda\cdot\widehat{j}(k,s)^*\mathrm{e}^{-\mathrm{i}\omega s}a(k,\lambda) \\ &\quad + e_\lambda\cdot\widehat{j}(k,s)\mathrm{e}^{\mathrm{i}\omega s}a^*(k,\lambda)\big) + \frac{1}{2}\mathrm{i}\mathrm{Im}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}s'\;\Theta(s-s') \\ &\quad \times \sum_{\lambda=1,2}\int \mathrm{d}^3k(2\omega)^{-1}(e_\lambda\cdot\widehat{j}(k,s))(e_\lambda\cdot\widehat{j}(k,s'))^*\mathrm{e}^{\mathrm{i}\omega(s-s')}\Big] \end{aligned} \tag{19.7}$$

with $\Theta(s) = 1$ for $s \geq 0$, $\Theta(s) = -1$ for $s < 0$.

Let us first examine the case where the charge travels at constant velocity, i.e. $j(x,t) = e\varphi(x-vt)v$, $|v| < 1$, and the initial $\psi = \Omega$. Classically, the current would build up the charge soliton; compare with (4.31), (4.32). There is no accompanying radiation. The quantum wave function $\psi(t) = U(t,0)\Omega$ is a coherent state of the Maxwell field. This implies that N_{f} has a Poisson distribution with

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average

$$\begin{aligned}&\langle\psi(t), N_{\mathrm{f}}\psi(t)\rangle_{\mathcal{F}}\\&= e^2\sum_{\lambda=1,2}\int \mathrm{d}^3k|\widehat{\varphi}|^2\omega^{-1}(e_\lambda\cdot v)^2(\omega-k\cdot v)^{-2}\big(1-\cos((\omega-v\cdot k)t)\big)\\&\cong v^2\log t\end{aligned}\tag{19.8}$$

for large t. On the other hand, $\langle\psi(t), H_{\mathrm{f}}\psi(t)\rangle_{\mathcal{F}}$ stays bounded because of the extra factor of ω from the definition of the energy. Also, in every bounded region in position space and in any region in momentum space avoiding the origin, the number of photons is Poisson-distributed with a finite mean. The photons in (19.8) are bound to the charge, i.e. virtual in the usual parlance. For the Pauli–Fierz Hamiltonian virtual photons can be probed only indirectly, e.g. through the effective dynamics discussed in chapter 16. As long as the energy remains finite no qualitative changes are expected, as confirmed by the fact that the g-factor and the effective mass are infrared convergent at least to order e^2.

As a second example let us study the generation of photons through accelerated motion. We prescribe the trajectory q_t with velocity v_t of the classical charge, and thus the current $j(x,t) = e\varphi(x-q_t)v_t$. The scattering process is captured most conveniently through the S-matrix defined by

$$S = \lim_{t\to\infty} U(t,0)^*\mathrm{e}^{2\mathrm{i}H_{\mathrm{f}}t}U(0,-t).\tag{19.9}$$

From (19.7) we conclude

$$\begin{aligned}S = \exp\Big[-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\sum_{\lambda=1,2}\int \mathrm{d}^3k(2\omega)^{-1/2}e_\lambda\cdot v_t\big(e\widehat{\varphi}^*\mathrm{e}^{-\mathrm{i}(\omega t-k\cdot q_t)}a(k,\lambda)\\+e\widehat{\varphi}\mathrm{e}^{\mathrm{i}(\omega t-k\cdot q_t)}a^*(k,\lambda)\big)-\frac{1}{2}\mathrm{i}\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}s'\Theta(s-s')\\\times\sum_{\lambda=1,2}\int \mathrm{d}^3k e^2|\widehat{\varphi}|^2(2\omega)^{-1}(e_\lambda\cdot v_s)(e_\lambda\cdot v_{s'})\mathrm{e}^{\mathrm{i}(\omega s-\omega s'-k\cdot q_s+k\cdot q_{s'})}\Big].\end{aligned}\tag{19.10}$$

Note that for constant velocity, $v_t = v$, $|v| < 1$, the time-integration yields the δ-function $\delta(\omega - k\cdot v)$ and therefore the S-matrix is trivial, $S = 1$. For the sake of an example let us assume that there are no incoming photons. Then the scattering

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state of interest is $S\Omega$, which is a coherent state with average number of photons

$$\langle S\Omega, N_{\mathrm{f}} S\Omega\rangle_{\mathcal{F}} = e^2 \sum_{\lambda=1,2} \int \mathrm{d}^3k |\widehat{\varphi}|^2 (2\omega)^{-1} | \int \mathrm{d}t\, (e_\lambda \cdot v_t) \mathrm{e}^{\mathrm{i}(\omega t - k\cdot q_t)}|^2. \quad (19.11)$$

In standard scattering $v_t \to v_\pm$ for $t \to \pm\infty$. If $v_+ = v_-$, from the previous argument one concludes that $\langle S\Omega, N_{\mathrm{f}} S\Omega\rangle_{\mathcal{F}} < \infty$. However if $v_+ \neq v_-$, then from the time-integration a factor $|k|^{-2}$ appears which together with the factor $1/\omega$ makes the integral in (19.11) logarithmically divergent at small k. As before, $\langle S\Omega, H_{\mathrm{f}} S\Omega\rangle_{\mathcal{F}} < \infty$. Also the number of photons is finite in any region of the form $\{k\,|\,|k| > \delta\}$ with $\delta > 0$.

If an electron is scattered by, say, a short-range electrostatic potential then in the collision process a large number of infrared photons is generated. Strictly speaking, there is no channel with elastic scattering. Since the total energy of scattered photons is bounded, the collision cross-section is slightly modified but remains finite. These infrared photons are however somewhat elusive objects. For example, for the state $S\Omega$ the photon density in position space decays as $|x|^{-3}$ for large $|x|$, which means that there is a small probability for the photons to have been created very far away from the source. A real detector necessarily makes a cutoff in the energy range and in position, thus necessarily misses the infrared part.

19.2 Energy renormalization in Nelson's scalar field model

On the classical level we consider a scalar wave field and couple it to a mechanical particle in such a way that the interaction is linear in the field, local, and translation invariant. This fixes the Hamiltonian function to be of the form

$$H = \frac{1}{2m} p^2 + \frac{1}{2} \int \mathrm{d}^3x \big(\pi(x)^2 + (\nabla\phi(x))^2 + m_{\mathrm{ph}}^2 \phi(x)^2\big) + e\phi_\varphi(q). \quad (19.12)$$

Here q, p are the position and momentum of the particle with bare mass m and $\pi(x)$ is the momentum field canonically conjugate to the scalar wave field $\phi(x)$. The wave speed c is set equal to one. e is the coupling strength, and $m_{\mathrm{ph}} \geq 0$ is the mass of the bosons. The equations of motion read

$$\partial_t^2 \phi(x,t) = (\Delta - m_{\mathrm{ph}}^2)\phi(x,t) - e\varphi(x - q_t), \quad (19.13)$$

$$m\ddot{q}_t = -e\nabla\phi_\varphi(q_t). \quad (19.14)$$

The solutions to (19.13) and (19.14) bear a fair qualitative similarity to the Abraham model, in particular, our discussion of the energy–momentum relation, the radiation reaction, and the center manifold could be repeated almost word for word.

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The quantization of (19.12) is straightforward. $\pi(x)$ and $\phi(x)$ become a scalar Bose field with commutation relations

$$[\phi(x), \pi(x')] = \mathrm{i}\delta(x - x'), \tag{19.15}$$

setting $\hbar = 1$. It is convenient to introduce the scalar creation and annihilation operators $a^*(k)$, $a(k)$ in momentum space. Then

$$\phi(x) = \int \mathrm{d}^3k \frac{1}{\sqrt{2\omega}} (2\pi)^{-3/2} \big(\mathrm{e}^{\mathrm{i}k\cdot x} a(k) + \mathrm{e}^{-\mathrm{i}k\cdot x} a^*(k)\big), \tag{19.16}$$

$$\pi(x) = \int \mathrm{d}^3k \sqrt{\omega/2} (2\pi)^{-3/2} \big(-\mathrm{i}\mathrm{e}^{\mathrm{i}k\cdot x} a(k) + \mathrm{i}\mathrm{e}^{-\mathrm{i}k\cdot x} a^*(k)\big) \tag{19.17}$$

with $\omega(k) = (k^2 + m_{\mathrm{ph}}^2)^{1/2}$. The quantized Hamiltonian reads

$$H = \frac{1}{2m} p^2 + H_{\mathrm{f}} + e\phi_\varphi(x), \tag{19.18}$$

where the momentum operator $p = -\mathrm{i}\nabla_x$ is canonically conjugate to the position x and

$$\phi_\varphi(x) = \int \mathrm{d}^3k \widehat{\varphi}(k) \frac{1}{\sqrt{2\omega}} \big(\mathrm{e}^{\mathrm{i}k\cdot x} a(k) + \mathrm{e}^{-\mathrm{i}k\cdot x} a^*(k)\big) \tag{19.19}$$

with $\widehat{\varphi}$ assumed to be real. H acts on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$; we call it the *Nelson Hamiltonian*. If $\int \mathrm{d}^3k |\widehat{\varphi}|^2 (\omega^{-2} + 1) < \infty$, then the interaction $e\phi_\varphi(x)$ is infinitesimally bounded with respect to $\frac{1}{2m} p^2 + H_{\mathrm{f}}$ and, by the Kato–Rellich theorem, H is self-adjoint with domain $D((p^2/2m) + H_{\mathrm{f}})$.

Since H is invariant under translations, the total momentum

$$P = p + P_{\mathrm{f}}, \quad P_{\mathrm{f}} = \int \mathrm{d}^3k k a^*(k) a(k), \tag{19.20}$$

is conserved. As in section 15.2, H can be unitarily transformed to fixed total momentum with the result

$$H(P) = \frac{1}{2m} (P - P_{\mathrm{f}})^2 + H_{\mathrm{f}} + e\phi_\varphi \tag{19.21}$$

and $\phi_\varphi = \phi_\varphi(0)$. The ground state energy of (19.21) defines the energy–momentum relation $E(P)$. If one sets

$$H(0) = \frac{1}{2m} P_{\mathrm{f}}^2 + H_{\mathrm{f}} + e\phi_\varphi, \tag{19.22}$$

then the effective mass is given by

$$\frac{m}{m_{\mathrm{eff}}} = 1 - \frac{2}{3m} \langle \psi_{\mathrm{g}}, P_{\mathrm{f}} \cdot \frac{1}{H(0) - E(0)} P_{\mathrm{f}} \psi_{\mathrm{g}} \rangle_{\mathcal{F}}, \tag{19.23}$$

where ψ_{g} is the ground state of $H(0)$, i.e. $H(0)\psi_{\mathrm{g}} = E(0)\psi_{\mathrm{g}}$.

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With this somewhat rapid introduction the problem under consideration is whether the Nelson Hamiltonian (19.18) remains well-defined in the point-charge limit $\varphi(x) \to \delta(x)$. Following the usual convention to denote the ultraviolet cutoff in momentum space by Λ, the point-charge limit means scaling the form factor as

$$\varphi_\Lambda(x) = \Lambda^3 \varphi(\Lambda x), \quad \text{respectively} \quad \widehat{\varphi}_\Lambda(k) = \widehat{\varphi}(k/\Lambda) \tag{19.24}$$

with $\Lambda \to \infty$.

The interaction $\phi_\varphi(x)$ is bounded relative to H_f only if $\int \mathrm{d}^3k |\widehat{\varphi}|^2/\omega^2 < \infty$. At $\Lambda = \infty$ this condition is violated indicating that the limit $\Lambda \to \infty$ is singular. To find out how singular we compute the ground state energy to second order in e^2, regarding in (19.22) $e\phi_\varphi$ as a perturbation. Then

$$E(0) = -e^2 \int \mathrm{d}^3k |\widehat{\varphi}_\Lambda(k)|^2 \frac{1}{2\omega} \Big(\omega + \frac{1}{2m}k^2\Big)^{-1} + \mathcal{O}(e^4) \tag{19.25}$$

which diverges as $-\log \Lambda$ for $\Lambda \to \infty$. Physically only energy differences count and one may want to subtract $E(0)$ from $H(0)$. After all, in the definition of H_f an infinite zero-point energy was already subtracted. There are two caveats to this. First, $E(0)$ from (19.25) is only a second-order perturbation and a priori one does not know which energy to subtract. More importantly, it must be ensured that physical properties are not distorted as $\Lambda \to \infty$. In the classical model the effective mass is the relevant indicator and we adopt the same criterion here. From (19.23) we compute, compare with (15.36),

$$\frac{m}{m_\mathrm{eff}} = 1 - \frac{2}{3m} e^2 \int \mathrm{d}^3k |\widehat{\varphi}_\Lambda(k)|^2 \frac{1}{2\omega} k^2 \Big(\omega + \frac{1}{2m}k^2\Big)^{-3} + \mathcal{O}(e^4), \tag{19.26}$$

which stays finite as $\Lambda \to \infty$, at least to second order, fostering our hope that $H(0) - E(0)$ is a well-defined Hamiltonian as $\Lambda \to \infty$.

The Nelson model has the simplifying feature that the energy renormalization can be made explicit through a unitary transformation originally introduced by E. P. Gross. It is constructive to work out the case of N charges coupled to the Bose field. The Hamiltonian (19.18) then generalizes to

$$H_N = \sum_{j=1}^N \frac{1}{2m_j} p_j^2 + H_\mathrm{f} + \sum_{j=1}^N e_j \phi_\varphi(x_j). \tag{19.27}$$

Here the j-th particle has position x_j, momentum $p_j = -\mathrm{i}\nabla_{x_j}$, mass m_j, and charge e_j. We define

$$T = -\sum_{j=1}^N e_j \int \mathrm{d}^3k \widehat{\varphi} \frac{1}{\sqrt{2\omega}} \beta_j \big(\mathrm{e}^{\mathrm{i}k\cdot x_j} a(k) - \mathrm{e}^{-\mathrm{i}k\cdot x_j} a^*(k)\big) \tag{19.28}$$

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with $\beta_j = (\omega + \frac{1}{2m_j}k^2)^{-1}$. e^{-T} is the Gross transformation. Since $\int \mathrm{d}^3k\beta_j^2/\omega < \infty$ provided $m_{\mathrm{ph}} > 0$, e^{-T} is unitary and well defined in $\mathcal{H}$ even at $\Lambda = \infty$. Let us set

$$A^-_{\varphi j}(x) = -\int \mathrm{d}^3k\widehat{\varphi}\frac{1}{\sqrt{2\omega}}k\beta_j \mathrm{e}^{\mathrm{i}k\cdot x}a(k)\,,$$
$$A^-_{\varphi j}(x)^* = A^+_{\varphi j}(x)\,, \quad A_{\varphi j}(x) = A^+_{\varphi j}(x) + A^-_{\varphi j}(x)\,. \tag{19.29}$$

◇ We here use on purpose the same notation as for the transverse vector potential, since through the Gross transformation $A_{\varphi j}(x)$ appears in the Hamiltonian in the same way as the transverse vector potential does for the Pauli–Fierz model. However $A_{\varphi j}(x)$ is longitudinal and $[p, A_{\varphi j}(x)] \neq 0$. It is better behaved at small x because the factor β_j gains one extra power in decay at large k. *Only* for section 19.2, A_φ is defined through (19.29). ◇

e^{-T} acts as

$$\mathrm{e}^{T}p_j\mathrm{e}^{-T} = p_j - e_jA^-_{\varphi j}(x_j) - e_jA^+_{\varphi j}(x_j), \quad \mathrm{e}^{T}x_j\mathrm{e}^{-T} = x_j,$$
$$\mathrm{e}^{T}a(k)\mathrm{e}^{-T} = a(k) - \sum_{j=1}^{N} e_j\widehat{\varphi}\frac{1}{\sqrt{2\omega}}\beta_j\mathrm{e}^{-\mathrm{i}k\cdot x_j},$$
$$\mathrm{e}^{T}a^*(k)\mathrm{e}^{-T} = a^*(k) - \sum_{j=1}^{N} e_j\widehat{\varphi}\frac{1}{\sqrt{2\omega}}\beta_j\mathrm{e}^{\mathrm{i}k\cdot x_j}. \tag{19.30}$$

When normally ordered, the Gross-transformed Hamiltonian becomes

$$\begin{aligned}\mathrm{e}^{T}H_N\mathrm{e}^{-T} = &\sum_{j=1}^{N}\frac{1}{2m_j}\big(p_j^2 - 2e_jp_j\cdot A^-_{\varphi j}(x_j) - 2e_jA^+_{\varphi j}(x_j)\cdot p_j\\ &+ e_j^2A^-_{\varphi j}(x_j)^2 + e_j^2A^+_{\varphi j}(x_j)^2 + 2e_j^2A^+_{\varphi j}(x_j)\cdot A^-_{\varphi j}(x_j)\big) + H_{\mathrm{f}}\\ &- \sum_{i\neq j=1}^{N} e_ie_j\int \mathrm{d}^3k|\widehat{\varphi}|^2\frac{1}{2\omega}(\beta_i + \beta_j - \omega\beta_i\beta_j)\mathrm{e}^{\mathrm{i}k\cdot(x_i-x_j)}\\ &- \sum_{j=1}^{N} e_j^2\int \mathrm{d}^3k|\widehat{\varphi}|^2\frac{1}{2\omega}\beta_j.\end{aligned} \tag{19.31}$$

Note that $A_{\varphi j}(x_j)$ does not commute with p_j. The last term in (19.31) is the energy renormalization, granted for a moment that the remainder is a well-defined Hamiltonian with energy bounded from below. The energy renormalization coincides with $E(0)$ as computed from second-order perturbation theory, compare with (19.25), and diverges as $-N\log\Lambda$.

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The next to last term in (19.31) is the instantaneous interaction between the particles which dominates their dynamics at small velocities; see section 20.2. Let us set $m_{\rm ph} = 0$ and $\Lambda = \infty$. Then the interaction potential for particles i and j is

$$V_{ij}(x) = -e_i e_j \int \mathrm{d}^3k \frac{1}{\omega}(\beta_i + \beta_j - \omega\beta_i\beta_j)\mathrm{e}^{\mathrm{i}k\cdot x} \tag{19.32}$$

as a function of their relative distance. $V_{ij}(x) \cong -e_i e_j/4\pi|x|$ for large $|x|$, and $V_{ij}(x) \cong e_i e_j \log|x|$ for small $|x|$. Even in the point-charge limit the interaction deviates from a strict Coulomb law at distances on the scale of the Compton wavelength for particles i, j. This confirms our previous findings that it is natural to regard the Compton wavelength as an effective size of the charged particles in the quantized theory. Even more importantly, the sign of the interaction is $-e_i e_j$. In the scalar theory particles of equal charge attract, those of opposite charge repel each other. Thus particles of opposite charge tend to segregate and a big cluster of one sign would be separated from a big cluster of the opposite sign. There could not be the delicate balance between nuclei (ions) and electrons which is responsible for the formation of atoms and molecules. If the photons were spinless, the world would have no similarity to the one we know.

We are left with the first piece of (19.31). Since it is additive in the particles, for notational simplicity we return to $N = 1$ and rewrite it as

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \mathrm{e}^{T}\Big(H + e^2 \int \mathrm{d}^3k |\widehat{\varphi}_\Lambda|^2 \frac{1}{2\omega}\beta\Big)\mathrm{e}^{-T} \\
&\quad = \frac{1}{2m}p^2 - \frac{e}{m}(p\cdot A^-(x) + A^+(x)\cdot p) \\
&\qquad + \frac{e^2}{2m}\big(A^-(x)^2 + A^+(x)^2 + 2A^+(x)\cdot A^-(x)\big) + H_{\rm f} \\
&\quad = \widetilde{H}_{\rm ren}.
\end{aligned} \tag{19.33}$$

Here, using $\widehat{\varphi}_\Lambda(0) = (2\pi)^{-3/2}$, we have

$$\begin{aligned}
A^-(x) &= -\int \mathrm{d}^3k \frac{1}{\sqrt{2\omega}} k\beta (2\pi)^{-3/2}\mathrm{e}^{\mathrm{i}k\cdot x} a(k)\,, \\
A^+(x) &= A^-(x)^*, \quad A(x) = A^+(x) + A^-(x).
\end{aligned} \tag{19.34}$$

$\widetilde{H}_{\rm ren}$ is the physical Hamiltonian in the point-charge limit. The splitting into A^- and A^+ results from normal ordering. The A-field is longitudinal but otherwise plays a role very similar to the vector potential in the Pauli–Fierz model.

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In the following it will be convenient to rewrite $\widetilde{H}_{\text{ren}}$ in dimensionless form. Through the canonical transformation (13.88) one obtains

$$\begin{aligned}\widetilde{H}_{\text{ren}} &= m H_{\text{ren}} \\ &= m\Big(\frac{1}{2}p^2 - e(p\cdot A^-(x) + A^+(x)\cdot p) \\ &\quad + \frac{1}{2}e^2\big(A^-(x)^2 + A^+(x)^2 + 2A^+(x)\cdot A^-(x)\big) + H_{\text{f}}\Big) \\ &= m(H_0 + H_{\text{int}}) \end{aligned} \tag{19.35}$$

with $\omega = (k^2 + (m_{\text{ph}}/m)^2)^{1/2}$, $\beta = (\omega + \frac{1}{2}k^2)^{-1}$, and $A^-(x)$ as in (19.34). We repeat the relative form bound estimates from section 13.3 with the result

$$|\langle\psi, H_{\text{int}}\psi\rangle| \le \Big(3e^2(2\pi)^{-3}\int \mathrm{d}^3k k^2\beta^2\omega^{-2}\Big)\langle\psi, H_0\psi\rangle. \tag{19.36}$$

If

$$3e^2(2\pi)^{-3}\int \mathrm{d}^3k k^2\beta^2\omega^{-2} < 1, \tag{19.37}$$

then H_{int} is H_0-form bounded with a bound less than 1, which implies that H_{ren} is a self-adjoint operator bounded from below.

The total momentum transforms as $\mathrm{e}^T(p + P_{\text{f}})\mathrm{e}^{-T} = p + P_{\text{f}} = P$ and

$$[H_{\text{ren}}, P] = 0, \tag{19.38}$$

as can also be checked directly. For fixed total momentum H_{ren} becomes

$$\begin{aligned}H_{\text{ren}}(P) &= \frac{1}{2}(P - P_{\text{f}})^2 - e\big((P - P_{\text{f}})\cdot A^- + A^+\cdot(P - P_{\text{f}})\big) \\ &\quad + \frac{1}{2}e^2\big(A^+\cdot A^+ + A^-\cdot A^- + 2A^+\cdot A^-\big) + H_{\text{f}}\end{aligned} \tag{19.39}$$

as acting on $\mathcal{F}$ with the shorthand $A = A(0)$.

The expression in (19.37) is finite also for $m_{\text{ph}} = 0$. Thus H_{ren} and $H_{\text{ren}}(P)$ are well-defined Hamiltonians even for massless bosons with infrared and ultraviolet cutoffs removed. However, e^{-T} is unitarily implemented only for $m_{\text{ph}} > 0$. At $m_{\text{ph}} = 0$, H and H_{ren} are not unitarily equivalent. As can be seen from (19.30) the Gross-transformed ϕ-field has a vacuum expectation which decays as $-e/4\pi|x|$ for large x and thus singles out the $P = 0$ representation; compare with our discussion in section 19.1. $H_{\text{ren}}(0)$ has a ground state in Fock space, whereas $H_{\text{ren}}(P)$, $P \neq 0$, has no ground state in Fock space, just as is the case for the Pauli–Fierz model.

H_{ren} is the result of a mathematical limit procedure and it is not automatically guaranteed that the limit Hamiltonian inherits the physically desired properties.

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For a modest check we compute the self-energy, the effective mass, and the binding energy for hydrogen-like atoms in low-order perturbation theory. While these quantities are well defined, it is not known whether they can be expanded around $e = 0$. Only if the bosons had the strictly positive mass $m_{\rm ph} > 0$, $H_{\rm ren}(P)$ for small $|P|$ and $H_{\rm ren} - Ze^2 V_{\rm coul}$ have a gap between their ground state and the continuous spectrum, which implies a convergent Taylor expansion at $e = 0$.

(i) Self-energy

We expand $E_{\rm ren}(0)$ in powers of e^2. $H_{\rm ren}(0)$ is split as

$$H_{\rm ren}(0) = H_0 + eH_1 + \frac{1}{2}e^2 H_2 \tag{19.40}$$

with $H_0 = \frac{1}{2}P_{\rm f}^2 + H_{\rm f}$, $H_1 = P_{\rm f} \cdot A^- + A^+ \cdot P_{\rm f}$, $H_2 = A^- \cdot A^- + A^+ \cdot A^+ + 2A^+ \cdot A^-$. The unperturbed ground state is Ω with energy 0. The expansion is written as

$$E_{\rm ren}(0) = \frac{1}{2}e^2 E^{(2)} + \frac{1}{4!}e^4 E^{(4)} + \frac{1}{6!}e^6 E^{(6)} + \mathcal{O}(e^8). \tag{19.41}$$

$E^{(2)} = 0$, since $H_1\Omega = 0$. The next order is

$$\begin{aligned} \frac{1}{4!}e^4 E^{(4)} &= -e^4 \frac{1}{2}\langle \Omega, A^- \cdot A^- \frac{1}{H_0} A^+ \cdot A^+ \Omega\rangle_{\mathcal{F}} \\ &= -e^4 (2\pi)^{-6} \int \mathrm{d}^3 k_1 \int \mathrm{d}^3 k_2 \frac{1}{2\omega_1}\beta_1^2 \frac{1}{2\omega_2}\beta_2^2 (k_1 \cdot k_2)^2 \frac{1}{E_{12}} \end{aligned} \tag{19.42}$$

with $\omega_i = \omega(k_i)$, $\beta_i = \left(\omega(k_i) + \frac{1}{2}k_i^2\right)^{-1}$, $i = 1, 2$, and $E_{12} = \omega_1 + \omega_2 + \frac{1}{2}(k_1 + k_2)^2$.

For the discussion below we still need the sixth order, which is given by

$$\begin{aligned} e^6 \frac{1}{6!}E^{(6)} = e^6 \frac{1}{4}\Big(&- \langle \Omega, A^- \cdot A^- \frac{1}{H_0} P_{\rm f} \cdot A^- \frac{1}{H_0} A^+ \cdot P_{\rm f} \frac{1}{H_0} A^+ \cdot A^+ \Omega\rangle_{\mathcal{F}} \\ &- \langle \Omega, A^- \cdot A^- \frac{1}{H_0} A^+ \cdot P_{\rm f} \frac{1}{H_0} P_{\rm f} \cdot A^- \frac{1}{H_0} A^+ \cdot A^+ \Omega\rangle_{\mathcal{F}} \\ &+ \langle \Omega, A^- \cdot A^- \frac{1}{H_0} A^+ \cdot A^- \frac{1}{H_0} A^+ \cdot A^+ \Omega\rangle_{\mathcal{F}} \Big). \end{aligned} \tag{19.43}$$

The integrals appearing in the expressions for $E^{(4)}$ and $E^{(6)}$ are convergent.

(ii) Effective mass

From the definition (15.23) and (19.39) one concludes

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}\langle \psi_{\rm g}, (P_{\rm f} + eA) \cdot (H_{\rm ren}(0) - E_{\rm ren}(0))^{-1} (P_{\rm f} + eA)\psi_{\rm g}\rangle_{\mathcal{F}} \tag{19.44}$$

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with $H_{\mathrm{ren}}(0)\psi_{\mathrm{g}} = E_{\mathrm{ren}}(0)\psi_{\mathrm{g}}$. m/m_{eff} is even in e and, provided $m_{\mathrm{ph}} > 0$, analytic for small e. Expanding in (19.44) to order e^2 by the scheme already explained, one finds

$$\frac{m_{\mathrm{eff}}}{m} = 1 + \frac{2}{3}e^2(2\pi)^{-3}\int \mathrm{d}^3k(k^2\beta^3/2\omega) + \mathcal{O}(e^4) \tag{19.45}$$

which agrees with (19.26) in the limit $\Lambda \to \infty$. For $m_{\mathrm{ph}} = 0$,

$$m_{\mathrm{eff}} = m\Big(1 + \frac{1}{6\pi^2}e^2 + \mathcal{O}(e^4)\Big) \tag{19.46}$$

is obtained. Since the mass renormalization is finite, the relation (19.46) allows us in principle to obtain the bare mass m from an acceleration experiment at small velocities which measures m_{eff} according to our discussion in section 16.6.

(iii) Binding energy

We consider two charges, a nucleus of charge Ze of infinite mass nailed down at the origin and a "meson" of charge e. According to (19.31) the renormalized Hamiltonian for $\Lambda \to \infty$ reads then

$$H = H_{\mathrm{ren}} - \frac{Ze^2}{4\pi|x|} \tag{19.47}$$

in units of m. For sufficiently small e, $e \neq 0$, H has a ground state. Denoting its ground state energy by E, by definition the (positive) binding energy is

$$E_{\mathrm{bin}} = m\big(E_{\mathrm{ren}}(0) - E\big), \tag{19.48}$$

since $mE_{\mathrm{ren}}(0)$ is the energy of the meson far away from the nucleus. E_{bin} is even in e and proportional to the bare mass m. Physically the natural units for E_{bin} are $m_{\mathrm{eff}}c^2$ and we write

$$E_{\mathrm{bin}} = m_{\mathrm{eff}}h_{\mathrm{bin}}(e^2), \tag{19.49}$$

which is regarded as a definition of h_{bin}.

We expand E in powers of e^2. To better follow the subtraction of the self-energy we first transform to the total momentum representation. Then the split-up for H is

$$\begin{aligned} H &= H_{\mathrm{at}} + H_{\mathrm{f}} + \frac{1}{2}P_{\mathrm{f}}^2 - p\cdot P_{\mathrm{f}} + e\big((P_{\mathrm{f}} - p)\cdot A^- + A^+\cdot(P_{\mathrm{f}} - p)\big) \\ &\quad + \frac{1}{2}e^2(A^-\cdot A^- + A^+\cdot A^+ + 2A^+\cdot A^-) \\ &= H_0 + eH_1 + \frac{1}{2}e^2H_2. \end{aligned} \tag{19.50}$$

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The atomic Hamiltonian is $H_{\rm at} = \frac{1}{2}p^2 - Ze^2/4\pi|x|$ with ground state energy $E_{\rm at} = -\frac{1}{2}(Ze^2/4\pi)^2$ and ground state $\psi_{\rm at}(x) = (\pi r_{\rm B}^3)^{-1/2}{\rm e}^{-|x|/r_{\rm B}}$, $r_{\rm B} = 4\pi/Ze^2$. The unperturbed ground state is $\psi_{\rm at} \otimes \Omega$ with energy $E_{\rm at}$. The perturbation expansion up to order e^6 is given as in (19.42), (19.43) with the corresponding substitutions for H_0, H_1. Let us first consider those terms not containing either $p \cdot A^-$ or $A^+ \cdot p$. The inverse operator $(H_{\rm at} + H_{\rm f} + \frac{1}{2}P_{\rm f}^2 - p \cdot P_{\rm f})^{-1}$ is expanded in $p \cdot P_{\rm f}$. Since $\langle \psi_{\rm at}, p^2\psi_{\rm at}\rangle_{L^2} = -2E_{\rm at}$, only the leading term contributes and all the self-energy terms cancel including order e^6. The only remaining contribution is

$$\begin{aligned} E_{\rm bin} &= -E_{\rm at} + e^2\langle \psi_{\rm at}\otimes\Omega,\ p\cdot A^-(H_0)^{-1}A^+\cdot p\psi_{\rm at}\otimes\Omega\rangle_{\mathcal{H}} + \mathcal{O}(e^8) \\ &= -E_{\rm at} + e^2(2\pi)^{-3}\int {\rm d}^3k \frac{1}{2\omega}\beta^2\langle \psi_{\rm at},\ p\cdot k \frac{1}{H_{\rm at} - E_{\rm at} + E_1 - p\cdot k} p\cdot k\psi_{\rm at}\rangle_{L^2} \\ &\quad + \mathcal{O}(e^8) \end{aligned} \tag{19.51}$$

with $E_1 = \beta^{-1}$. Expanding in $p \cdot k$ and in $H_{\rm at} - E_{\rm at}$ yields

$$\begin{aligned} E_{\rm bin} &= -E_{\rm at} + \frac{1}{3}e^2(2\pi)^{-3}\int {\rm d}^3k \frac{1}{2\omega}\beta^2k^2\langle \psi_{\rm at}, (E_1)^{-1}p^2\psi_{\rm at}\rangle_{L^2} + \mathcal{O}(e^8) \\ &= -E_{\rm at}\Big(1 + \frac{2}{3}e^2(2\pi)^{-3}\int {\rm d}^3k(k^2\beta^3/2\omega)\Big) + \mathcal{O}(e^8). \end{aligned} \tag{19.52}$$

Note that in (19.51) the Taylor coefficient of order e^{10} is infrared divergent, which implies that $E_{\rm bin}$ cannot be analytic at $e = 0$.

As a final step, we carry out the mass renormalization to order e^2 as required according to (19.49). The corrections $\mathcal{O}(e^6)$ cancel and

$$E_{\rm bin} = -m_{\rm eff}E_{\rm at} + \mathcal{O}(e^8). \tag{19.53}$$

$h_{\rm bin}$ acquires a radiative correction at least as small as $\mathcal{O}(e^8)$, which confirms the conventional picture. For small coupling the predictions of the one-particle theory are reliable. The coupling to the field generates to leading order the attractive Coulomb potential. Further effects of the interaction with the scalar field are small. Having no compelling incentive, the strong coupling regime of $H_{\rm ren}$ is apparently little explored. It is conceivable that for large e the kinetic energy of the meson cannot balance the singular Coulomb attraction. If so, H of (19.50) would no longer be bounded from below.

19.3 Ultraviolet limit, energy and mass renormalization

The ultraviolet limit of the Pauli–Fierz model is a poorly understood subject. All we can do is to explain the few hints available, which in their optimistic

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interpretation indicate that the ultraviolet cutoff may be removed at the expense of a renormalization in energy and mass.

As we learned from the Nelson model, the indicative quantities are the self-energy, the effective mass, and the binding energy of the electron. To study these properties in the point-charge (= ultraviolet) limit, it is convenient to switch to relativistic units as explained at the end of section 13.4. To repeat, for constant total momentum p one has

$$H(p) = \frac{1}{2}\big(p - P_{\mathrm{f}} - \sqrt{4\pi\alpha}A_\varphi\big)^2 + H_{\mathrm{f}}, \tag{19.54}$$

where

$$H_{\mathrm{f}} = \sum_{\lambda=1,2}\int \mathrm{d}^3k|k|a^*(k,\lambda)a(k,\lambda), \quad P_{\mathrm{f}} = \sum_{\lambda=1,2}\int \mathrm{d}^3k k a^*(k,\lambda)a(k,\lambda),$$
$$A_\varphi = \sum_{\lambda=1,2}\int \mathrm{d}^3k\widehat{\varphi}(k/\Lambda\lambda_{\mathrm{c}})\frac{1}{\sqrt{2|k|}}\big(a(k,\lambda) + a^*(k,\lambda)\big). \tag{19.55}$$

Here $\alpha = e^2/4\pi\hbar c$ is the fine-structure constant written in Heaviside–Lorentz units, $\lambda_{\mathrm{c}} = \hbar/mc$ the Compton wavelength, and Λ the large k cutoff, $\Lambda \to \infty$ eventually. Energies are measured in units of mc^2, momenta in units of mc. In the case of the hydrogen atom with the nucleus pinned down at the origin, in relativistic units the Hamiltonian reads

$$H = \frac{1}{2}\big(-\mathrm{i}\nabla_x - \sqrt{4\pi\alpha}A_\varphi(x)\big)^2 + H_{\mathrm{f}} - \frac{\alpha Z}{|x|}, \tag{19.56}$$

where we ignored the smearing of the Coulomb potential by φ; compare with (13.89).

19.3.1 Self-energy

Since $E(p)$ has its minimum at $p = 0$, the self-energy is given by

$$E_{\mathrm{self}} = mc^2 E_\Lambda, \quad E_\Lambda = \inf_{\|\psi\|=1}\langle\psi, H(0)\psi\rangle_{\mathcal{F}}, \tag{19.57}$$

and the first task is to get some idea of how E_Λ diverges as $\Lambda \to \infty$. Of course, the self-energy has no observable consequences. Still, it is a sort of theoretical test which must be passed before more difficult problems can be tackled. We

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normal-order $H(0)$ as

$$\begin{aligned} H(0) &= \frac{1}{2}P_{\mathrm{f}}^2 + H_{\mathrm{f}} + e(P_{\mathrm{f}}\cdot A_\varphi^- + A_\varphi^+\cdot P_{\mathrm{f}}) + \frac{1}{2}e^2(A_\varphi^+\cdot A_\varphi^+ + A_\varphi^-\cdot A_\varphi^- \\ &\quad + 2A_\varphi^+\cdot A_\varphi^-) + 4\pi\alpha\int \mathrm{d}^3k|\widehat{\varphi}(k/\Lambda\lambda_{\mathrm{c}})|^2\frac{1}{2|k|} \\ &= H_0 + eH_1 + \frac{1}{2}e^2H_2 \ + E_0, \end{aligned} \tag{19.58}$$

where A_φ is the transverse vector potential split as $A_\varphi = A_\varphi^+ + A_\varphi^-,\ A_\varphi^+ = (A_\varphi^-)^*$. E_0 is the lowest order of the self-energy and diverges as Λ^2. The next order is computed as in the Gross-transformed Nelson Hamiltonian with the result

$$\begin{aligned} E_\Lambda &= E_0 - (4\pi\alpha)^2\frac{1}{2}\langle\Omega,\, A_\varphi^-\cdot A_\varphi^-\frac{1}{H_0}A_\varphi^+\cdot A_\varphi^+\Omega\rangle_{\mathcal{F}} \\ &= E_0 - (4\pi\alpha)^2\int \mathrm{d}^3k_1\int \mathrm{d}^3k_2|\widehat{\varphi}(k_1/\Lambda\lambda_{\mathrm{c}})|^2|\widehat{\varphi}(k_2/\Lambda\lambda_{\mathrm{c}})|^2\big((2|k_1|)(2|k_2|) \\ &\quad \times 4(|k_1|+|k_2|+\frac{1}{2}(k_1+k_2)^2)\big)^{-1}\big(1+(\widehat{k}_1\cdot\widehat{k}_2)^2\big) + \mathcal{O}(\alpha^3). \end{aligned} \tag{19.59}$$

The order α^2 diverges also as Λ^2 with a negative prefactor, however. Thus in contrast to the Nelson model, mere perturbation theory does not tell of the self-energy. If the electron spin were included, there are cancellations between E_0 and the spin contribution which yields a divergence proportional to Λ.

A second attempt is to guess a variational wave function. Variation over coherent states leads to the trivial minimizer $\psi = \Omega$, which reflects that for $p = 0$ the transverse vector field vanishes classically. A more ingenious approach is due to Lieb and Loss. They give up the zero total momentum restriction and consider

$$H = \frac{1}{2}\big(-\mathrm{i}\nabla_x - \sqrt{4\pi\alpha}A_\varphi(x)\big)^2 + H_{\mathrm{f}}. \tag{19.60}$$

The minimum of H equals the self-energy E_Λ; see section 15.2. The variational wave function is taken to be of the Pekar form $\psi = \phi\otimes\Phi$, with $\Phi\in\mathcal{F}$ and $\phi(x)$ a real function. Therefore

$$\begin{aligned} E_\Lambda &\le \langle\psi, H\psi\rangle_{\mathcal{H}} \\ &= \frac{1}{2}\int \mathrm{d}^3x|\nabla\phi(x)|^2 + 2\pi\alpha\int \mathrm{d}^3x\phi(x)^2\langle\Phi, A_\varphi(x)^2\Phi\rangle_{\mathcal{F}} + \langle\Phi, H_{\mathrm{f}}\Phi\rangle_{\mathcal{F}}, \end{aligned} \tag{19.61}$$

since the cross-term has average zero. For Φ we choose the ground state of

$$H_\phi = 2\pi\alpha\int \mathrm{d}^3x\phi(x)^2A_\varphi(x)^2 + H_{\mathrm{f}}. \tag{19.62}$$

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H_ϕ is a quadratic Hamiltonian and thus its ground state energy is given by

$$E_\phi = \frac{1}{2}\mathrm{tr}\big[\big(Q_\perp\widehat{\varphi}(-\Delta + 4\pi\alpha\phi(x)^2)\widehat{\varphi}Q_\perp\big)^{1/2} - \big(Q_\perp\widehat{\varphi}(-\Delta)\widehat{\varphi}Q_\perp\big)^{1/2}\big]. \quad (19.63)$$

Here the trace is over $L^2(\mathbb{R}^3,\mathbb{R}^3)$, $Q_\perp$ is the projection onto transverse vector fields, $\widehat{\varphi}$ is regarded as a multiplication operator in momentum space, and $-\Delta + 4\pi\alpha\phi(x)^2$ is diagonal with respect to the vector indices. Combining (19.61) and (19.63) we obtain

$$E_\Lambda \leq \frac{1}{2}\int \mathrm{d}^3x|\nabla\phi(x)|^2 + E_\phi \quad (19.64)$$

as a nonlinear variational bound for E_Λ.

The difference of square roots is unpleasant. To simplify we use that $\mathrm{tr}[\sqrt{A+B} - \sqrt{A} - \sqrt{B}] \leq 0$. Then

$$E_\phi \leq \sqrt{\pi\alpha}\,\mathrm{tr}[(Q_\perp\widehat{\varphi}\phi^2\widehat{\varphi}Q_\perp)^{1/2}] \leq \sqrt{\pi\alpha}\,\mathrm{tr}[(\widehat{\varphi}\phi^2\widehat{\varphi})^{1/2}], \quad (19.65)$$

since the square root is increasing. In spirit, the bound (19.65) equals $\mathrm{tr}[\widehat{\varphi}\phi] = (2\pi)^{3/2}\varphi(0)\widehat{\phi}(0)$. To actually achieve it, one sets $\phi(x) = \phi_K(x) = K^{3/2}\phi_\mathrm{s}(Kx)$ with scaling parameter $K \cong \Lambda^{6/7}$, such that $\widehat{\phi}_\mathrm{s}$ has support in a ball of radius 1 and $\|\phi_\mathrm{s}\| = 1$. Let us choose $\widehat{\varphi}_\Lambda(k) = \chi(|k|/\Lambda\lambda_\mathrm{c})$, $\chi(|k|) = (2\pi)^{-3/2}$ for $|k| \leq 1$, $\chi = 0$ for $|k| > 1$. If $K < \Lambda\lambda_\mathrm{c}$, then $\widehat{\varphi}_{2\Lambda}\widehat{\phi}_K\widehat{\varphi}_\Lambda = \widehat{\phi}_K\widehat{\varphi}_\Lambda$. Thus

$$\begin{aligned}\mathrm{tr}[(\widehat{\varphi}_\Lambda\phi_K^2\widehat{\varphi}_\Lambda)^{1/2}] &= \mathrm{tr}[(\widehat{\varphi}_\Lambda\phi_K\widehat{\varphi}_{2\Lambda}\phi_K\widehat{\varphi}_\Lambda)^{1/2}] \leq \mathrm{tr}[(\widehat{\varphi}_{2\Lambda}\phi_K\widehat{\varphi}_{2\Lambda}\phi_K\widehat{\varphi}_{2\Lambda})^{1/2}] \\ &= \mathrm{tr}[\widehat{\varphi}_{2\Lambda}\phi_K] = (2\pi)^{-1/2}(2/3\pi^3)(\Lambda\lambda_\mathrm{c})^3\int \mathrm{d}^3x\,\phi_K(x). \quad (19.66)\end{aligned}$$

Hence

$$E_\Lambda \leq \frac{1}{2}\int \mathrm{d}^3x|\nabla\phi_K(x)|^2 + \sqrt{2\alpha}(2\Lambda\lambda_\mathrm{c})^3(1/3\pi^3)\int \mathrm{d}^3x\,\phi_K(x). \quad (19.67)$$

One can choose ϕ_s such that $\widehat{\phi}_\mathrm{s}(0) > 0$. Then

$$E_\Lambda \leq c_1K^2 + c_2\sqrt{\alpha}(\Lambda\lambda_\mathrm{c})^3K^{-3/2} \quad (19.68)$$

with $c_1, c_2 > 0$. Optimizing with respect to K yields, for $\Lambda\lambda_\mathrm{c}$ sufficiently large,

$$E_\mathrm{self} \leq c_+(\Lambda\lambda_\mathrm{c})^{12/7}mc^2. \quad (19.69)$$

The guess is that 12/7 is the correct power. The best available lower bound is of order $(\Lambda\lambda_\mathrm{c})^{3/2}mc^2$.

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19.3.2 Effective mass

We turn to the effective mass, which is defined by

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3}\langle \psi_{\text{g}}, \left(P_{\text{f}} + \sqrt{4\pi\alpha}A_{\varphi}\right) \cdot \frac{1}{H(0) - E(0)} \left(P_{\text{f}} + \sqrt{4\pi\alpha}A_{\varphi}\right)\psi_{\text{g}}\rangle_{\mathcal{F}}, \tag{19.70}$$

where ψ_{g} is the ground state of $H(0)$, $H(0)\psi_{\text{g}} = E(0)\psi_{\text{g}}$, setting $p = 0$ in (19.54). m_{eff}/m is an even function of e. The issue of interest is its cutoff dependence for fixed e. Clearly the right-hand side depends only on $\Lambda\lambda_{\text{c}}$, compare with (19.55). This allows us to write

$$\frac{m_{\text{eff}}}{m} = h_{\text{mas}}(\hbar\Lambda/mc), \tag{19.71}$$

which defines h_{mas}. h_{mas} depends on α with $h_{\text{mas}} \geq 1$ and $h_{\text{mas}}(0) = 1$.

If h_{mas} has a finite limit as $\Lambda \to \infty$, then

$$m^{\star}_{\text{eff}} = m h_{\text{mas}}(\infty), \tag{19.72}$$

where $m^{\star}_{\text{eff}}$ is the effective mass in the model with removed ultraviolet cutoff. This situation is realized for the Nelson Hamiltonian (19.18). On the other hand, if asymptotically h_{mas} increases linearly in Λ, i.e. $h_{\text{mas}}(\lambda) = b\lambda$, $b > 0$, for large λ, then

$$m^{\star}_{\text{eff}} = \lim_{\Lambda\to\infty} m_{\Lambda} h_{\text{mas}}(\hbar\Lambda/m_{\Lambda}c) = \infty \tag{19.73}$$

for any choice of $m = m_{\Lambda}$ as long as $m_{\Lambda} > 0$, which is required by a stable theory. Such a linear dependence we found for the classical Abraham model, where in the point-charge limit the electron becomes infinitely heavy with no counterbalancing mechanism.

The most intriguing case, presumably realized in the Pauli–Fierz model, is

$$h_{\text{mas}}(\lambda) \cong b_0\lambda^{\gamma} \tag{19.74}$$

for large λ with $0 < \gamma < 1$ and γ possibly depending on α. Then

$$m_{\text{eff}} = b_0(c^{-1}\hbar\Lambda)^{\gamma} m^{1-\gamma}. \tag{19.75}$$

Setting now

$$m = (c^{-1}\hbar\Lambda)^{-\gamma/(1-\gamma)} b_1^{1/(1-\gamma)}, \tag{19.76}$$

we obtain

$$m^{\star}_{\text{eff}} = b_0 b_1. \tag{19.77}$$

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Thus, as $\Lambda \to \infty$ simultaneously we have to let $m \to 0$ in accordance with (19.76), recall that $\gamma < 1$. The effective mass $m^\star_{\rm eff}$ stays finite in this limit. Such a limiting procedure is the standard *mass renormalization*. b_0 is dimensionless and defined through (19.70). b_1 has the dimension of mass and is a free scaling parameter adjustable to the effective mass $m^\star_{\rm eff}$ as supplied from sources outside of theory, e.g. from an experiment. Note that, in contrast to a finite mass renormalization, the bare mass m has disappeared from the scene.

At present the only way of deciding whether $\gamma < 1$ is a sort of consistency check by expanding $m_{\rm eff}$ in powers of α. We use the normal-ordered H_0 from (19.58) and follow the scheme outlined in the case of Nelson's model.

The order α is straightforward, since the approximations $\psi_{\rm g} = \Omega$ and $E(0) = 0$ suffice, giving the result

$$\frac{m}{m_{\rm eff}} = 1 - \frac{2}{3}(4\pi\alpha) \int {\rm d}^3k |\widehat{\varphi}(k/\lambda_{\rm c})|^2 \Big(k^2\Big(1 + \frac{1}{2}|k|\Big)\Big)^{-1} + \mathcal{O}(\alpha^2). \quad (19.78)$$

The conventional sharp ultraviolet cutoff is made through the choice $\widehat{\varphi}(k) = (2\pi)^{-3/2}$ for $|k| \le \Lambda$ and $\widehat{\varphi}(k) = 0$ for $|k| > \Lambda$. Inserting in (19.78) we obtain

$$\begin{aligned} \frac{m}{m_{\rm eff}} &= 1 - \frac{4\alpha}{3\pi} \int_0^{\Lambda\lambda_{\rm c}} {\rm d}k \Big(1 + \frac{1}{2}k\Big)^{-1} + \mathcal{O}(\alpha^2) \\ &= 1 - \frac{8\alpha}{3\pi} \log\Big(1 + \frac{1}{2}\Lambda\lambda_{\rm c}\Big) + \mathcal{O}(\alpha^2). \end{aligned} \quad (19.79)$$

To order α, $m_{\rm eff}$ diverges as $\log\Lambda$ in contrast to the classical Abraham model which has a divergence proportional to Λ. Equation (19.79) suggests that

$$\frac{m_{\rm eff}}{m} = (\Lambda\lambda_{\rm c})^{8\alpha/3\pi} \quad (19.80)$$

for small α and large Λ. If so, $\gamma = 8\alpha/3\pi$. If the electron spin is included, then there is an extra contribution from the fluctuating magnetic field, see Eq. (15.68), and $8\alpha/3\pi$ is increased to $16\alpha/3\pi$.

The order α^2 requires more effort. The normalized ground state is needed up to order e^3 and is given by

$$\psi_{\rm g} = \Big(1 - e^2\frac{1}{2}\frac{1}{H_0}A^+_\varphi \cdot A^+_\varphi + e^3\frac{1}{H_0}(P_{\rm f} \cdot A^-_\varphi + A^+_\varphi \cdot P_{\rm f})\frac{1}{H_0}A^+_\varphi \cdot A^+_\varphi\Big)\,\Omega. \quad (19.81)$$

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Expanding $(H(0) - E(0))^{-1}$ results in six terms proportional to α^2. The details are lengthy and not particularly illuminating. We obtain

$$\begin{aligned}\frac{m}{m_{\text{eff}}} = 1 &- \frac{2}{3}(4\pi\alpha)\int \mathrm{d}^3k_1 |\widehat{\varphi}(k_1/\lambda_{\text{c}})|^2 \frac{1}{2|k_1|}\frac{2}{E_1} \\ &- \frac{2}{3}(4\pi\alpha)^2 \int \mathrm{d}^3k_1 |\widehat{\varphi}(k_1/\lambda_{\text{c}})|^2 \frac{1}{2|k_1|}\int \mathrm{d}^3k_2 |\widehat{\varphi}(k_2/\lambda_{\text{c}})|^2 \frac{1}{2|k_2|} \\ &\times \Big\{ -\Big(\frac{1}{E_1}+\frac{1}{E_2}\Big)\frac{1}{E_{12}}(1+s) + \frac{1}{2(E_{12})^3}(k_1+k_2)^2(1+s) \\ &\times \Big(\frac{1}{E_1}+\frac{1}{E_2}\Big)\frac{1}{(E_{12})^2}(k_1\cdot k_2)(-1+s) - \frac{1}{E_1}\frac{1}{E_2}\big(1+s\big) \\ &+ \Big(\frac{k_1^2}{E_1^2}+\frac{k_2^2}{E_2^2}\Big)\frac{1}{E_{12}}(1-s) + \frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_{12}}(k_1\cdot k_2)(-1+s)\Big\} + \mathcal{O}(\alpha^3)\end{aligned} \tag{19.82}$$

with the shorthand

$$E_i = |k_i| + \frac{1}{2}k_i^2, \quad i = 1, 2, \quad E_{12} = |k_1| + |k_2| + \frac{1}{2}(k_1+k_2)^2, \quad s = (\widehat{k}_1 \cdot \widehat{k}_2)^2. \tag{19.83}$$

The conventional wisdom is to take the lowest-order approximation seriously and to make the ansatz

$$\frac{m_{\text{eff}}}{m} = (\Lambda\lambda_{\text{c}})^{((8\alpha/3\pi)+b\alpha^2)}. \tag{19.84}$$

Expanding in α yields

$$\frac{m_{\text{eff}}}{m} = 1 + \frac{8\alpha}{3\pi}\log(\Lambda\lambda_{\text{c}}) + \frac{1}{2}\Big(\frac{8\alpha}{3\pi}\log(\Lambda\lambda_{\text{c}})\Big)^2 + b\alpha^2\log(\Lambda\lambda_{\text{c}}) + \mathcal{O}(\alpha^3). \tag{19.85}$$

To be consistent, the $(\log(\Lambda\lambda_{\text{c}}))^2$ term must have the correct prefactor, whereas the $\log(\Lambda\lambda_{\text{c}})$ term identifies the as yet unknown coefficient b. Indeed, inserting in (19.82) the sharp cutoff $\widehat{\varphi}$ results in terms which diverge as $\log(\Lambda\lambda_{\text{c}})$ and $(\log(\Lambda\lambda_{\text{c}}))^2$. Only the second term inside the curly brackets diverges as $(\Lambda\lambda_{\text{c}})^{1/2}$. This would suggest $h_{\text{mas}}(\lambda) = \sqrt{\lambda}$ for large λ and $\gamma = \frac{1}{2}$ independent of α, at least for small α. Whether this is an artifact of our method remains to be understood.

To have an intuitive picture why in the ultraviolet limit the Pauli–Fierz model can behave so differently from its classical relative, it is useful to turn to the functional integral (14.51) with the Maxwell field already integrated out. First note that the self-energy is automatically cancelled by the normalizing partition function. Also, since we study the ultraviolet limit, to be definite we may set $t = 1$, $V = 0$,

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and pin the Brownian motion at both ends, $q_{-1} = 0 = q_1$. $m \to 0$ means that in (14.51) the underlying Wiener measure has local fluctuations diverging as $1/\sqrt{m}$. They fight the singular behavior of $W(q_s - q_{s'}, s - s')$ near the diagonal $\{s = s'\}$. If successfully, the two effects balance each other such that the limit measure locally looks like Brownian motion with effective diffusivity $1/m^{\star}_{\text{eff}}$.

19.3.3 Binding energy

With $E^{\text{coul}}_{\Lambda}$ denoting the ground state energy of H from (19.56), the binding energy is defined by

$$E_{\text{bin}} = mc^2\big(E_{\Lambda} - E^{\text{coul}}_{\Lambda}\big). \tag{19.86}$$

Since $m \to 0$, it is mandatory to take the binding energy in units of m_{eff}, and we write

$$E_{\text{bin}} = m_{\text{eff}}\frac{m}{m_{\text{eff}}}h_{\text{bin}}(\Lambda\lambda_{\text{c}}) = m_{\text{eff}}\Big(\frac{h_{\text{bin}}(\Lambda\lambda_{\text{c}})}{h_{\text{mas}}(\Lambda\lambda_{\text{c}})}\Big). \tag{19.87}$$

The scaling function h_{bin} depends on α, $h_{\text{bin}} \geq 0$. For the binding energy to remain finite (and nonzero) in the limit $\Lambda \to \infty$, assuming already the validity of (19.74), it is required that

$$h_{\text{bin}}(\lambda) = b_0'\lambda^{\gamma'} \quad \text{and} \quad \gamma = \gamma', \tag{19.88}$$

for large λ. If (19.88) holds, then

$$E_{\text{bin}} = m^{\star}_{\text{eff}}c^2(b_0'/b_0) \tag{19.89}$$

in the limit $\Lambda \to \infty$. The ratio b_0'/b_0 is a consequence of the theory. To have agreement with experiments, on top of (19.88) one should have

$$b_0'/b_0 \cong (\alpha Z)^2/2, \tag{19.90}$$

at least for small α.

As before, a minimal control is provided by perturbation theory. The atomic Hamiltonian is

$$H_{\text{at}} = -\tfrac{1}{2}\Delta - \alpha Z/|x| \tag{19.91}$$

with eigenvalues E^{at}_n and eigenfunctions ψ_n, $H_{\text{at}}\psi_n = E^{\text{at}}_n\psi_n$, $n = 1, 2, \ldots$, ground state $\psi_1 = \psi_{\text{at}}$. $E^{\text{at}}_1 = E_{\text{at}} = -(\alpha Z)^2/2$ is the atomic ground state energy. The computation proceeds in perfect analogy with the Nelson model. Replacing m by $m_{\text{eff}}(m/m_{\text{eff}})$ to order α removes the large k divergence of the matrix element

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for the perturbed energy. In the limit $\Lambda \to \infty$ the net result is

$$E_{\text{bin}} = -m_{\text{eff}}c^2\Big[E_1^{\text{at}} + 4\pi\alpha\frac{2}{3}(2\pi)^{-3}\int \mathrm{d}^3k\frac{1}{2\omega}\Big(\omega + \frac{1}{2}k^2\Big)^{-1} \times \langle p\psi_{\text{at}}, (H_{\text{at}} - E_1^{\text{at}})\Big(H_{\text{at}} - E_1^{\text{at}} + \omega + \frac{1}{2}k^2\Big)^{-1} \cdot p\psi_{\text{at}}\rangle_{L^2}\Big] + \text{h.o.}, \tag{19.92}$$

where h.o. stands for higher orders in α. Of course, the hope is that through mass renormalization the cancellation is so precise that h.o. really means smaller than the leading correction.

To compute the matrix element in (19.92) we switch to atomic units through the replacements $x \rightsquigarrow x/\alpha$, $p \rightsquigarrow p\alpha$, which implies $H \rightsquigarrow \alpha^2 H$. Let us denote by $\mu(\mathrm{d}\lambda)$ the spectral measure of $Z^{-1}(p^2)^{1/2}\psi_{\text{at}}$ in atomic units. It is normalized as $Z^{-2}\langle\psi_{\text{at}}, p^2\psi_{\text{at}}\rangle_{L^2} = 1$ and has a support starting at $E_2^{\text{at}} - E_1^{\text{at}} = Z^2(1/2)(3/4)$. With this notation (19.92) becomes

$$E_{\text{bin}} = -m_{\text{eff}}c^2E_1^{\text{at}}\Big[1 - \frac{8}{3\pi}\alpha^3\int_{3Z^2/8}^{\alpha}\mu(\mathrm{d}\lambda)\lambda \times \int_0^{\infty}\mathrm{d}k(2+k)^{-1}(\alpha^2\lambda + k + \frac{1}{2}k^2)^{-1}\Big]. \tag{19.93}$$

Because of the coupling to the radiation field the binding energy is reduced. The shift is, however, rather small, $\alpha^3|\log\alpha|$ in relative and $\alpha^5|\log\alpha|$ in absolute order. Evaluating the integral in (19.93) yields a shift which is only a few percent away from the experimental value of 8173 MHz, which should be compared with the ionization energy of 3×10^9 MHz for the unperturbed hydrogen atom.

In addition the upper bound

$$\gamma' \leq 6/7 \tag{19.94}$$

is available. While the bound could be far from truth, the crucial point is its being less than one. The proof of (19.94) is based on the operator bound

$$-\frac{1}{|x|} \geq -\kappa|-\mathrm{i}\nabla_x + A(x)| \tag{19.95}$$

which holds for any vector field A. The numerical coefficient is $\kappa = \pi Z/2 + 2.22Z^{2/3} + 1.04$. Setting $T = (-\mathrm{i}\nabla_x - \sqrt{4\pi\alpha}A_\varphi(x))^2/2$ we obtain for H of (19.56) with $Z = 1$

$$H \geq T + H_{\text{f}} - \kappa\alpha\sqrt{2T} \geq T + H_{\text{f}} - \kappa\alpha\sqrt{2(T + H_{\text{f}})}. \tag{19.96}$$

Let now ψ be the ground state of H. Then by Jensen's inequality and with the abbreviation $f(x) = x - \kappa\alpha\sqrt{2x}$

$$E_\Lambda^{\text{coul}} \geq \langle\psi, f(T + H_{\text{f}})\psi\rangle \geq f(\langle\psi, (T + H_{\text{f}})\psi\rangle). \tag{19.97}$$

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f attains its minimum at $x_{\min} = \frac{1}{2}(\kappa\alpha)^2$. If $E_\Lambda \geq x_{\min}$, which is the case for sufficiently large Λ, then

$$E_\Lambda - E_\Lambda^{\mathrm{coul}} \leq \kappa\alpha\sqrt{2E_\Lambda}. \tag{19.98}$$

Therefore by (19.69)

$$E_{\mathrm{bin}} \leq \widetilde{c}_+(\Lambda\lambda_{\mathrm{c}})^{6/7}mc^2 \tag{19.99}$$

and

$$h_{\mathrm{bin}}(\lambda) \leq \widetilde{c}_+\lambda^{6/7} \tag{19.100}$$

for large λ.

19.3.4 Lamb shift and line width

As explained in chapter 17, through the coupling to the quantized radiation field the energy levels of the hydrogen atom are shifted and acquire a finite lifetime which is measured by the inverse width of the spectral line. The expressions (17.35), (17.36) are derived for an N-level atom in the dipole approximation. For the removal of the ultraviolet cutoff, retardation effects are of importance and the translation-invariant coupling must be used. Thus the arguments of chapter 17 have to be adapted to the Hamiltonian (19.56), which could be easily done. An alternative, for our purposes equivalent, route is to use perturbation theory for the level shift. The lifetime then follows from a Kramers–Kronig relation, since both quantities are linked to the same spectral measure; compare with Eq. (17.34).

We follow this second route. The computation is basically identical to that leading to (19.92) and uses the virial theorem $\langle\psi_n, p^2\psi_n\rangle = -2E_n^{\mathrm{at}}$. If δE_n denotes the level shift relative to E_Λ, the net result reads

$$\begin{aligned}\delta E_n = m_{\mathrm{eff}}c^2\Big[E_n^{\mathrm{at}} + 4\pi\alpha\int \mathrm{d}^3k|\widehat{\varphi}(k/\Lambda\lambda_{\mathrm{c}})|^2\frac{1}{2\omega}\Big(\omega+\frac{1}{2}k^2\Big)^{-1}\langle\nabla_x\psi_n, (H_{\mathrm{at}}-E_n^{\mathrm{at}})\\ \times\Big(H_{\mathrm{at}}-E_n^{\mathrm{at}}+\omega+\frac{1}{2}k^2\Big)^{-1}\cdot Q_\perp(k)\nabla_x\psi_n\rangle_{L^2}\Big] + \mathrm{h.o.}\end{aligned} \tag{19.101}$$

For large k the matrix element decays as $|k|^{-2}$, which makes the integral (19.101) ultraviolet convergent. The Lamb shift refers to the frequency of emitted radiation and is therefore an energy difference. In fact, experimentally the splitting between the $2S_{1/2}$ and $2P_{1/2}$ levels is 1058 MHz, in comparison to the unperturbed ground state energy of 3×10^9 MHz, and is mostly due to the coupling to the quantized radiation field. Evaluating numerically the intergrals in (19.101) at $\Lambda=\infty$ and taking into account the coupling of the electron to the quantized magnetic field,

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a few percent effect only, yields a Lamb shift which is 2.5% lower than the true value.

19.3.5 g-factor of the electron

The gyromagnetic ratio was discussed in section 16.6 through investigating the motion of an electron in a homogeneous weak magnetic field. Here we point out that the g-factor can also be obtained from the Zeeman splitting of the ground state energy at total momentum $P = 0$, which is the basis of the high-precision Penning trap experiment. For a constant external magnetic field B, in relativistic units, the Hamiltonian reads

$$H_B = m\Big(\frac{1}{2}\big(\sigma \cdot (p - eA_\varphi(x))\big)^2 + H_{\rm f} \\ -\frac{e}{2m^2}\sigma \cdot B - \frac{e}{2m^2}\big((p - eA_\varphi(x)) \times x\big) \cdot B + \frac{e^2}{8m^2}(x \times B)^2\Big). \quad (19.102)$$

Let ψ be an approximate ground state for $H = H_{B=0}$. Then the linear Zeeman splitting, ΔE, is given through first-order perturbation theory in B as

$$\Delta E = \frac{1}{m}B \cdot \Big(-\frac{e}{2}\langle\psi, \sigma\psi\rangle_{\mathcal{H}} - \frac{e}{2}\langle\psi, (p - eA_\varphi(x)) \times x\psi\rangle_{\mathcal{H}}\Big). \quad (19.103)$$

Next we write

$$\big(H - E(0)\big)x\psi = [H - E(0), x]\psi = -\mathrm{i}(p - eA_\varphi(x))\psi. \quad (19.104)$$

In this form the total momentum can be fixed at $P = 0$. Then H becomes

$$H(0) = \frac{1}{2}(P_{\rm f} + eA_\varphi)^2 - \frac{e}{2}\sigma \cdot B_\varphi + H_{\rm f} \quad (19.105)$$

with ψ the ground state $\psi_{\rm g}$ of $H(0)$. Hence

$$\Delta E = \frac{1}{m_{\rm eff}}\frac{m_{\rm eff}}{m}B \cdot \Big(-\frac{e}{2}\langle\psi_{\rm g}, \sigma\psi_{\rm g}\rangle_{\mathbb{C}^2\otimes\mathcal{F}} \\ + \frac{e}{2}\mathrm{i}\langle\psi_{\rm g}, (P_{\rm f} + eA_\varphi)\frac{1}{H(0) - E(0)} \times (P_{\rm f} + eA_\varphi)\psi_{\rm g}\rangle_{\mathbb{C}^2\otimes\mathcal{F}}\Big) \\ = |B|\frac{e}{2m_{\rm eff}}g. \quad (19.106)$$

We orient the B-field along the z-axis and take as ground state the one with total angular momentum pointing parallel to B; compare with section 16.6. Since

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$(\sigma_3 + 2J_{\mathrm{f}3} + 2S_{\mathrm{f}3})\psi_{\mathrm{g}+} = \psi_{\mathrm{g}+}$, the g-factor is thus given by

$$\frac{1}{2}g = \Big(1 - \frac{2}{3}\langle\psi_{\mathrm{g}+}, (P_{\mathrm{f}} + eA_\varphi)(H(0) - E(0))^{-1}(P_{\mathrm{f}} + eA_\varphi)\psi_{\mathrm{g}+}\rangle_{\mathbb{C}^2\otimes\mathcal{F}}\Big)^{-1} \times \Big(1 - 2\langle\psi_{\mathrm{g}+}, (J_{\mathrm{f}3} + S_{\mathrm{f}3})\psi_{\mathrm{g}+}\rangle_{\mathbb{C}^2\otimes\mathcal{F}} - 2\mathrm{Im}\langle\psi_{\mathrm{g}+}, (P_{\mathrm{f}} + eA_\varphi)_2(H(0) - E(0))^{-1}(P_{\mathrm{f}} + eA_\varphi)_1\psi_{\mathrm{g}+}\rangle_{\mathbb{C}^2\otimes\mathcal{F}}\Big), \tag{19.107}$$

in agreement with (19.104).

We recall that, to second order and with no cutoffs, the g-factor is computed to

$$\frac{1}{2}g = 1 + \frac{8}{3}\left(\frac{\alpha}{2\pi}\right) + \mathcal{O}(\alpha^2), \tag{19.108}$$

which is 0.2% away from the true value. For fixed, small e both numerator and denominator in (19.107) are expected to tend to 0 as $\Lambda \to \infty$ in such a way that their ratio is close to 1.

Notes and references

Section 19.1

The infrared behavior of radiative corrections to scattering was first studied by Bloch and Nordsieck (1937), Nordsieck (1937), and Pauli and Fierz (1938). Within the framework of the massless scalar Nelson model of section 19.2, Fröhlich (1973) constructs the one-particle shell and investigates the scattering theory. Further progress in this direction is Pizzo (2000). In his thesis Chen (2001) establishes the infrared limit of the energy–momentum relation. In contrast to the Pauli–Fierz model the scalar Nelson model is infrared divergent also at $p = 0$. The Gross transformation (19.28) switches to the representation corresponding to $p = 0$. In fact it implements the shift $\phi(x) - eV_{\varphi\mathrm{coul}}(x)$. We refer to Arai (2001), Lőrinczi, Minlos and Spohn (2002b) and Hirokawa, Hiroshima and Spohn (2002). The quantized Maxwell field coupled to a classical current is a standard textbook example (Kibble 1968; Thirring 1958). The representation theory for coherent states is developed by Klauder, McKenna and Woods (1966) with follow-ups within the algebraic framework (Emch 1972; Dubin 1974; Bratteli and Robinson 1987, 1997).

Section 19.2

The scalar field model is studied in solid state physics and includes a large body of experimental work. It describes an electron coupled to the optical mode of a polar crystal and is known as a polaron (Landau 1933; Fröhlich 1954). In the standard approximation the dispersion of the field is $\omega(k) = \omega_0$ and the coupling function

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$\widehat{\varphi}(k) = |k|^{-1}$. The ground state energy is well approximated by the variational approach of Feynman (1955). Upper and lower bounds are proved by Lieb and Yamazaki (1958). A large coupling theory is available (Pekar 1954). A rigorous proof of the Pekar limit can be found in Donsker and Varadhan (1983) and Lieb and Thomas (1997). The effective mass is studied in Spohn (1987) who also provides an extensive list of references. The Pekar limit of the effective mass still remains an open problem. Useful reviews are Devreese and Peeters (1984) and Gerlach and Löwen (1991). Gross (1976) develops systematic corrections to the large coupling theory. Nelson (1964a) studies the scalar model through functional integration; see chapter 14. Nelson (1964b) uses the transformation of Gross (1962), itself inspired by Lee, Low and Pines (1953), to control the removal of the ultraviolet cutoff. Nelson's analysis is pushed much further in Fröhlich (1973, 1974). The discussion of chapter 14 transcribes word for word to the Nelson model with the welcome simplification that stochastic Ito integrals become Riemann integrals. We refer to Lőrinczi and Minlos (2001), Lőrinczi *et al.* (2002a), and Betz *et al.* (2002). For sufficiently small coupling the existence of a ground state for H of (19.18) is proved in Hirokawa *et al.* (2002). On the one-particle level, $H = \sqrt{p^2 + m^2} - e^2/4\pi|x|$ is not bounded from below for large e. Since for the Nelson model $E(p) \simeq |p|$ for large p, the same instability could be present for the Hamiltonian (19.18). Hainzl, Hirokawa and Spohn (2003) provide upper and lower bounds on E_{bin} which establish (19.52) with an error $\mathcal{O}(e^7 \log e)$.

Section 19.3

The estimates of the ground state energy are taken from Lieb and Loss (2000, 2002), who study in addition the case of many particles and semirelativistic models. The scaling (19.80) follows also from a perturbative one-loop renormalization (Chen 1996; Bugliaro *et al.* 1996). The effective mass to order α^2 seems to be novel. Details of the perturbative computation leading to (19.82) can be found in Hiroshima and Spohn (2003). Fröhlich argues that the effective mass depends nonanalytically on α and therefore the interchange of limits, $\alpha \to 0$ and $\Lambda \to \infty$, leads to erroneous results. In a more proper treatment one should successively eliminate the interaction at high momenta. The resulting renormalization group flow equations yield a plausible outcome and, indeed, reflect the nonanalytic dependence in α. Moniz and Sharp (1974, 1977) and Grotch *et al.* (1982) claim cutoff dependences of the effective mass which are in contradiction to our findings. The bound for γ' is from Lieb and Loss (2002), which is based on the lower operator bound (19.95) for the Coulomb potential as proved in Lieb, Loss and Siedentop (1996). The famous calculation of the Lamb shift by Bethe (1947) is based on the dipole approximation and has a divergence as $\log \Lambda$. As pointed out immediately (Kroll and Lamb 1949), the shift becomes ultraviolet convergent in a

relativistic theory which necessarily includes positrons. The role of retardation is mentioned by Kroll (1965). An accurate calculation is Au and Feinberg (1974), ignoring spin however. It is included in Grotch (1981), from which our numbers are taken. Quantum electrodynamic effects are most dominant for the 1S Lamb shift, which experimentally is determined with higher accuracy than the $2S_{1/2} - 2P_{1/2}$ splitting. We refer to Weitz *et al.* (1994). For small coupling quantitative estimates on the binding energy are available. One constructs upper and lower bounds with the leading terms given by formal perturbation theory, which is not directly applicable because of the missing spectral gap; see Catto and Hainzl (2004), Chen, Voulgater and Vulgater (2003), Hainzl (2002, 2003), Hainzl, Seiringer (2002), and Hainzl, Voulgater, Vulgater (2003). At present, one obstacle is that no corresponding result for the effective mass is available. Since physically energies are calibrated through $m_{\mathrm{eff}}c^2$, such a bound is mandatory. The g-factor as based on the shift in energy is computed in Grotch and Kazes (1977) to second order in e. The derivation of the nonperturbative expression (19.107) seems to be new.