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Abstract

We consider the problem of deforming simultaneously a *pair* of given structures. We show that such deformations are governed by an L_∞-algebra, which we construct explicitly. Our machinery is based on Voronov's derived bracket construction. In this paper we consider only geometric applications, including deformations of coisotropic submanifolds in Poisson manifolds, of twisted Poisson structures, and of complex structures within generalized complex geometry. These applications cannot be, to our knowledge, obtained by other methods such as operad theory.

Contents

Introduction

Deformation theory was developed in the 1950s by Kodaira–Kuranishi–Spencer for complex structures [KS58a, KS58b, KS60, Kur65] and by Gerstenhaber for associative algebras [Ger64]. Nijenhuis and Richardson then gave an interpretation of deformations in terms of graded Lie algebras [NR66, NR67], which was later promoted by Deligne: *deformations of a given algebraic or geometric structure* Δ *are governed by a differential graded Lie algebra (DGLA) or, more generally, by an* L_∞*-algebra*.

For example, given a vector space V, Gerstenhaber in [Ger64] introduced a graded Lie algebra $(L, [-,-])$ such that an associative algebra structure on V is given by $\Delta \in L_1$ such that $[\Delta, \Delta] = 0$. A *deformation* of Δ is an element $\Delta + \tilde{\Delta}$ such that $\tilde{\Delta} \in L_1$ and

$$0 = [\Delta + \tilde{\Delta}, \Delta + \tilde{\Delta}] = 2[\Delta, \tilde{\Delta}] + [\tilde{\Delta}, \tilde{\Delta}] = 2(d_\Delta \tilde{\Delta} + \tfrac{1}{2}[\tilde{\Delta}, \tilde{\Delta}]). \quad (1)$$

Therefore, the DGLA $(L, d_\Delta, [\cdot\,,\cdot])$ governs deformations of the associative algebra (V, Δ).

It is usually a hard task to show that the deformations of a given structure are governed by an L_∞-algebra, and even harder to construct explicitly the L_∞-algebra. When one succeeds in doing so, as a reward one gets the cohomology theory, analogues of Massey products, and a natural equivalence relation on the space of deformations. Moreover, quasi-isomorphic L_∞-algebras govern equivalent deformation problems, a result with non-trivial applications to quantization (see [Kon03]).

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In this work we consider *simultaneous* deformations of *two* (interrelated) structures. A typical example is given by the simultaneous deformations of (Δ, Φ), where Δ denotes a pair of associative algebras and Φ is an algebra morphism between them. These deformations are characterized by a cubic equation (unlike (1), which is quadratic) and are therefore governed by an L_∞-algebra with non-trivial l_3-term.

Our main result, Theorem 3 in § 1.4, constructs explicitly L_∞-algebras governing such simultaneous deformation problems.

Outline of the paper. *L_∞-algebras*, introduced by Lada and Stasheff [LS93], consist of collections $\{l_i\}_{i \geqslant 1}$ of 'multibrackets' satisfying higher Jacobi identities. They can be built out of what we call *V-data* $(L, P, \mathfrak{a}, \Delta)$ via *derived bracket constructions* due to Voronov [Vor05a, Vor05b], which extend those of Kosmann-Schwarzbach [Kos04] (see Theorems 1 and 2). Our main contribution is to determine L_∞-algebras governing *simultaneous* deformation problems (Theorem 3), by recognizing that they arise as in Voronov's Theorem 2. These results are collected in § 1.

In the companion paper [FZ13], we find algebraic applications to the study of simultaneous deformations of algebras and morphisms in the following categories: Lie, L_∞, Lie bi- and associative algebras, and more generally in any category of algebras over Koszul operads. These results can alternatively be obtained by operadic methods, see for example [FMY09, MV09], but our techniques have the advantage of not assuming any knowledge of the operadic machinery and of easily delivering explicit formulae. Recently, using our techniques, Ji studied simultaneous deformations in the category of Lie algebroids [Ji14].

The main novelty concerning applications, and the focus of this paper, is in geometry. In § 2, we determine L_∞-algebras governing simultaneous deformations of:

- coisotropic submanifolds of Poisson manifolds;
- Dirac structures in Courant algebroids (with twisted Poisson structures as a special case);
- generalized complex structures in Courant algebroids (with complex structures as a special case).

We also describe explicitly the equivalence relation on the space of twisted Poisson structures.

None of these examples, to our knowledge, falls under the scope of the operadic methods, and one should have in mind that in this geometric setting, no tool such as Koszul duality gives for nothing the graded Lie algebra L we need as part of the V-data.

Outlook: deformation quantization of symmetries. It is known from [BFFLS77] that the quantization of a mechanical system (Poisson manifold) can be understood as a deformation of the algebra of smooth functions 'in the direction' of the Poisson structure, the first-order term of the Taylor expansion of this deformation.

One can associate to any Poisson structure such a quantization [Kon03]: Poisson structures and their quantizations are Maurer–Cartan elements for suitable L_∞-algebras (Schouten and Gerstenhaber algebras, respectively), so it suffices to build a L_∞-morphism between these two L_∞-algebras (formality theorem). This morphism sends Maurer–Cartan elements to Maurer–Cartan elements, i.e. associates a quantization to any Poisson structure.

Our long term goal is to apply this approach to symmetries. The notion of symmetry of a mechanical system $(C^\infty(M), \{-,-\})$ can be understood as a Lie algebra map $(\mathfrak{g}, [-,-]) \to (C^\infty(M), \{-,-\})$. This map can be extended, in the category of Poisson algebras, to $(S\mathfrak{g}, \{-,-\})$, the Poisson algebra of polynomial functions on $\mathfrak{g}^*$. Its graph is a coisotropic submanifold of the Poisson manifold $\mathfrak{g}^* \times M$. Therefore, our first step towards this long term goal is to construct in § 2.1 an L_∞-algebra governing simultaneous deformations of Poisson tensors and their coisotropic submanifolds. This L_∞-algebra plays the role of the Schouten

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algebra in the presence of symmetries. It extends the L_∞-algebras governing deformations of coisotropic submanifolds of Poisson manifolds considered by Oh and Park [OP05] and Cattaneo and Felder [CF07], since, in their settings, the Poisson structure was kept fixed.

1. L_∞-algebras via derived brackets and Maurer–Cartan elements

The purpose of this section is to establish Theorem 3, which produces the L_∞-algebras appearing in the rest of the paper. Therefore, we first review some basic material about L_∞-algebras in § 1.1; then we recall in § 1.2 Voronov's constructions, which will be used to establish Theorem 3 in § 1.4. Our proof is a direct computation, but we also provide a conceptual argument in terms of tangent cohomology, building on § 1.3. We conclude by justifying in § 1.5 why no convergence issues arise in our machinery, and discussing equivalences in § 1.6.

1.1 Background on L_∞-algebras

We start defining (differential) graded Lie algebras, which are special cases of L_∞-algebras.

Definition 1.1. A *graded Lie algebra* is a $\mathbb{Z}$-graded vector space $L = \bigoplus_{n\in\mathbb{Z}} L_n$ equipped with a degree-preserving bilinear bracket $[\cdot\,,\cdot] : L \otimes L \longrightarrow L$ which satisfies:

(1) graded antisymmetry: $[a, b] = -(-1)^{|a||b|}[b, a]$;
(2) graded Leibniz rule: $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$.

Here a, b, c are homogeneous elements of L and the degree $|x|$ of an homogeneous element $x \in L_n$ is by definition n.

Definition 1.2. A *differential graded Lie algebra* (DGLA for short) is a graded Lie algebra $(L, [\cdot\,,\cdot])$ equipped with a homological derivation $d : L \to L$ of degree 1. In other words:

(1) $|da| = |a| + 1$ (d of degree 1);
(2) $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$ (derivation);
(3) $d^2 = 0$ (homological).

In order to formulate the definition of an L_∞-algebra, a notion due to Lada and Stasheff [LS93], let us give two notations. Given two elements v_1, v_2 in a graded vector space V, let us define the *Koszul sign* of the transposition $\tau_{1,2}$ of these two elements by

$$\epsilon(\tau_{1,2}, v_1, v_2) := (-1)^{|v_1||v_2|}.$$

We then extend multiplicatively this definition to an arbitrary permutation using a decomposition into transpositions. We will often abuse the notation $\epsilon(\sigma, v_1, \ldots, v_n)$ by writing $\epsilon(\sigma)$, and we define $\chi(\sigma) := \epsilon(\sigma)(-1)^\sigma$.

We will also need *unshuffles*: $\sigma \in S_n$ is called an $(i, n-i)$-unshuffle if it satisfies $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. The set of $(i, n-i)$-unshuffles is denoted by $S_{(i,n-i)}$. After [LM95, Definition 2.1], we have the following definition.

Definition 1.3. An *L_∞-algebra* is a $\mathbb{Z}$-graded vector space V equipped with a collection ($k \geqslant 1$) of linear maps $l_k : \otimes^k V \longrightarrow V$ of degree $2 - k$ satisfying, for every collection of homogeneous elements $v_1, \ldots, v_n \in V$:

(1) graded antisymmetry: for every $\sigma \in S_n$,

$$l_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \chi(\sigma) l_n(v_1, \ldots, v_n);$$

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(2) relations: for all $n \geqslant 1$,

$$\sum_{\substack{i+j=n+1\\ i,j\geqslant 1}} (-1)^{i(j-1)} \sum_{\sigma\in S_{(i,n-i)}} \chi(\sigma) l_j(l_i(v_{\sigma(1)},\dots,v_{\sigma(i)}),v_{\sigma(i+1)},\dots,v_{\sigma(n)})=0.$$

In a *curved* L_∞-*algebra*, one additionally allows for an element $l_0\in V_2$, one allows i and j to be zero in the relations (2), and one adds the relation corresponding to $n=0$.

Notice that when all l_k vanish except for $k=2$, we obtain graded Lie algebras.

In Definition 1.3, the multibrackets are graded antisymmetric and l_k has degree $2-k$, whereas in the next definition they are graded symmetric and all of degree 1.

Definition 1.4. An $L_\infty[1]$-*algebra* is a graded vector space W equipped with a collection ($k\geqslant 1$) of linear maps $m_k:\otimes^k W \longrightarrow W$ of degree 1 satisfying, for every collection of homogeneous elements $v_1,\dots,v_n\in W$:

(1) graded symmetry: for every $\sigma\in S_n$,

$$m_n(v_{\sigma(1)},\dots,v_{\sigma(n)})=\epsilon(\sigma)m_n(v_1,\dots,v_n);$$

(2) relations: for all $n\geqslant 1$,

$$\sum_{\substack{i+j=n+1\\ i,j\geqslant 1}} \sum_{\sigma\in S_{(i,n-i)}} \epsilon(\sigma) m_j(m_i(v_{\sigma(1)},\dots,v_{\sigma(i)}),v_{\sigma(i+1)},\dots,v_{\sigma(n)})=0.$$

In a *curved* $L_\infty[1]$-*algebra*, one additionally allows for an element $m_0\in W_1$ (which can be understood as a bracket with zero arguments), one allows i and j to be zero in the relations (2), and one adds the relation corresponding to $n=0$.

Remark 1.5. There is a bijection between L_∞-algebra structures on a graded vector space V and $L_\infty[1]$-algebra structures on $V[1]$, the graded vector space defined by $(V[1])_i := V_{i+1}$ [Vor05a, Remark 2.1]. The multibrackets are related by applying the décalage isomorphisms

$$(\otimes^n V)[n]\cong \otimes^n(V[1]),\quad v_1\cdots v_n\mapsto v_1\cdots v_n\cdot(-1)^{(n-1)|v_1|+\cdots+2|v_{n-2}|+|v_{n-1}|}, \tag{2}$$

where $|v_i|$ denotes the degree of $v_i\in V$. The bijection extends to the curved case.

From now on, for any $v\in V$, we denote by $v[1]$ the corresponding element in $V[1]$ (which has degree $|v|-1$). Also, we denote the multibrackets in $L_\infty[1]$-algebras by $\{\cdots\}$, we denote by $d:=m_1$ the unary bracket, and in the curved case we denote $\{\emptyset\}:=m_0$ (the bracket with zero arguments).

Definition 1.6. Given an $L_\infty[1]$-algebra W, a *Maurer–Cartan element* is a degree-zero element α satisfying the Maurer–Cartan equation

$$\sum_{n=1}^{\infty}\frac{1}{n!}\{\underbrace{\alpha,\dots,\alpha}_{n\text{ times}}\}=0. \tag{3}$$

One denotes by $MC(W)$ the set of its Maurer–Cartan elements.

If W is a curved $L_\infty[1]$-algebra, one defines Maurer–Cartan elements by adding $m_0\in W_1$ to the left-hand side of (3) (i.e. by letting the sum in (3) start at $n=0$).

There is an issue with the above definition: the left-hand side of (3) is generally an infinite sum. In this paper, we solve this issue by considering *filtered* $L_\infty[1]$-algebras (see Definition 1.16), for which the above infinite sum automatically converges.

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1.2 Voronov's constructions of L_∞-algebras as derived brackets

In this subsection, we introduce V-data and recall how Voronov associates $L_\infty[1]$-algebras to a V-data.

DEFINITION 1.7. A *V-data* consists of a quadruple $(L, \mathfrak{a}, P, \Delta)$, where:

- L is a graded Lie algebra (we denote its bracket by $[\cdot\,,\cdot]$);
- $\mathfrak{a}$ is an abelian Lie subalgebra;
- $P : L \to \mathfrak{a}$ is a projection whose kernel is a Lie subalgebra of L;
- $\Delta \in \mathrm{Ker}(P)_1$ is an element such that $[\Delta, \Delta] = 0$.

When Δ is an arbitrary element of L_1 instead of $\mathrm{Ker}(P)_1$, we refer to $(L, \mathfrak{a}, P, \Delta)$ as a *curved V-data*.

THEOREM 1 [Vor05a, Theorem 1, Corollary 1]. *Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data. Then $\mathfrak{a}$ is a curved $L_\infty[1]$-algebra for the multibrackets $\{\emptyset\} := P\Delta$ and $(n \geqslant 1)$*

$$\{a_1, \ldots, a_n\} = P[\ldots[[\Delta, a_1], a_2], \ldots, a_n]. \tag{4}$$

We obtain a $L_\infty[1]$-algebra exactly when $\Delta \in \mathrm{Ker}(P)$.

When $\Delta \in \mathrm{Ker}(P)$, there is actually a larger $L_\infty[1]$-algebra, which contains $\mathfrak{a}$ as in Theorem 1 as an $L_\infty[1]$-subalgebra.

THEOREM 2 [Vor05b, Theorem 2]. *Let $V := (L, \mathfrak{a}, P, \Delta)$ be a V-data and denote $D := [\Delta, \cdot] : L \to L$. Then the space $L[1] \oplus \mathfrak{a}$ is a $L_\infty[1]$-algebra for the differential*

$$d(x[1], a) := (-(Dx)[1], P(x + Da)), \tag{5}$$

the binary bracket

$$\{x[1], y[1]\} = [x, y][1](-1)^{|x|} \in L[1], \tag{6}$$

and, for $n \geqslant 1$,

$$\{x[1], a_1, \ldots, a_n\} = P[\ldots[x, a_1], \ldots, a_n] \in \mathfrak{a}, \tag{7}$$

$$\{a_1, \ldots, a_n\} = P[\ldots[Da_1, a_2], \ldots, a_n] \in \mathfrak{a}. \tag{8}$$

Here $x, y \in L$ and $a_1, \ldots, a_n \in \mathfrak{a}$. Up to permutation of the entries, all the remaining multibrackets vanish.

Notation 1.8. We will denote by

$$\mathfrak{a}^P_\Delta$$

and by

$$(L[1] \oplus \mathfrak{a})^P_\Delta \quad \text{or sometimes } \mathfrak{g}(V)$$

the $L_\infty[1]$-algebras produced by Theorems 1 and 2.

Given a curved V-data, assume that $\Phi \in \mathfrak{a}_0$ is such that $e^{[\cdot,\Phi]}$ is well defined (see Proposition 1.18 for a sufficient condition), giving an automorphism of $(L, [\cdot\,,\cdot])$. We will consider

$$P_\Phi := P \circ e^{[\cdot,\Phi]} : L \to \mathfrak{a}. \tag{9}$$

Notice that P_Φ is a projection, since $e^{[\cdot,\Phi]}|_{\mathfrak{a}} = \mathrm{Id}_{\mathfrak{a}}$ by the abelianity of $\mathfrak{a}$.

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Remark 1.9. Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data and $\Phi \in \mathfrak{a}_0$ as above. Then Φ is a Maurer–Cartan element of $\mathfrak{a}^P_\Delta$ if and only if

$$P_\Phi \Delta = 0 \tag{10}$$

or, equivalently $\Delta \in \ker(P_\Phi)$. This follows immediately from (4) and will be used repeatedly in the proof of Theorem 3.

Remark 1.10. Let L' be a graded Lie subalgebra of L preserved by D (for example, $L' = \mathrm{Ker}(P)$). Then $L'[1] \oplus \mathfrak{a}$ is stable under the multibrackets of Theorem 2. We denote by $(L'[1] \oplus \mathfrak{a})^P_\Delta$ the induced $L_\infty[1]$-structure.

Remark 1.11. Voronov's Theorem 2 [Vor05b] is actually formulated for any degree 1 derivation D of L preserving $\mathrm{Ker}(P)$ and satisfying $D \circ D = 0$. We restrict ourselves to inner derivations for the sake of simplicity and since all the derivations that appear in our examples are of this kind.

A 'semidirect product' $L_\infty[1]$-algebra similar to the one in Theorem 2 appeared in [BFLS98, FM07].

1.3 The tangent complex within Voronov's theory

In this subsection we study how Voronov's $L_\infty[1]$-algebras behave under twisting. We will use this in § 1.4 to provide an alternative argument for Theorem 3.

It is well known [Get09, Proposition 4.4] that one can twist an $L_\infty[1]$-algebra $\mathfrak{g}$ by one of its Maurer–Cartan elements α. One obtains a new $L_\infty[1]$-algebra $\mathfrak{g}_\alpha$, sometimes called the *tangent complex at* α. Its nth multibracket is

$$\{\cdots\}^\alpha_n = \{\cdots\}_n + \{\alpha, \ldots\}_{n+1} + \frac{1}{2!}\{\alpha, \alpha, \ldots\}_{n+2} + \cdots, \tag{11}$$

where $\{\cdots\}_j$ denotes the jth multibracket of $\mathfrak{g}$.

A property of the tangent complex $\mathfrak{g}_\alpha$ is that its Maurer–Cartan elements are in one to one correspondence with the deformations of α, i.e.

$$\alpha + \tilde{\alpha} \in MC(\mathfrak{g}) \Leftrightarrow \tilde{\alpha} \in MC(\mathfrak{g}_\alpha) \tag{12}$$

(by [LV12, Proposition 12.2.33] or direct computation). We express the notion of tangent complex in the setting of Voronov's theory (recall that the notation $\mathfrak{g}(V)$ was defined in § 1.2).

Lemma 1.12. *Let $V := (L, \mathfrak{a}, P, \Delta)$ be a filtered V-data and let $\alpha := (\Delta'[1], \Phi')$ be a Maurer–Cartan element of $\mathfrak{g}(V)$. Then*

$$\mathfrak{g}(V)_\alpha = \mathfrak{g}(V_\alpha)$$

with $V_\alpha := (L, \mathfrak{a}, P_{\Phi'}, \Delta + \Delta')$.

This lemma is a generalization of the remark by Domenico Fiorenza that $((L[1]\oplus\mathfrak{a})^P_0)_{(\Delta[1],0)} = (L[1] \oplus \mathfrak{a})^P_\Delta$. We do not need to prove that V_α is a V-data since, as a twist of an $L_\infty[1]$-algebra, $\mathfrak{g}(V_\alpha)$ is automatically an $L_\infty[1]$ algebra.

Proof. Let $n > 2$ and $\alpha := (\Delta'[1], \Phi') \in L[1] \oplus \mathfrak{a}$. The kth summand ($k \geqslant 0$) of the right-hand side of (11), applied to elements $x_i[1] + a_i$, can be rewritten as

$$\frac{1}{k!}\{\underbrace{\alpha, \ldots, \alpha}_{k}, x_1[1] + a_1, \ldots, x_n[1] + a_n\}_{n+k} = A_k + B_k + C_k$$

with

$$A_k = \frac{1}{(k-1)!} P[\dots[\dots[\Delta', \underbrace{\Phi'], \dots, \Phi'}_{k-1}], a_1, \dots, a_n], \tag{13}$$

$$B_k = \frac{1}{k!} \sum_{i=1}^{n} P[\dots[\dots[x_i, \underbrace{\Phi'], \dots, \Phi'}_{k}], a_1, \dots, \hat{a}_i, \dots, a_n], \tag{14}$$

$$C_k = \frac{1}{k!} P[\dots[\dots[\Delta, \underbrace{\Phi'], \dots, \Phi'}_{k}], a_1, \dots, a_n], \tag{15}$$

defining $A_0 := 0$. Notice that (13) and (14) come from (7) (encoding the $L[1]$-components of α and $x_i[1] + a_i$, respectively) and (15) comes from (8).

On the other hand, the brackets of $(L[1] \oplus \mathfrak{a})^{P_{\Phi'}}_{\Delta+\Delta'}$ for $n > 2$ read

$$\{x_1[1] + a_1, \dots, x_n[1] + a_n\}_n = A + B + C,$$

where

$$\begin{aligned} A &= P_{\Phi'}[\dots[\Delta', a_1], \dots, a_n], \\ B &= \sum_{i=1}^{n} P_{\Phi'}[\dots[x_i, a_1], \dots, \hat{a}_i, \dots, a_n], \\ C &= P_{\Phi'}[\dots[\Delta, a_1], \dots, a_n]. \end{aligned}$$

Since $e^{[\cdot,\Phi']}$ is a morphism of graded Lie algebras and $e^{[\cdot,\Phi']}|_{\mathfrak{a}} = \mathrm{Id}_{\mathfrak{a}}$, we have

$$e^{[\cdot,\Phi']}[\dots[x, a_1], \dots, a_n] = [\dots[e^{[\cdot,\Phi']}x, a_1], \dots, a_n]$$

for all $x \in L$. Expanding $e^{[\cdot,\Phi']}$ as a series gives

$$A = \sum_k A_k, \quad B = \sum_k B_k, \quad C = \sum_k C_k,$$

therefore showing that the nth multibrackets agree for $n > 2$. Similar computations give the cases $n = 1, 2$. □

1.4 The main tool

Given a V-data $(L, \mathfrak{a}, P, \Delta)$, we fix a Maurer–Cartan Φ of $\mathfrak{a}^P_\Delta$ and study the deformations of Δ and Φ.

In what follows, the assumption *filtered* is there to ensure the convergences of the infinite sums appearing, and can be neglected on a first reading. We will address convergence issues in § 1.5.

Lemma 1.13. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered V-data and let $\Phi \in MC(\mathfrak{a}^P_\Delta)$. Then $(L, \mathfrak{a}, P_\Phi, \Delta)$ is also a V-data.*

Proof. The projection P_Φ is well defined in Proposition 1.18 in § 1.5. The subspace $\mathrm{Ker}(P_\Phi) = e^{[\cdot,-\Phi]}(\mathrm{Ker}(P))$ is a Lie subalgebra of L, since $e^{[\cdot,-\Phi]}$ is a Lie algebra automorphism of L and $\ker(P)$ is a Lie subalgebra. Further, $\Delta \in \ker(P_\Phi)$ by Remark 1.9. Hence, $(L, \mathfrak{a}, P_\Phi, \Delta)$ is a V-data. □

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The following is the main tool used in the rest of the paper. It says that the deformations of Δ and Φ are governed by $(L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$. In the applications, Φ will be the object of interest, as it will correspond to morphisms, subalgebras, and so on.

Theorem 3. *Let $(L,\mathfrak{a},P,\Delta)$ be a filtered V-data and let $\Phi\in MC(\mathfrak{a}^P_\Delta)$. Then, for all $\tilde\Delta\in L_1$ and $\tilde\Phi\in\mathfrak{a}_0$:*

$$\begin{cases}[\Delta+\tilde\Delta,\Delta+\tilde\Delta]=0\\ \Phi+\tilde\Phi\in MC(\mathfrak{a}^P_{\Delta+\tilde\Delta});\end{cases}\quad\Leftrightarrow\quad(\tilde\Delta[1],\tilde\Phi)\in MC((L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta).\tag{16}$$

In this case, $\mathfrak{a}^P_{\Delta+\tilde\Delta}$ is a curved $L_\infty[1]$-algebra. It is a $L_\infty[1]$-algebra exactly when $\tilde\Delta\in\mathrm{Ker}(P)$.

Proof. By Lemma 1.13, we can apply Theorem 2 to obtain the $L_\infty[1]$-algebra $(L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$, whose multibrackets we denote by $\{\cdots\}$. We compute each summand appearing in the left-hand side of the Maurer–Cartan equation for $(\tilde\Delta[1],\tilde\Phi)$ in $(L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$, which reads

$$\sum_{n=1}^{\infty}\frac{1}{n!}\{(\tilde\Delta[1],\tilde\Phi),\ldots,(\tilde\Delta[1],\tilde\Phi)\}.\tag{17}$$

We have

$$\begin{aligned}\{(\tilde\Delta[1],\tilde\Phi)\}&=(-[\Delta,\tilde\Delta][1],\ P_\Phi\tilde\Delta+P_\Phi[\Delta,\tilde\Phi]),\\ \{(\tilde\Delta[1],\tilde\Phi),(\tilde\Delta[1],\tilde\Phi)\}&=(-[\tilde\Delta,\tilde\Delta][1],\ 2\cdot P_\Phi[\tilde\Delta,\tilde\Phi]+P_\Phi[[\Delta,\tilde\Phi],\tilde\Phi]),\\ \{\underbrace{(\tilde\Delta[1],\tilde\Phi),\ldots,(\tilde\Delta[1],\tilde\Phi)}_{n\text{ times}}\}&=(\ 0\ ,\ n\cdot P_\Phi[[[\tilde\Delta,\underbrace{\tilde\Phi],\ldots],\tilde\Phi}_{n-1\text{ times}}]+P_\Phi[[[[\Delta,\underbrace{\tilde\Phi],\tilde\Phi],\ldots],\tilde\Phi}_{n\text{ times}}]).\end{aligned}$$

The last line refers to the nth term for $n\geqslant 3$, and holds since the higher brackets with two or more entries in $L[1]\oplus\{0\}$ vanish.

Hence, the $L[1]$-component of (17) is just $-\frac{1}{2}[\Delta+\tilde\Delta,\Delta+\tilde\Delta][1]$. The $\mathfrak{a}$-component of (17) is

$$\begin{aligned}&P_\Phi(e^{[\cdot,\tilde\Phi]}\tilde\Delta+(e^{[\cdot,\tilde\Phi]}-1)\Delta)\\ &\quad=P_\Phi e^{[\cdot,\tilde\Phi]}(\Delta+\tilde\Delta)\\ &\quad=Pe^{[\cdot,\Phi+\tilde\Phi]}(\Delta+\tilde\Delta),\end{aligned}$$

which, by Remark 1.9, is the left-hand side of the Maurer–Cartan equation in $\mathfrak{a}^P_{\Delta+\tilde\Delta}$ for $\Phi+\tilde\Phi$. Here in the first equation we used Remark 1.9.

The last two statements follow from Theorem 1. □

We end this subsection by presenting an alternative, more conceptual proof of Theorem 3. It is given by

$$\begin{aligned}(\tilde\Delta[1],\tilde\Phi)\in MC((L[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta)&\Leftrightarrow((\Delta+\tilde\Delta)[1],\Phi+\tilde\Phi)\in MC((L[1]\oplus\mathfrak{a})^P_0)\\ &\Leftrightarrow\begin{cases}[\Delta+\tilde\Delta,\Delta+\tilde\Delta]=0\\ \Phi+\tilde\Phi\in MC(\mathfrak{a}^P_{\Delta+\tilde\Delta}).\end{cases}\end{aligned}$$

The first equivalence is the conjunction of Lemma 1.12 (applied to $V=(L,\mathfrak{a},P,0)$ and $\alpha=(\Delta[1],\Phi)$) and of property (12). The second equivalence comes from the fact that the only non-vanishing brackets of $(L[1]\oplus\mathfrak{a})^P_0$ are given by $d(x[1])=Px$ for $x\in L$, by (6) and (7).

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1.5 Convergence issues

The left-hand side of the Maurer–Cartan equation (3) is generally an infinite sum. In this subsection, we review Getzler's notion of a filtered L_∞-algebra [Get10a], which guarantees that the above infinite sum converges. We show that simple assumptions on V-data ensure that the Maurer–Cartan equations of the (curved) $L_\infty[1]$-algebras we construct in Theorem 3 (and Lemma 1.12) do converge.

DEFINITION 1.14. Let V be a graded vector space. A *complete filtration* is a descending filtration by graded subspaces

$$V = \mathcal{F}^{-1}V \supset \mathcal{F}^0 V \supset \mathcal{F}^1 V \supset \cdots$$

such that the canonical projection $V \to \varprojlim V/\mathcal{F}^n V$ is an isomorphism. Here

$$\varprojlim V/\mathcal{F}^n V := \{\vec{x} \in \Pi_{n \geqslant -1} V/\mathcal{F}^n V : P_{i,j}(x_j) = x_i \text{ when } i < j\},$$

where $P_{i,j} : V/\mathcal{F}^j V \longrightarrow V/\mathcal{F}^i V$ is the canonical projection induced by the inclusion $\mathcal{F}^j V \subset \mathcal{F}^i V$.

Remark 1.15. If V can be written as a direct product of subspaces $V = \prod_{k \geqslant -1} V^k$, then $\{\mathcal{F}^n V\}_{n \geqslant -1}$ is a complete filtration of V, where $\mathcal{F}^n V := \prod_{k \geqslant n} V^k$.

DEFINITION 1.16. Let W be a curved $L_\infty[1]$-algebra. We say that W is *filtered*[1] if there exists a complete filtration on the vector space W such that all multibrackets $\{\cdots\}$ have filtration degree -1.

Notice that for an element $\Phi \in W$ of filtration degree 1, we have $\{\Phi, \ldots, \Phi\}_n \in \mathcal{F}^{n-1} W$ for all n, so the infinite sum

$$\sum_{n=0}^{\infty} \frac{1}{n!} \{\Phi, \ldots, \Phi\}_n \tag{18}$$

converges in W by the completeness of the filtration. Indeed, setting $w_i := \sum_{n=0}^{i} 1/n! \{\Phi, \ldots, \Phi\}_n$ mod $\mathcal{F}^i W$ for all i defines an element $\vec{w} \in \Pi_{n \geqslant -1} W/\mathcal{F}^n W$, which turns out to belong to $\varprojlim W/\mathcal{F}^n W \cong W$.

We define *Maurer–Cartan elements* to be $\Phi \in W_0 \cap \mathcal{F}^1 W$ for which the infinite sum (18) vanishes, and we write $MC(W)$ for the set of Maurer–Cartan elements.

DEFINITION 1.17. Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data (Definition 1.7). We say that this curved V-data is *filtered* if there exists a complete filtration on the graded vector space L such that:

(a) the Lie bracket has filtration degree zero, i.e. $[\mathcal{F}^i L, \mathcal{F}^j L] \subset \mathcal{F}^{i+j} L$ for all $i, j \geqslant -1$;

(b) $\mathfrak{a}_0 \subset \mathcal{F}^1 L$;

(c) the projection P has filtration degree zero, i.e. $P(\mathcal{F}^i L) \subset \mathcal{F}^i L$ for all $i \geqslant -1$.

PROPOSITION 1.18. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered, curved V-data. Then, for every $\Phi \in MC(\mathfrak{a}^P_\Delta) \subset \mathfrak{a}_0$:*

(1) *the projection $P_\Phi := P \circ e^{[\cdot, \Phi]} : L \to \mathfrak{a}$ is well defined and has filtration degree zero;*

(2) *the curved $L_\infty[1]$-algebra $\mathfrak{a}^{P_\Phi}_\Delta$ given by Theorem 1 is filtered by $\mathcal{F}^n \mathfrak{a} := \mathcal{F}^n L \cap \mathfrak{a}$. Further, the sum (18) converges for any degree-zero element a of $\mathfrak{a}$;*

[1] Our definition differs from Getzler's, which requires that $W = \mathcal{F}^0 W$ and that the multibrackets have filtration degree zero except for the zeroth bracket, which has filtration degree 1.

(3) *if* $\Delta \in \ker(P)$, *the* $L_\infty[1]$*-algebra* $(L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ *given by Theorem 2 is filtered by* $\mathcal{F}^n(L[1] \oplus \mathfrak{a}) := (\mathcal{F}^n L)[1] \oplus \mathcal{F}^n \mathfrak{a}$. *Further, the sum (18) converges for any degree-zero element element* $(x[1], a)$ *of* $L[1] \oplus \mathfrak{a}$.

Proof. (1) For every $x \in L$, say $x \in \mathcal{F}^i L$, by Definition 1.17(a) and (b) we have

$$[[\dots[x, \underbrace{\Phi], \dots], \Phi]}_{n \text{ times}} \in \mathcal{F}^{i+n} L.$$

Hence, the completeness of the filtration on L implies that $e^{[\cdot,\Phi]}$ is a well-defined endomorphism of L. The above also shows that $e^{[\cdot,\Phi]}$ has filtration degree zero and, since P does by Definition 1.17(c), we conclude that the projection P_Φ has filtration degree zero.

(2) We first check that $\{\mathcal{F}^n \mathfrak{a}\}_{n \geqslant -1}$ is a complete filtration of the vector space $\mathfrak{a}$.

The map $\mathfrak{a} \to \varprojlim \mathfrak{a}/\mathcal{F}^n \mathfrak{a}$ is surjective. Indeed, take an element of $\varprojlim \mathfrak{a}/\mathcal{F}^n \mathfrak{a}$ and consider its image under the canonical embedding $\varprojlim \mathfrak{a}/\mathcal{F}^n \mathfrak{a} \hookrightarrow \varprojlim W/\mathcal{F}^n W$. It is a sequence of elements $\{a_i \bmod \mathcal{F}^i W\}_{i \geqslant -1}$, where $a_i \in \mathfrak{a}$. The surjectivity of $W \to \varprojlim W/\mathcal{F}^n W$ implies that there is an element $w \in W$ such that $a_i \bmod \mathcal{F}^i W = w \bmod \mathcal{F}^i W$ for all i, which implies that $w \in \mathcal{F}^i W + \mathfrak{a}$ for all i and hence $w \in \cap_i(\mathcal{F}^i W + \mathfrak{a})$. Since $\cap_i(\mathcal{F}^i W) = \{0\}$ (by the injectivity of $W \to \varprojlim W/\mathcal{F}^n W$), this means that $w \in \mathfrak{a}$.

The map $\mathfrak{a} \to \varprojlim \mathfrak{a}/\mathcal{F}^n \mathfrak{a}$ is injective. Indeed, an element $a \in \mathfrak{a}$ is sent to 0 if and only if $a \in \cap_i(\mathcal{F}^i \mathfrak{a})$. But $\cap_i(\mathcal{F}^i \mathfrak{a}) \subset \cap_i(\mathcal{F}^i W)$, which is $\{0\}$ as seen above.

The multibracket of $\mathfrak{a}^{P_\Phi}_\Delta$ is given by $P_\Phi[\dots[[\Delta, \bullet], \bullet], \dots, \bullet]$ (see Theorem 1). Using (1) and Definition 1.17(a), we see that this multibracket has filtration degree -1.

For the last statement, notice that $\mathfrak{a}_0 \subset \mathcal{F}^1 \mathfrak{a}$ by Definition 1.17(b).

(3) $\{(\mathcal{F}^n L)[1] \oplus \mathcal{F}^n \mathfrak{a}\}_{n \geqslant -1}$ is a complete filtration of the vector space $L[1] \oplus \mathfrak{a}$ because the two summands are complete filtrations of $L[1]$ and $\mathfrak{a}$, respectively (by assumption and by (2), respectively). The multibrackets of $(L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ are given in Theorem 2, and all have filtration degree -1 by (1) and Definition 1.17(a).

For the last statement, notice that the non-vanishing multibrackets of $(L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ accept at most two entries from $L[1]$, and use again $\mathfrak{a}_0 \subset \mathcal{F}^1 \mathfrak{a}$. $\square$

A version of Proposition 1.18 in which the curved V-data is not assumed to be filtered, and working in the formal setting, is given in [FZ13].

1.6 Equivalences of Maurer–Cartan elements

Let W be an $L_\infty[1]$-algebra. On $MC(W)$, the set of Maurer–Cartan elements, there is a canonical involutive (singular) distribution $\mathcal{D}$ which induces an equivalence relation on $MC(W)$ known as *gauge equivalence*. More precisely, each $z \in W_{-1}$ defines a vector field $\mathcal{Y}^z$ on W_0, whose value at $m \in W_0$ is[2]

$$\mathcal{Y}^z|_m := dz + \{z, m\} + \frac{1}{2!}\{z, m, m\} + \frac{1}{3!}\{z, m, m, m\} + \cdots. \tag{19}$$

This vector field is tangent to $MC(W)$. The distribution at the point $m \in MC(W)$ is defined as $\mathcal{D}|_m = \{\mathcal{Y}^z|_m : z \in W_{-1}\}$.

[2] The infinite sum (19) is guaranteed to converge if W is filtered and $W_{-1} \subset \mathcal{F}^1 W$; see § 1.5. For the example we consider in § 2.3, this sum is actually finite.

Remark 1.19. We give a justification of the above statements; see also [KS, § 3.4.2], [Mer00, § 2.5], and [Fuk03, § 2.2]. Suppose that W is finite dimensional, so that the $L_\infty[1]$-algebra structure is encoded[3] by a degree 1, self-commuting vector field Q on W [Vor05a, Example 4.1]. We recall the following fact, which holds for any vector field X on W_0 and any element $m \in W_0$ (which defines a constant vector field m on W_0):

$$X|_m = (e^{[m,\cdot]}X)|_0. \tag{20}$$

Indeed, both sides equal $((\phi_{-1})_*X)|_0$, where ϕ denotes the time-one flow of m (translation by m). Equation (20) applied to $X = Q$ implies immediately that a point $m \in W_0$ is a zero of Q if and only if $-m$ satisfies the Maurer–Cartan equation (3).

View $z \in W_{-1}$ as a constant (degree -1) vector field on W. Then $[Q, z]$ is a degree-zero vector field. As $\mathcal{L}_{[Q,z]}Q = [[Q, z], Q] = 0$, the flow of $[Q, z]$ preserves the set of zeros of Q and hence $[Q, z]$ is tangent to this set. Equation (20) applied to $X = [Q, Z]$ implies that $[Q, z]|_{W_0}$ is the pushforward by $-\mathrm{Id}_{W_0}$ of $\mathcal{Y}^z$; therefore, $\mathcal{Y}^z$ is tangent to $MC(W)$.

A computation shows that $\mathcal{D}$ can also be described in terms of all degree -1 vector fields: $\mathcal{D}|_m = \{[Q, Z]|_m : Z \in \chi_{-1}(W)\}$ for all $m \in MC(W)$. Since $[[Q, Z], [Q, Z']] = [Q, [[Q, Z], Z']]$, it follows that $\mathcal{D}$ is involutive.

We will display explicitly the equivalence relation induced on twisted Poisson structures in § 2.3 and show that in this case the equivalence classes coincide with the orbits of a group action.

2. Applications to Poisson geometry

In this section, we apply the machinery developed in § 1 to examples arising from Poisson geometry. We study deformations of Poisson manifolds and coisotropic submanifolds in § 2.1. We consider deformations of Courant algebroids and Dirac structures in § 2.2, focusing on the special case of twisted Poisson structures (and discussing equivalences) in § 2.3. Finally, we consider deformations of Courant algebroids and generalized complex structures in § 2.4, discussing the case of complex structures in § 2.5.

2.1 Coisotropic submanifolds of Poisson manifolds

In this subsection, we consider deformations of Poisson structures on a manifold M and deformations of coisotropic submanifolds. We build on work of Oh and Park [OP05], who realized that deformations of a coisotropic submanifold of a fixed symplectic manifold are governed by an $L_\infty[1]$-algebra, and on work of Cattaneo and Felder [CF07], who associated an $L_\infty[1]$-algebra to any coisotropic submanifold of a Poisson manifold.

Our main reference for this deformation problem is [Sch09, § 3.2], which is based on [OP05, CF07]. Recall that a *Poisson structure* on M is a bivector field π on M such that $[\pi, \pi] = 0$, where the bracket denotes the Schouten bracket, and that a submanifold $C \subset (M, \pi)$ is *coisotropic* if $\pi^\sharp TC^\circ \subset TC$, where $TC^\circ := \{\xi \in T^*M|_C : \xi|_{TC} = 0\}$ and $\pi^\sharp : T^*M \to TM$ is the contraction with π [CW99].

Let M be a manifold. Let $C \subset M$ be a submanifold. Fix an embedding of the normal bundle $\nu C := TM|_C/TC$ into a tubular neighborhood of C in M, such that the embedding and its derivative are the identity on C. In the following, we will identify νC with its image in M.

[3] The multibrackets on W are recovered from Q by applying Theorem 1 to the V-data $(L = \chi(W), \mathfrak{a} = \{\text{constant vector fields on } W\}, P(X) = X|_0, Q)$.

We say that a vector field on νC is *fiberwise polynomial* if it preserves the fiberwise polynomial functions on the vector bundle νC. Such a vector field X has *polynomial degree* n (denoted $|X|_{\mathrm{pol}} = n$) if its action on fiberwise polynomial functions raises their degree (as polynomials) at most by n. Locally, choose local coordinates on C and linear coordinates along the fibers of νC, which we denote collectively by x and p, respectively. Then the fiberwise polynomial vector fields are exactly those which are sums of expressions $f_1(x)F_1(p)(\partial/\partial x)$ and $f_2(x)F_2(p)(\partial/\partial p)$, where $f_i \in C^\infty(C)$ and the F_i are polynomials. The polynomial degrees of the two vector fields exhibited here are $\deg(F_1)$ and $\deg(F_2) - 1$, respectively.

Consider $\chi^\bullet(\nu C)$, the space of multivector fields on the total space νC, and denote by $\chi^\bullet_{fp}(\nu C)$ the sums of products of fiberwise polynomial vector fields. The space $\chi^\bullet(\nu C))[1]$ is a graded Lie algebra when endowed with the Schouten bracket $[\cdot,\cdot]$, and $\chi^\bullet_{fp}(\nu C)[1]$ is a graded Lie subalgebra. The notion of polynomial degrees carries on to fiberwise polynomial multivector fields, by $|X_1 \wedge \cdots \wedge X_k|_{\mathrm{pol}} = \sum_i |X_i|_{\mathrm{pol}}$. The Schouten bracket preserves the polynomial degree (this is clear if we think of multivector fields as acting on tuples of functions).

Sections in $\Gamma(\wedge \nu C)$ can be regarded as elements of $\chi^\bullet_{fp}(\nu C)$ which are vertical (tangent to the fibers) and fiberwise constant. A *fiberwise polynomial Poisson* bivector field on νC is an element $\pi \in \chi^2_{fp}(\nu C)$ such that $[\pi,\pi] = 0$. Notice that the associated Poisson bracket raises the degree of fiberwise polynomial functions on νC by at most $|\pi|_{\mathrm{pol}}$.

Remark 2.1. The condition that a Poisson structure be fiberwise polynomial is quite strong. The results of this subsection are extended in [SZ13] to Poisson structures in a neighborhood $U \subset \nu C$ of the zero section which are 'fiberwise entire', in the following sense: the Poisson bracket of two fiberwise polynomial functions, restricted to $U \cap \nu_x C$, is given by a converging power series (for any $x \in C$).

Lemma 2.2. *Let π be a fiberwise polynomial Poisson structure on νC. The following quadruple forms a curved V-data:*

- *the graded Lie algebra $L := \chi^\bullet_{fp}(\nu C)[1]$;*
- *its abelian subalgebra $\mathfrak{a} := \Gamma(\wedge \nu C)[1]$;*
- *the natural projection $P : L \to \mathfrak{a}$ given by restriction to C and projection along $\wedge T(\nu C)|_C \to \wedge \nu C$;*
- *$\Delta := \pi$;*

hence, by Theorem 1, we obtain a curved $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$.

Its Maurer–Cartan equation reads

$$P \sum_{n=0}^{|\pi|_{\mathrm{pol}}+2} \frac{1}{n!}[[\ldots[\pi, \underbrace{\Phi], \ldots], \Phi]}_{n \text{ times}} = 0, \tag{21}$$

where $\Phi \in \Gamma(\nu C)[1]$ is seen as a vertical vector field on νC. Here $\Phi \in \Gamma(\nu C)[1]$ is a Maurer–Cartan element in $\mathfrak{a}^P_\Delta$ if and only if $\mathrm{graph}(-\Phi)$ *is a coisotropic submanifold of $(\nu C, \pi)$.*

Further, the above quadruple forms a V-data if and only if C is a coisotropic submanifold of $(\nu C, \pi)$.

Proof. The fact that the above quadruple forms a curved V-data is essentially the content of [CF07, § 2.6]. For a more detailed proof, we refer to [Sch09, Lemma 3.3 in § 3.3], use that $\chi^\bullet_{fp}(\nu C)$ is a graded Lie subalgebra of $\chi^\bullet(\nu C)$, and use that $[\pi,\pi] = 0$ by the definition of Poisson structure.

To prove (21), we argue as follows. Elements $a_i \in \mathfrak{a}_0 = \Gamma(\nu C)[1]$, seen as vertical vector fields on νC, have polynomial degree -1 (in coordinates they read $f(x)(\partial/\partial p)$). Since the Schouten

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bracket preserves the polynomial degree, $[[\dots[\pi, a_1], \dots], a_n]$ has polynomial degree $|\pi|_{\mathrm{pol}} - n$. Since the polynomial degree of a non-vanishing bivector field is $\geqslant -2$, we conclude that the above iterated brackets vanish for $n > |\pi|_{\mathrm{pol}} + 2$.

The equivalence[4] between $\Phi \in \Gamma(\nu C)[1]$ being a Maurer–Cartan element and $\mathrm{graph}(-\Phi)$ being a coisotropic submanifold of $(\nu C, \pi)$ is proven as follows. Denote by $\psi : \nu C \to \nu C$ the time-one flow of the vector field Φ (so ψ is just translation by Φ). In particular, $\psi(\mathrm{graph}(-\Phi)) = C$. The pushforward bivector field by ψ satisfies $\psi_*(\pi) = e^{[\cdot, \Phi]}\pi$. Hence, $\mathrm{graph}(-\Phi)$ is coisotropic (with respect to π) if and only if C satisfies the coisotropicity condition with respect to $e^{[\cdot,\Phi]}\pi$, which is just (21). To show that $\psi_*(\pi) = e^{[\cdot,\Phi]}\pi$, let f, g be fiberwise polynomial functions on νC. We have $\psi^* f = e^{\Phi} f$ using the Taylor expansion of f on each fiber. Hence,

$$\begin{aligned}(\psi_*\pi)(f, g) &= (\psi^{-1})^*(\pi(\psi^* f, \psi^* g)) = e^{-\Phi}(\pi(e^{\Phi} f, e^{\Phi} g)) \\ &= e^{[\cdot,\Phi]}[[\pi, e^{[\Phi,\cdot]} f], e^{[\Phi,\cdot]} g] = [[e^{[\cdot,\Phi]}\pi, f], g] = (e^{[\cdot,\Phi]}\pi)(f, g).\end{aligned}$$

For the last statement, use Theorem 1 and notice that C is coisotropic if and only if we can write $\pi = \sum_j X_j \wedge Y_j$ with X_j tangent to C, i.e. if and only if $\pi \in \ker(P)$. □

Hence, we can apply Theorem 3 (with $\Delta = \pi = 0$ and $\Phi = 0$).

COROLLARY 2.3. *Let C be a submanifold of a manifold and consider a tubular neighborhood νC. For all $\tilde{\pi} \in \chi^2_{fp}(\nu C)$ and $\tilde{\Phi} \in \Gamma(\nu C)$:*

$$\begin{cases} \tilde{\pi} \text{ is a Poisson structure} \\ \mathrm{graph}(-\tilde{\Phi}) \text{ is a coisotropic submanifold of } (\nu C, \tilde{\pi}); \end{cases}$$

$\Leftrightarrow$ $(\tilde{\pi}[2], \tilde{\Phi}[1])$ *is a Maurer–Cartan element of the L_∞[1]-algebra* $\chi^\bullet_{fp}(\nu C)[2] \oplus \Gamma(\wedge \nu C)[1]$.

The above $L_\infty[1]$-algebra structure is given by the multibrackets (all others vanish)

$$\begin{aligned} d(X[1]) &= PX, \\ \{X[1], Y[1]\} &= [X, Y][1](-1)^{|X|}, \\ \{X[1], a_1, \dots, a_n\} &= P[\dots[X, a_1], \dots, a_n] \quad \text{for all } n \geqslant 1, \end{aligned}$$

where $X, Y \in \chi^\bullet_{fp}(\nu C)[1]$, $a_1, \dots, a_n \in \Gamma(\wedge \nu C)[1]$, and $[\cdot\,,\cdot]$ denotes the Schouten bracket on $\chi^\bullet_{fp}(\nu C)[1]$.

Remark 2.4. (1) The formulae for the multibrackets in Corollary 2.3 show that the Maurer–Cartan equation for $(\tilde{\pi}[2], \tilde{\Phi}[1])$ has at most $|\tilde{\pi}|_{\mathrm{pol}} + 2$ terms, by the same argument as in Lemma 2.2.

(2) It is known that the deformation problem of coisotropic submanifolds in Poisson (even symplectic) manifolds is formally obstructed [OP05]. Corollary 2.3 is used in [SZ13] to show that the same applies to the simultaneous deformation problem of coisotropic submanifolds and fiberwise entire Poisson structures.

We now display the $L_\infty[1]$-algebra governing the deformations of a Poisson structure π and of a coisotropic submanifold C.

COROLLARY 2.5. *Let (M, π) be a Poisson manifold and C a coisotropic submanifold. Identify a tubular neighborhood of C in M with the normal bundle νC in such a way that π is fiberwise polynomial. There is an $L_\infty[1]$-algebra structure on $\chi^\bullet_{fp}(\nu C)[2] \oplus \Gamma(\wedge \nu C)[1]$ whose Maurer–Cartan elements are exactly pairs $(\tilde{\pi}[2], \tilde{\Phi}[1])$, where $\tilde{\pi} \in \chi^2_{fp}(\nu C)$ and $\tilde{\Phi} \in \Gamma(\nu C)$ are*

[4] See [Sch09, Example 3.2 in § 4.3] for an example where π is not fiberwise polynomial and the correspondence fails.

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such that $\pi + \tilde{\pi}$ *is a Poisson structure and* $\mathrm{graph}(-\Phi)$ *is a coisotropic submanifold with respect to* $\pi + \tilde{\pi}$.

Its non-vanishing multibrackets $\{\cdots\}^{\pi}$ *are given as follows:*

$$\begin{aligned}
d^{\pi}(X[1]) &= (-[\pi, X][1], P(X)),\\
d^{\pi}(a) &= (0, P([\pi, a])),\\
\{X[1], Y[1]\}_2^{\pi} &= (-1)^{|X|}[X, Y][1],\\
\{a_1, \ldots, a_n\}_n^{\pi} &= P([\ldots[\pi, a_1], \ldots, a_n]) \quad \textit{for all } n \geqslant 1,\\
\{X[1], a_1, \ldots, a_n\}_{n+1}^{\pi} &= P([\ldots[X, a_1], \ldots, a_n]) \quad \textit{for all } n \geqslant 1,
\end{aligned}$$

where $X, Y \in \chi^{\bullet}_{fp}(\nu C)[1]$ *and* $a_1, \ldots, a_n \in \Gamma(\wedge \nu C)[1]$.

Proof. By (12), in order to obtain an $L_\infty[1]$-algebra whose Maurer–Cartan elements are those specified in the statement of the present corollary, we can twist the $L_\infty[1]$-algebra of Corollary 2.3 using the Maurer–Cartan element $(\pi[2], 0)$. Notice that $(\pi[2], 0)$ is really a Maurer–Cartan element, since π is a Poisson structure and C is coisotropic with respect to π.

The multibrackets $\{\cdots\}^{\pi}$ are computed as in (11). Notice that the particular form of the multibrackets appearing in Corollary 2.3 forces all terms on the right-hand side of (11) to be zero, except possibly for the first two. $\square$

2.2 Dirac structures and Courant algebroids

In this subsection, we consider a Courant algebroid structure on a fixed vector bundle and a Dirac subbundle A. We study deformations of the Courant algebroid structure (with the constraint that the symmetric pairing remains unchanged) and of the Dirac subbundle A. Deformations of Dirac subbundles within a fixed Courant algebroid were studied by Liu *et al.* [LWX97] and by Bursztyn *et al.* [BCŠ05]. We will make use of facts from [BCŠ05, § 3] and Roytenberg [Roy02b, § 3, Roy99, § 3, Roy02a]. We refer to [Sch09, § 1.4] or [CS11] for some basic facts on graded geometry.

Recall that a *Courant algebroid* consists of a vector bundle $E \to M$ with a non-degenerate symmetric pairing on the fibers, a bilinear operation $[\![\cdot\,,\cdot]\!]$ on $\Gamma(E)$, and a bundle map $\rho : E \to TM$ satisfying compatibility conditions; see for instance [Roy02a, Definition 4.2]. An example is $TM \oplus T^*M$ with the natural pairing, $[\![X + \xi, Y + \eta]\!] := [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi$, and $\rho(X + \xi) = X$ (this is sometimes called the standard Courant algebroid). A *Dirac structure* is a subbundle $L \subset E$ such that L equals its orthogonal with respect to the pairing, and so that $\Gamma(L)$ is closed under $[\![\cdot\,,\cdot]\!]$; see [Cou90]. Examples of Dirac structures for the standard Courant algebroid are provided by graphs of closed 2-forms and of Poisson bivector fields.

Fix a Courant algebroid $E \to M$ and maximal isotropic subbundles A and K (not necessarily involutive) so that $E = A \oplus K$ as a vector bundle. Identify $K \cong A^*$ via the pairing on the fibers of E. Consider the maps

$$\begin{aligned}
\Gamma(\wedge^2 A^*) \to \Gamma(A), &\quad \eta_1 \wedge \eta_2 \mapsto \mathrm{pr}_A([\![(0, \eta_1), (0, \eta_2)]\!]),\\
\Gamma(\wedge^2 A) \to \Gamma(A^*), &\quad a_1 \wedge a_2 \mapsto \mathrm{pr}_{A^*}([\![(a_1, 0), (a_2, 0)]\!]),
\end{aligned}$$

and view them as elements $\psi \in \Gamma(\wedge^3 A)$ and $\varphi \in \Gamma(\wedge^3 A^*)$, respectively. Denote by d_A the degree 1 derivation of $\Gamma(\wedge A^*)$ given by the bracket $[\cdot\,,\cdot]_A := \mathrm{pr}_A([\![\cdot\,,\cdot]\!]|_A)$ on $\Gamma(A)$ and the bundle map $\rho|_A : A \to TM$. Similarly, denote by d_{A^*} the degree 1 derivation of $\Gamma(\wedge A)$ given by the bracket $[\eta_1, \eta_2]_{A^*} := \mathrm{pr}_{A^*}([\![(0, \eta_1), (0, \eta_2)]\!])$ on $\Gamma(A^*)$ and the bundle map $\rho|_{A^*} : A^* \to TM$. The data given by ψ, φ, $(A, [\cdot\,,\cdot]_A, \rho|_A)$, and $(A^*, [\cdot\,,\cdot]_{A^*}, \rho|_{A^*})$ forms a *proto-bialgebroid*. From these data, one can reconstruct the Courant algebroid structure on E: the bilinear operation is recovered as

$$\begin{aligned}&[\![(a_1,\eta_1),(a_2,\eta_2)]\!]\\&\quad=([a_1,a_2]_A+\mathcal{L}_{\eta_1}a_2-\iota_{\eta_2}d_{A^*}a_1+\psi(\eta_1,\eta_2,\cdot),\\&\qquad[\eta_1,\eta_2]_{A^*}+\mathcal{L}_{a_1}\eta_2-\iota_{a_2}d_A\eta_1+\varphi(a_1,a_2,\cdot))\end{aligned}\tag{22}$$

and the anchor as $\rho_A+\rho_{A^*}:A\oplus A^*\to TM$ [Roy99, § 3.8]; see also [Kos05, § 3.2].

Recall that Courant algebroids are in bijective correspondence with degree 2 symplectic graded manifolds $\mathcal{M}$ together with a degree 3 function $\Delta\in C(\mathcal{M})$ satisfying $\{\Delta,\Delta\}=0$ [Roy02a, Theorem 4.5]. (Here $\{\cdot,\cdot\}$ denotes the degree -2 Poisson bracket on $C(\mathcal{M})$ induced by the symplectic structure.) The Courant algebroid E corresponds to

$$(\mathcal{M}:=T^*[2]A[1],\Delta=-\varphi+h_{d_A}+F^*(h_{d_{A^*}})-\psi)$$

with the canonical symplectic structure, by [Roy99, Theorem 3.8.2]. Here we view $\psi\in\Gamma(\wedge^3A)$ and $\varphi\in\Gamma(\wedge^3A^*)$ as elements of $C_3(\mathcal{M})$. Further, $h_{d_A}\in C_3(\mathcal{M})$ is the fiberwise linear function induced by d_A, the function $h_{d_{A^*}}\in C_3(T^*[2]A^*[1])$ is defined similarly, and $F:T^*[2]A[1]\to T^*[2]A^*[1]$ is the canonical symplectomorphism known as *Legendre transformation* [Roy99, § 3.4]. We denote by pr the cotangent projection $\mathcal{M}\to A[1]$.

LEMMA 2.6. *Fix a Courant algebroid $E\to M$ and maximal isotropic subbundles A and K so that $E=A\oplus K$ as a vector bundle. The following quadruple forms a curved V-data:*
- *the graded Lie algebra $L:=C(\mathcal{M})[2]$ with Lie bracket*[5] $\{\cdot,\cdot\}$;
- *its abelian subalgebra $\mathfrak{a}:=\mathrm{pr}^*(C(A[1]))[2]\cong\Gamma(\wedge A^*)[2]$;*
- *the natural projection $P:L\to\mathfrak{a}$ given by evaluation on the base $A[1]$;*
- $\Delta=-\varphi+h_{d_A}+F^*(h_{d_{A^*}})-\psi$;

hence, by Theorem 1, we obtain a curved $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$. For every $\Phi\in\Gamma(\wedge^2A^*)$, we have that $\Phi[2]$ is a Maurer–Cartan element of $\mathfrak{a}^P_\Delta$ if and only if

$$\mathrm{graph}(-\Phi):=\{(X-\iota_X\Phi):X\in A\}\subset A\oplus A^*=E$$

is a Dirac structure.

Further, the above quadruple forms a V-data if and only if A is a Dirac structure of E.

Proof. Since $\{\cdot,\cdot\}$ is the canonical Poisson bracket on the cotangent bundle, the cotangent fibers and the base $A[1]$ are Lagrangian submanifolds. Hence, $\mathfrak{a}$ is an abelian Lie subalgebra of L and $\ker(P)$, which consists of a function on $T^*[2]A[1]$ vanishing on the base, is a Lie subalgebra. We have $\{\Delta,\Delta\}=0$ since Δ induces a Courant algebroid structure on $A\oplus A^*$. Hence, the above quadruple is a curved V-data and, by Theorem 1, we obtain a curved $L_\infty[1]$-algebra structure $\mathfrak{a}^P_\Delta$.

We compute the Maurer–Cartan equation of $\mathfrak{a}^P_\Delta$. Let $\Phi\in\mathfrak{a}_0=\Gamma(\wedge^2A^*)[2]$. We have $\{\emptyset\}=P\Delta=-\varphi$. Notice that $-\varphi$ does not appear in the remaining terms of the Maurer–Cartan equation, since $\{-\varphi,\Phi\}=0$, for both entries belong to the abelian subalgebra $\mathfrak{a}$. From the expression in coordinates for $F^*(h_{d_{A^*}})$, it follows that $\{F^*(h_{d_{A^*}}),\Phi\}$ and $\{-\psi,\Phi\}$ vanish on the base $A[1]$. So,

$$P\{\Delta,\Phi\}=\{h_{d_A},\Phi\}=d_A\Phi\in\Gamma(\wedge^3A^*),$$

where we used [Roy99, Lemma 3.3.1 1)]. Further, $\{\{h_{d_A},\Phi\},\Phi\}=0$ since both $\{h_{d_A},\Phi\}$ and Φ lie in the abelian Lie subalgebra $\mathrm{pr}^*(C(A[1]))$, and in coordinates it is clear that $\{\{-\psi,\Phi\},\Phi\}$ vanishes on the base $A[1]$. So,

$$P\{\{\Delta,\Phi\},\Phi\}=\{\{F^*(h_{d_{A^*}}),\Phi\},\Phi\}=-[\Phi,\Phi]_{A^*},$$

[5] $\{\cdot,\cdot\}$, as a bracket on L, has degree zero. Hence, $(L,\{\cdot,\cdot\})$ is a graded Lie algebra.

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where we used [Roy99, Lemma 3.6.2]. Further,

$$P\{\{\{\Delta,\Phi\},\Phi\},\Phi\}=\{\{\{-\psi,\Phi\},\Phi\},\Phi\}=(\Phi^{\sharp}\wedge\Phi^{\sharp}\wedge\Phi^{\sharp})\psi\in\Gamma(\wedge^{3}A^{*}),$$

where $\Phi^{\sharp}:A\to A^{*}, v\mapsto\iota_v\Phi$ is the contraction in the first component (see [Kos05, § 4.1.2]). All the other terms of the Maurer–Cartan equation vanish. Hence, we conclude that the Maurer–Cartan equation is

$$-\varphi+d_A\Phi-\tfrac{1}{2}[\Phi,\Phi]_{A^*}+\wedge^3\,\tilde{\Phi}(\psi)=0, \tag{23}$$

where $\wedge^3\tilde{\Phi}$ is defined as in § 2.3.

We show that Φ satisfies (23) if and only if graph$(-\Phi)$ is a Dirac structure, using the results of [Roy02b, § 4]. There, Roytenberg considered the time-one flow F_Φ of the hamiltonian vector field of Φ on $\mathcal{M}$. As F_Φ is a symplectomorphism of $\mathcal{M}$, it corresponds to an isomorphism of Courant algebroids between E' and E, where E' is the Courant algebroid[6] corresponding to the degree 3 function $F_\Phi^*(\Delta)$ on $\mathcal{M}$. The pullback function $F_\Phi^*(\Delta)$ splits into a sum according to bi-degree, and the component lying in $\Gamma(\wedge^3A^*)$ is

$$-\varphi-d_A\Phi-\tfrac{1}{2}[\Phi,\Phi]_{A^*}-\wedge^3\tilde{\Phi}(\psi); \tag{24}$$

see [Roy02b, (4.2)]. The subbundle A is a Dirac structure in E' if its image under the isomorphism $E'\cong E$, which is graph(Φ), is a Dirac structure in E. On the other hand, as one sees easily from [Roy99, Theorem 3.8.2], A is a Dirac structure in E' if and only if the component of $F_\Phi^*(\Delta)$ lying in $\Gamma(\wedge^3A^*)$ (which is given by (24)) vanishes or, equivalently, if $-\Phi$ satisfies the Maurer–Cartan equation (23). Putting together these two statements proves the claim.

Finally, notice that $\Delta\in\ker(P)$ if and only if its component in the bi-degree corresponding to $\Gamma(\wedge^3A^*)$, which is $-\varphi$, vanishes. Using again [Roy99, Theorem 3.8.2], we see that the quadruple $(L,\mathfrak{a},P,\Delta)$ forms a V-data if and only if A is a Dirac structure of E. ◻

Corollary 2.7. *Fix a Courant algebroid $E\to M$, a Dirac structure A, and a complementary isotropic subbundle K. Let $(L,\mathfrak{a},P,\Delta)$ be as in Lemma 2.6 for all $\tilde{\Delta}\in C(\mathcal{M})_3$ and $\tilde{\Phi}\in\Gamma(\wedge^2A^*)$:*

$$\begin{cases}\Delta+\tilde{\Delta}\ \textit{defines a new Courant algebroid}\\ \textit{structure on the vector bundle } E;\\ \mathrm{graph}(-\tilde{\Phi})\ \textit{is a Dirac structure there}\end{cases}\quad\Leftrightarrow\quad(\tilde{\Delta}[3],\tilde{\Phi}[2])\in MC((L[1]\oplus\mathfrak{a})^P_\Delta).$$

Proof. The quadruple $(L,\mathfrak{a},P,\Delta)$ is a V-data by the last statement of Lemma 2.6; hence, we can apply Theorem 3 with $\Phi=0$. Use again Lemma 2.6 to phrase the conclusions of Theorem 3 in terms of Courant algebroids and Dirac structures. ◻

Remark 2.8. We check that the V-data $(L,\mathfrak{a},P,\Delta)$ is filtered (Definition 1.17). The graded manifold $T^*[2]A^*[1]$ is a vector bundle over $A^*[1]$, so we can denote by $C^k(T^*[2]A^*[1])$ the functions which are polynomials of degree k on each fiber. Using the Legendre transformation F to identify $\mathcal{M}=T^*[2]A[1]$ with $T^*[2]A^*[1]$, we obtain a direct product decomposition $L=\prod_{k\geqslant -1}L^k$, where $L^k:=C^{k+1}(T^*[2]A^*[1])$. Notice that an element of $\mathrm{pr}^*(C_{k+1}(A[1]))[2]\cong\Gamma(\wedge^{k+1}A^*)[2]$ lies in L^k. By Remark 1.15, $\mathcal{F}^nL:=\prod_{k\geqslant n}L^k$ is a complete filtration of the vector space L. The remaining items of Definition 1.17 are easily checked.

[6] Both Courant algebroids have the same underlying vector bundle and the same symmetric pairing.

2.3 Twisted Poisson structures

In this subsection, we present a special case of the situation studied in § 2.2. We apply Corollary 2.7 to the standard Courant algebroid over a manifold M and $A = T^*M$. We obtain an $L_\infty[1]$-algebra whose Maurer–Cartan elements consist of closed 3-forms and twisted Poisson structures [ŠW01], recovering the $L_\infty[1]$-algebra recently displayed by Getzler [Get10b], and study their equivalences. Further, given a closed 3-form H and an H-twisted Poisson structure, we describe the $L_\infty[1]$-algebra governing the deformations of the pair (H, π). Twisted Poisson structures appeared in relation to deformations also in [Par01, § 3].

We will need the following notation: for $\pi \in \wedge^a TM$ and $a \geqslant 1$, we define

$$\pi^\sharp : T^*M \to \wedge^{a-1} TM, \quad \xi \to \iota_\xi \pi,$$

and we define $\pi^\sharp \equiv 0$ if $a = 0$. We also need an extension of the above to several multivectors: for $\pi_1 \in \wedge^{a_1} TM, \dots, \pi_n \in \wedge^{a_n} TM$ $(n \geqslant 1, a_i \geqslant 1)$, we define

$$\begin{aligned} \pi_1^\sharp \wedge \cdots \wedge \pi_n^\sharp : \wedge^n T^*M &\to \wedge^{a_1+\cdots+a_n-n} TM, \\ \xi_1 \wedge \cdots \wedge \xi_n &\mapsto \sum_{\sigma \in S_n} (-1)^\sigma \pi_1^\sharp(\xi_{\sigma(1)}) \wedge \cdots \wedge \pi_n^\sharp(\xi_{\sigma(n)}), \end{aligned}$$

where $\xi_i \in T^*M$ and $(-1)^\sigma$ is the sign of the permutation σ.

Recall that, given a bivector field π and a closed 3-form H, one says that π is an *H-twisted Poisson structure* [ŠW01, (1)] if and only if

$$[\pi, \pi] = 2 \wedge^3 \tilde{\pi}(H),$$

where $\wedge^3 \tilde{\pi} = \frac{1}{6}(\pi^\sharp \wedge \pi^\sharp \wedge \pi^\sharp)$ and $[\cdot\,,\cdot]$ is the Schouten bracket of multivector fields.

COROLLARY 2.9. *Let M be a manifold. There is an $L_\infty[1]$-algebra structure on*

$$\mathfrak{L} := \Omega^{\bullet \geqslant 1}(M)[3] \oplus \chi^\bullet(M)[2]$$

whose only non-vanishing multibrackets are:

(a) *minus the de Rham differential on differential forms;*
(b) $\{\pi_1, \pi_2\} = [\pi_1, \pi_2](-1)^{a_1+1}$, *where $\pi_i \in \chi^{a_i}(M)$;*
(c) $\{H, \pi_1, \dots, \pi_n\} = (-1)^{\sum_{i=1}^n a_i(n-i)}(\pi_1^\sharp \wedge \cdots \wedge \pi_n^\sharp)H$ *for all $n \geqslant 1$, where $H \in \Omega^n(M)$ and $\pi_1 \in \chi^{a_1}(M), \dots, \pi_n \in \chi^{a_n}(M)$ with all $a_i \geqslant 1$.*

Its Maurer–Cartan elements are exactly pairs $(H[3], \pi[2])$, where $H \in \Omega^3(M)$ and $\pi \in \chi^2(M)$ are such that $dH = 0$ and π is an H-twisted Poisson structure.

Remark 2.10. The graded vector space $\mathfrak{L} = \Omega^{\bullet \geqslant 1}(M)[3] \oplus \chi^\bullet(M)[2]$ is concentrated in degrees $\{-2, \dots, \dim(M) - 2\}$, and its degree i component is $\Omega^{i+3}(M) \oplus \chi^{i+2}(M)$.

Proof. We apply Corollary 2.7 to the standard Courant algebroid $TM \oplus T^*M$ (defined at the beginning of § 2.2), to $A = T^*M$, and to $K = TM$. Notice that it corresponds to the Lie bialgebroid (A, K), where A has the zero structure and $K = TM$ has its canonical Lie algebroid structure.

We use the following notation for the canonical local coordinates on $\mathcal{M} := T^*[2]T^*[1]M$: we denote by x_j arbitrary local coordinates on M and by p_j the canonical coordinates on the fibers of $T^*[1]M$ (so the degrees are $|x_j| = 0, |p_j| = 1$, for $j = 1, \dots, \dim(M)$). By P_j, v_j, we denote

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the conjugate coordinates on the fibers of $\mathcal{M} \to T^*[1]M$ with degrees $|P_j| = 2, |v_j| = 1$. One has $\{P_j, x_k\} = \delta_{jk}$ and $\{p_j, v_k\} = \delta_{jk}$. The element of $C_3(\mathcal{M})$ corresponding to the standard Courant algebroid is $\mathcal{S} := \sum_i P_i v_i$.

The quadruple appearing in Lemma 2.6 reads:

- $L := C(T^*[2]T^*[1]M)[2]$, whose Lie bracket we denote by $\{\cdot\,,\cdot\}$;
- $\mathfrak{a} := C(T^*[1]M))[2] \cong \chi^\bullet(M)[2]$;
- the natural projection $P : L \to \mathfrak{a}$ given by evaluation on the base $T^*[1]M$, i.e. setting $P_j = 0, v_j = 0$ for all j;
- $\Delta = \sum_i P_i v_i$.

The multibrackets of the $L_\infty[1]$-algebra $(L[1] \oplus \mathfrak{a})^P_\Delta$ are given in Theorem 2. Notice that using the Legendre transformation F, we have

$$\Omega(M)[2] = C(T[1]M)[2] \subset C(T^*[2]T[1]M)[2] \cong L,$$

and $\Omega^{\bullet\geqslant 1}(M)[2] \subset \ker(P)$ is a Lie subalgebra preserved by $\{\Delta, \cdot\}$. So, by Remark 1.10, it follows that $\mathfrak{L} = \Omega^{\bullet\geqslant 1}(M)[3] \oplus \chi^\bullet(M)[2]$ is an $L_\infty[1]$-subalgebra of $(L[1] \oplus \mathfrak{a})^P_\Delta$. We justify why the restriction of the multibrackets to $\mathfrak{L}$ is the one described in the statement of this corollary. Type (a) follows from (5) and

$$\left\{\sum_i P_i v_i, F(x) v_{\epsilon(1)} \cdots v_{\epsilon(k)}\right\} = \sum_i \frac{\partial F}{\partial x_i} v_i v_{\epsilon(1)} \cdots v_{\epsilon(k)},$$

where $\epsilon(i) = 1, \ldots, \dim(M)$. Type (b) follows from (8) and [Roy99, Lemma 3.6.2]. Type (c) follows from (7) and a lengthy but straightforward computation in coordinates.

For the statement on Maurer–Cartan elements, we proceed as follows. Given $H \in \Omega^3(M)$, the degree 3 function $\sum_i P_i v_i + H$ on $\mathcal{M}$ defines a Courant algebroid structure (i.e. is self-commuting) if and only if H is closed, and in this case it induces the $(-H)$-twisted[7] Courant algebroid $(TM \oplus T^*M)_{-H}$ [Roy02a, § 4], [Zam12, § 8]. Hence, by Corollary 2.7, $(H[3], \pi[2])$ is a Maurer–Cartan element of $\mathfrak{L}$ if and only if H is closed and graph$(-\pi)$ is a Dirac structure in $(TM \oplus T^*M)_{-H}$. The latter condition is equivalent to $-\pi$ being a $(-H)$-twisted Poisson structure [ŠW01, § 3], that is, to π being an H-twisted Poisson structure. □

Given a closed 3-form H and an H-twisted Poisson structure π, we now describe the $L_\infty[1]$-algebra governing deformations of the pair (H, π) to pairs consisting of a closed 3-form and a correspondingly twisted Poisson structure.

Corollary 2.11. *Let M be a manifold, H a closed 3-form, and π an H-twisted Poisson structure. There is an $L_\infty[1]$-algebra structure on*

$$\Omega^{\bullet\geqslant 1}(M)[3] \oplus \chi^\bullet(M)[2]$$

whose Maurer–Cartan elements are exactly pairs $(\widetilde{H}[3], \widetilde{\pi}[2])$, where $\widetilde{H} \in \Omega^3(M)$ and $\widetilde{\pi} \in \chi^2(M)$ are such that $\widetilde{H}$ is closed and $\pi + \widetilde{\pi}$ is an $(H + \widetilde{H})$-twisted Poisson structure.

Its nth multibracket is $\{\cdots\}_n + \{\cdots\}'_n$, where $\{\cdots\}_n$ denotes the nth multibracket of the $L_\infty[1]$-algebra described in Corollary 2.9 and $\{\cdots\}'_n$ is defined as follows:

[7] Recall that the K-twisted Courant algebroid is $TM \oplus T^*M$ with bilinear operation $[\![X + \xi, Y + \eta]\!]_K := [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X K$.

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(i) *for* $n=1$*:*

$$\{(\widetilde{H}_1,\widetilde{\pi}_1)\}'_1 = -[\pi,\widetilde{\pi}_1] + \frac{1}{2}\{H,\pi,\pi,\widetilde{\pi}_1\}_4 + \frac{1}{\widetilde{n}_1!}\{\widetilde{H}_1,\pi,\dots,\pi\}_{\widetilde{n}_1+1};$$

(ii) *for* $n=2$*:*

$$\begin{aligned}\{(\widetilde{H}_1,\widetilde{\pi}_1),(\widetilde{H}_2,\widetilde{\pi}_2)\}'_2 &= \{H,\pi,\widetilde{\pi}_1,\widetilde{\pi}_2\}_4 + \delta_{\widetilde{n}_1,\geqslant 2}\{\widetilde{H}_1,\pi,\dots,\pi,\widetilde{\pi}_2\}_{\widetilde{n}_1+1}\frac{1}{(\widetilde{n}_1-1)!}\\ &\quad + \delta_{\widetilde{n}_2,\geqslant 2}\{\widetilde{H}_2,\pi,\dots,\pi,\widetilde{\pi}_1\}_{\widetilde{n}_2+1}\frac{(-1)^{(\widetilde{n}_2-1)\widetilde{a}_1}}{(\widetilde{n}_2-1)!};\end{aligned}$$

(iii) *for* $n=3$*:*

$$\begin{aligned}&\{(\widetilde{H}_1,\widetilde{\pi}_1),(\widetilde{H}_2,\widetilde{\pi}_2),(\widetilde{H}_3,\widetilde{\pi}_3)\}'_3\\ &\quad = \{H,\widetilde{\pi}_1,\widetilde{\pi}_2,\widetilde{\pi}_3\}_4 + \delta_{\widetilde{n}_1,\geqslant 3}\{\widetilde{H}_1,\pi,\dots,\pi,\widetilde{\pi}_2,\widetilde{\pi}_3\}_{\widetilde{n}_1+1}\frac{1}{(\widetilde{n}_1-2)!}\\ &\qquad + \delta_{\widetilde{n}_2,\geqslant 3}\{\widetilde{H}_2,\pi,\dots,\pi,\widetilde{\pi}_1,\widetilde{\pi}_3\}_{\widetilde{n}_2+1}\frac{(-1)^{(\widetilde{n}_2-1)\widetilde{a}_1}}{(\widetilde{n}_2-2)!}\\ &\qquad + \delta_{\widetilde{n}_3,\geqslant 3}\{\widetilde{H}_3,\pi,\dots,\pi,\widetilde{\pi}_1,\widetilde{\pi}_2\}_{\widetilde{n}_3+1}\frac{(-1)^{(\widetilde{n}_3-1)(\widetilde{a}_1+\widetilde{a}_2)}}{(\widetilde{n}_3-2)!};\end{aligned}$$

(iv) *for* $n\geqslant 4$*:*

$$\{(\widetilde{H}_1,\widetilde{\pi}_1),\dots,(\widetilde{H}_n,\widetilde{\pi}_n)\}'_n = \sum_{\substack{1\leqslant i\leqslant n\\ \widetilde{n}_i\geqslant n}}\{\widetilde{H}_i,\pi,\dots,\pi,\widetilde{\pi}_1,\dots,\widehat{\widetilde{\pi}}_i,\dots,\widetilde{\pi}_n\}_{\widetilde{n}_i+1}\frac{(-1)^{(\widetilde{n}_i-1)(\widetilde{a}_1+\cdots+\widetilde{a}_{i-1})}}{(\widetilde{n}_i-n+1)!},$$

where $(\widetilde{H}_i,\widetilde{\pi}_i)\in\Omega^{\widetilde{n}_i}(M)\oplus\chi^{\widetilde{a}_i}(M)$*,* δ *is the Kronecker delta (so* $\delta_{a,\geqslant b}=1$ *if* $a\geqslant b$ *and zero otherwise), and* $\widehat{\widetilde{\pi}}_i$ *denotes omission of the element* $\widetilde{\pi}_i$*.*

Proof. By (12), the sought $L_\infty[1]$-algebra is obtained by twisting the $L_\infty[1]$-algebra of Corollary 2.9 by the Maurer–Cartan element $(H[3],\pi[2])$. The nth twisted multibracket is computed as in (11), and we write it as $\{\cdots\}_n+\{\cdots\}'_n$. Notice that of the three types of multibrackets defined in Corollary 2.9, type (a) never appears while computing $\{\cdots\}'_n$, and type (b) appears only when $n=1$. Notice further that type (c) involves exactly one differential form and as many multivector fields as the degree of the form. (This explains for instance why H does not appear in the expression for $\{\cdots\}'_n$ when $n\geqslant 4$.) □

2.3.1 *Equivalences of twisted Poisson structures.* Consider the $L_\infty[1]$-algebra $\mathfrak{L}$ of Corollary 2.9. We make explicit the equivalence relation induced on its set of Maurer–Cartan elements.

Fix $(B,X)\in\mathfrak{L}_{-1}=\Omega^2(M)\oplus\chi(M)$. It defines a vector field $\mathcal{Y}^{(B,X)}$ on $\mathfrak{L}_0=\Omega^3(M)\oplus\chi^2(M)$. By (19) and Corollary 2.9, at the point (H,π) the vector field reads

$$\mathcal{Y}^{(B,X)}|_{(H,\pi)} = (-dB,[X,\pi]+\wedge^2\tilde{\pi}(B-\iota_X H)), \tag{25}$$

where $\wedge^2\tilde{\pi} := \frac{1}{2}(\pi^\sharp\wedge\pi^\sharp)$.

Remark 2.12. The binary bracket on $\mathfrak{L}_{-1}$ reduces to the Lie bracket of vector fields on $\chi(M)$, making $\mathfrak{L}_{-1}$ into a Lie algebra. The assignment $(B,H)\mapsto\mathcal{Y}^{(B,X)}$ is not a Lie algebra morphism. For instance, the bracket of $\mathcal{Y}^{(0,X)}$ and $\mathcal{Y}^{(0,\tilde{X})}$ differs from $\mathcal{Y}^{(0,[X,\tilde{X}])}$ for generic $X,\tilde{X}\in\chi(M)$, as one can check using the proof of Proposition 2.16 below.

For any diffeomorphism ϕ of M, we consider the vector bundle automorphism

$$TM \oplus T^*M, Y + \eta \mapsto \phi_* Y + (\phi^{-1})^* \eta,$$

which by abuse of notation we denote by ϕ_*. For any $B \in \Omega^2(M)$, we consider

$$e^B : TM \oplus T^*M, Y + \eta \mapsto Y + (\eta + \iota_Y B).$$

Recall that the vector bundle $TM \oplus T^*M$ is endowed with a canonical pairing on the fibers given by $\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \frac{1}{2}(\iota_{X_1}\xi_2 + \iota_{X_2}\xi_1)$.

Remark 2.13. The group of vector bundle automorphisms of $TM \oplus T^*M$ preserving the canonical pairing and preserving[8] the canonical projection $TM \oplus T^*M \to TM$ is given exactly by $\{\phi_* e^B : \phi \in \mathrm{Diff}(M), B \in \Omega^2(M)\}$. This follows by the same argument as for [Gua11, Proposition 2.5]. Further, notice that $e^B \phi_* = \phi_* e^{\phi^* B}$.

Abusing notation, for any bivector field π such that $1 + B^\flat \pi^\sharp : T^*M \to T^*M$ is invertible, we denote by $e^B \pi$ the unique bivector field whose graph is $e^B(\mathrm{graph}(\pi))$. Here $B^\flat$ is the contraction in the first component of B.

Consider the connected group $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$ (where the second factor denotes the diffeomorphisms isotopic to the identity), with multiplication

$$(B_1, \phi_1) \cdot (B_2, \phi_2) = (B_1 + (\phi_1^{-1})^* B_2, \phi_1 \circ \phi_2).$$

The partial[9] action of $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$ on $\Omega^3(M) \oplus \chi^2(M)$ by

$$(B, \phi) \cdot (H, \pi) = ((\phi^{-1})^*(H) - dB e^B \phi_* \pi)$$

preserves

$$MC(\mathfrak{L}) = \{(H, \pi) \in \Omega^3_{\mathrm{closed}}(M) \oplus \chi^2(M) : \pi \text{ is an } H\text{-twisted Poisson structure}\}.$$

This can easily be checked using the following two facts. First, for every $H \in H^3_{\mathrm{closed}}(M)$, the isomorphism $e^B \phi_* : TM \oplus T^*M \to TM \oplus T^*M$ maps the H-twisted Courant bracket to the $(\phi^{-1})^*(H) - dB$-twisted Courant bracket [Gua11, § 2.2]. Second, π is an H-twisted Poisson structure if and only if $\mathrm{graph}(\pi)$ is involutive with respect to the H-twisted Courant bracket.

Remark 2.14. The map $e^B \phi_*$ is an isomorphism of Courant algebroids (from the H-twisted to the $(\phi^{-1})^*(H) - dB$-twisted Courant algebroids) covering ϕ, by the above and Remark 2.13. As a consequence, twisted Poisson structures lying in the same orbit of the $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$ action share many properties. For instance, their associated Lie algebroids are isomorphic and, in particular, the underlying (singular) foliations are diffeomorphic.

Notice further that the $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$ action on $MC(\mathfrak{L})$ preserves the cohomology class of closed 3-forms. For instance, the orbit through $(H = 0, \pi = 0)$ is $\{(H', 0) : H' \text{ is exact}\}$.

We will show that the natural equivalence relation on $MC(\mathfrak{L})$ is given by the above $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$ action. To do so, we first need a technical lemma.

[8] In the sense that the projection $TM \oplus T^*M \to TM$ is equivariant with respect to the vector bundle automorphism and the derivative of its base map.

[9] The action is defined whenever $1 + B^\flat(\phi_*\pi)^\sharp$ is invertible.

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Lemma 2.15. *Let X be a vector field on a manifold M with flow ϕ^t defined for $t \in I \subset \mathbb{R}$, let $\{C_t\}_{t\in I}$ be a smooth family of 2-forms, and let π be a bivector field. Denote $\pi_t := (\phi_t)_*(e^{C_t}\pi)$. Then*

$$\frac{d}{dt}\pi_t = [X, \pi_t] + \wedge^2\tilde{\pi}_t\bigg((\phi_{-t})^*\bigg(\frac{d}{dt}C_t\bigg)\bigg). \tag{26}$$

Proof. We have

$$\frac{d}{dt}(e^{C_t}\pi) = \wedge^2\widetilde{(e^{C_t}\pi)}\bigg(\frac{d}{dt}C_t\bigg). \tag{27}$$

This follows from $(e^{C_t}\pi)^\sharp = \pi^\sharp(1 + C_t^\flat\pi^\sharp)^{-1}$ [ŠW01, § 4] and from $(d/dt)(e^{C_t}\pi)^\sharp = -(e^{C_t}\pi)^\sharp((d/dt)C_t)^\flat(e^{C_t}\pi)^\sharp$. Using (27) in the first equality, we obtain

$$\begin{aligned}\frac{d}{dt}\pi_t &= (\phi_t)_*\bigg(\frac{d}{dt}(e^{C_t}\pi) + [X, e^{C_t}\pi]\bigg)\\ &= (\phi_t)_*\bigg(\wedge^2\widetilde{(e^{C_t}\pi)}\bigg(\frac{d}{dt}C_t\bigg)\bigg) + (\phi_t)_*[X, e^{C_t}\pi],\end{aligned}$$

which equals the right-hand side of (26). □

Proposition 2.16. *The leaves of the involutive singular distribution*

$$\mathrm{span}\{\mathcal{Y}^{(B,X)} : (B, X) \in \mathfrak{L}_{-1} = \Omega^2(M) \oplus \chi(M)\} \tag{28}$$

on $MC(\mathfrak{L})$ coincide with the orbits of the partial action of $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$ on $MC(\mathfrak{L})$.

Proof. It suffices to show that (28) coincides with the singular distribution given by the infinitesimal action associated to the group action of $\Omega^2(M) \rtimes \mathrm{Diff}(M)_\circ$. Notice that the Lie algebra of this group is $\Omega^2(M) \oplus \chi(M)$, so take an element $(B, X) \in \Omega^2(M) \oplus \chi(M)$. We compute the corresponding generator of the action $\mathcal{Z}^{(B,X)}$ at a point $(H, \pi) \in MC(\mathfrak{L})$: we have

$$\mathcal{Z}^{(B,X)}|_{(H,\pi)} := \frac{d}{dt}\bigg|_{t=0}(tB, \phi_t)\cdot(H, \pi) = (-d(\iota_X H + B), [X, \pi] + \wedge^2\tilde{\pi}(B)), \tag{29}$$

where ϕ_t is the flow of X and where we use Lemma 2.15 to compute $(d/dt)|_{t=0}(\phi_t)_* e^{(\phi_t)^*(tB)}(\pi)$. Comparing this with (25), we see that

$$\mathcal{Z}^{(B-\iota_X H, X)}|_{(H,\pi)} = \mathcal{Y}^{(B,X)}|_{(H,\pi)}.$$

This shows that the two singular distributions agree at the point (H, π) and, repeating at every point of $MC(\mathfrak{L})$, we conclude that the two singular distributions agree on $MC(\mathfrak{L})$. □

We conclude by describing explicitly the flow on $MC(\mathfrak{L})$ induced by a fixed element (B, X) of $\mathfrak{L}_{-1}$.

Proposition 2.17. *Let $(B, X) \in \Omega^2(M) \oplus \chi(M)$. The integral curve of $\mathcal{Y}^{(B,X)}$ starting at the point $(H, \pi) \in \mathfrak{L}_0$ reads*

$$t \mapsto (H - t dB, (\phi_t)_* e^{C_t^H}\pi), \tag{30}$$

where ϕ denotes the flow of X and

$$C_t^H := D_t + \int_0^t (\phi_s^*)(B - \iota_X H)\, ds$$

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for D_t the unique solution with $D_0 = 0$ of

$$\frac{d}{dt} D_t = t(\phi_t^*)\iota_X \, dB.$$

(The above curve is defined as long as ϕ_t is defined and $1 + (C_t^H)^\flat \pi^\sharp$ is invertible.)

Proof. Fix $(H, \pi) \in \mathfrak{L}_0$ and consider the curve defined in (30). The curve is tangent to the vector field $\mathcal{Y}^{(B,X)}$ at all times t, by virtue of Lemma 2.15 and since

$$(\phi_{-t})^* \left(\frac{d}{dt} C_t^H \right) = (\phi_{-t})^*[t(\phi_t^*)\iota_X \, dB + (\phi_t^*)(B - \iota_X H)] = B - \iota_X (H - t \, dB).$$

Since at time $t = 0$ the curve is located at the point (H, π), we are done. $\square$

Remark 2.18. Let $(B, X) \in \Omega^2(M) \oplus \chi(M)$, where B is closed, and let $(H, \pi) \in \mathfrak{L}_0$. Then $D_t = 0$ and consequently $(\phi_t)_* e^{C_t^H}$ is a one-parameter *group* of orthogonal vector bundle automorphisms of $TM \oplus T^*M$ (see [Gua11, Proposition 2.6]). Hence, the second component of the integral curve of $\mathcal{Y}^{(B,X)}$ starting at (H, π) is the image of (the graph of) π under a one-parameter group of orthogonal vector bundle automorphisms of $TM \oplus T^*M$.

2.4 Generalized complex structures and Courant algebroids

In this subsection, we consider deformations of Courant algebroid structures on a fixed pseudo–Riemannian vector bundle and of their generalized complex structures. Deformations of generalized complex structures within a fixed Courant algebroid were studied by Gualtieri in [Gua11, § 5].

Fix a Courant algebroid $E \to M$ and a generalized almost complex structure J, i.e. a vector bundle map $J : E \to E$ with $J^2 = -\mathrm{Id}$ preserving the fiberwise pairing. The structure J can be equivalently encoded by a complex maximal isotropic subbundle $A \subset E \otimes \mathbb{C}$ transverse to the complex conjugate $\bar{A}$. The correspondence is as follows: given J, define A to be the $+i$-eigenbundle of the complexification of J. Given A, consider the complex endomorphism of $E \otimes \mathbb{C}$ with $+i$-eigenbundle A and $-i$-eigenbundle $\bar{A}$, and define J to be the restriction to E. Further, we have that J is a *generalized complex structure* (i.e. it satisfies a certain integrability condition [Hit03], [Gua11, Definition 3.1]) if and only if A is a *complex Dirac structure.*

Hence, we are in the situation of § 2.2, except that we consider *complex* maximal isotropic subbundles in the complexification $E \otimes \mathbb{C}$ of a (real) Courant algebroid. Notice that E does not have a preferred splitting into maximal isotropic subbundles. On the other hand, $E \otimes \mathbb{C}$ is a complex Courant algebroid with a splitting $E \otimes \mathbb{C} = A \oplus \bar{A}$ into complex maximal isotropic subbundles. The construction of [Roy02a, Theorem 4.5] leads to a *complex* graded manifold[10] with a degree 2 symplectic structure $\{\cdot, \cdot\}$, namely $\mathcal{N} = T^*[2]A[1]$. We denote its 'global functions', a graded commutative algebra over $\mathbb{C}$, by $C_{\mathbb{C}}(\mathcal{N})$.

Lemma 2.19. *Fix a Courant algebroid $E \to M$ and a generalized almost complex structure J, encoded by a complex maximal isotropic subbundle A transverse to $\bar{A}$. The following quadruple forms a curved V-data:*

- *the complex graded Lie algebra $L := C_{\mathbb{C}}(\mathcal{N})[2]$ with Lie bracket $\{\cdot, \cdot\}$;*
- *its complex abelian subalgebra $\mathfrak{a} := \mathrm{pr}^*(C_{\mathbb{C}}(A[1]))[2] \cong \Gamma(\wedge A^*)[2]$;*

[10] It is given by a sheaf of graded commutative algebras over $\mathbb{C}$ satisfying the usual locally triviality condition.

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- *the natural projection $P : L \to \mathfrak{a}$ given by evaluation on the base $A[1]$;*
- *$\Delta = -\varphi + h_{d_A} + F^*(h_{d_{A^*}}) - \psi$, defined analogously to § 2.2;*

hence, by Theorem 1, we obtain a complex[11] *curved $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$.*

For all $\Phi \in \Gamma(\wedge^2 A^*)$, we have that $\Phi[2]$ is a Maurer–Cartan element in $\mathfrak{a}^P_\Delta$ if and only if

$$\mathrm{graph}(-\Phi) := \{(X - \iota_X \Phi) : X \in A\} \subset A \oplus \bar{A} = E \otimes \mathbb{C}$$

is a complex Dirac structure in $E \otimes \mathbb{C}$.

Further, the above quadruple forms a V-data if and only if J is a generalized complex structure.

Proof. This is obtained exactly as in the proof of Lemma 2.6, but working over $\mathbb{C}$ and taking $K := \bar{A}$. □

As in § 2.2, let $\mathcal{M}$ be the (real) degree 2 symplectic manifold with self-commuting degree 3 function Δ corresponding to the Courant algebroid E. We have $C_{\mathbb{C}}(\mathcal{N}) = C(\mathcal{M}) \otimes \mathbb{C}$. Since Δ defines a complex Courant algebroid structure on $E \otimes \mathbb{C}$ which is the complexification of a (real) Courant algebroid structure on E, it follows that $\Delta \in C(\mathcal{M}) \subset C_{\mathbb{C}}(\mathcal{N})$. We are interested only in complex Courant algebroid structures on $E \otimes \mathbb{C}$ which are complexifications of Courant algebroid structures on E, so we deform Δ only within $C(\mathcal{M})$.

Corollary 2.20. *Fix a Courant algebroid $E \to M$ and a generalized complex structure J, encoded by a complex Dirac structure A. Let $\mathcal{M}$ and the V-data $(L, \mathfrak{a}, P, \Delta)$ be as in Lemma 2.19. Then there exists a (real) $L_\infty[1]$-algebra structure on $(C(\mathcal{M})[2])[1] \oplus \mathfrak{a}$ with the property that for all $\tilde{\Delta} \in C(\mathcal{M})_3$ and small enough $\tilde{\Phi} \in \Gamma(\wedge^2 A^*)$:*

$$\begin{cases} \Delta + \tilde{\Delta} \text{ defines a Courant algebroid structure on } E;\\ \mathrm{graph}(-\tilde{\Phi}) \text{ is the } +i\text{-eigenbundle of a generalized complex structure there} \end{cases}$$
$$\Leftrightarrow (\tilde{\Delta}[3], \tilde{\Phi}[2]) \text{ is a Maurer–Cartan element of } (C(\mathcal{M})[2])[1] \oplus \mathfrak{a}.$$

Proof. Apply Theorem 3 (which holds over $\mathbb{C}$ as well) with $\Phi = 0$ to obtain the complex $L_\infty[1]$-structure $(L[1] \oplus \mathfrak{a})^P_\Delta$. View the latter as a real $L_\infty[1]$-structure. Since $\Delta \in C(\mathcal{M})[2]$, it follows that $(C(\mathcal{M})[2])[1] \oplus \mathfrak{a}$ is an $L_\infty[1]$-subalgebra. Use Lemma 2.19 to phrase the conclusions of Theorem 3 in terms of Courant algebroids and generalized complex structures. □

Remark 2.21. To see that the above V-data is filtered, proceed exactly as in Remark 2.8.

2.5 Deformations of complex structures

In this subsection, we study a special case of the situation considered in § 2.4: we study deformations of a complex structure on M to generalized complex structures in the H-twisted Courant algebroids $TM \oplus T^*M$, where H ranges through all (real) closed 3-forms. Deformations of a complex structure within a fixed Courant algebroid (the standard one) were studied by Gualtieri in [Gua11, § 5.3].

Fix a complex structure I on a manifold M. It gives rise to a generalized complex structure $J_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}$ for the standard Courant algebroid $TM \oplus T^*M$, whose $+i$-eigenbundle is the complex Dirac subbundle $A := T_{0,1} \oplus T^*_{1,0}$ [Gua11, § 3]. Here $T_{1,0}$ and $T_{0,1}$ denote the holomorphic and anti-holomorphic tangent bundles of (M, I), respectively.

[11] Hence, the underlying graded vector space is complex and the multibrackets are $\mathbb{C}$-linear.

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Recall that $\Omega^k(M,\mathbb{R}) = \bigoplus_{p,q\geqslant 0, p+q=k} \Omega^{p,q}(M,\mathbb{R})$.

Consider $\Gamma(\wedge A^*) = \bigoplus_{r\geqslant 0, s\geqslant 0} \Omega^{0,r}(M, \wedge^s T_{1,0})$. Fix $p \geqslant 1$ and $\Theta_i \in \Omega^{0,r_i}(M, \wedge^{s_i} T_{1,0})$ with $s_i \geqslant 1$ (for $i = 1, \ldots, p$). We define a map

$$(\Theta_1^\sharp \wedge \cdots \wedge \Theta_p^\sharp) : \Omega^{p,q}(M,\mathbb{R}) \to \Omega^{0,q+\sum_i r_i}(M, \wedge^{-p+\sum_i s_i} T_{1,0})$$

similarly to the map defined at the beginning of § 2.3, but taking into account the differential form part of the Θ_i, which simply gets wedge-multiplied. More precisely, assume that $\Theta_i = \omega_i \otimes \pi_i$ with $\omega_i \in \Omega^{0,r_i}(M,\mathbb{C}), \pi_i \in \Gamma(\wedge^{s_i} T_{1,0})$ and let $\alpha \in \Omega^{p,0}(M,\mathbb{C})$ and $\sigma \in \Omega^{0,q}(M,\mathbb{C})$. Then the above map is given by

$$\alpha \wedge \sigma \mapsto \pm \sigma \wedge \omega_1 \wedge \cdots \wedge \omega_p \otimes ((\pi_1^\sharp \wedge \cdots \wedge \pi_p^\sharp)\alpha),$$

where the last expression on the right-hand side was defined at the beginning of § 2.3, and the sign $\pm$ is the parity of $pq + \sum_{i=1}^p (p + s_1 + \cdots + s_{i-1}) r_i$ (using the convention $s_0 := 0$).

Further, we define $[\Theta_1, \Theta_2] := (-1)^{s_1 r_2} \omega_1 \wedge \omega_2 \otimes [\pi_1, \pi_2]$.

Corollary 2.22. *Let (M, I) be a complex manifold. There is an $L_\infty[1]$-algebra structure on*

$$\mathfrak{C} := \Omega(M,\mathbb{R})[3] \oplus \bigoplus_{r\geqslant 0, s\geqslant 0} \Omega^{0,r}(M, \wedge^s T_{1,0})[2]$$

whose only non-vanishing multibrackets (up to permutations of the entries) are:

(a) *the differential, which maps (H, Θ) to $(-dH, \bar\partial\Theta + H^{0,p})$, where, when $H \in \Omega^p(M,\mathbb{R})$, $H^{0,p}$ denotes the component of H lying in $\Omega^{0,p}(M,\mathbb{R})$;*

(b) *$\{\Theta_1, \Theta_2\} = (-1)^{r_1+r_2+s_1+1}[\Theta_1, \Theta_2]$ for $\Theta_i \in \Omega^{0,r_i}(M, \wedge^{s_i} T_{1,0})$;*

(c) *$\{H, \Theta_1, \ldots, \Theta_p\} = (-1)^{\sum_{i=1}^p s_i(p-i)} (\Theta_1^\sharp \wedge \cdots \wedge \Theta_p^\sharp) H$ for all $p \geqslant 1$, where $H \in \Omega^{p,q}(M,\mathbb{R})$ and $\Theta_i \in \Omega^{0,r_i}(M, \wedge^{s_i} T_{1,0})$ with $s_i \geqslant 1$.*

Its Maurer–Cartan elements are exactly pairs $(H[3], \Theta[2])$, where

$$H \in \Omega^3(M,\mathbb{R}), \quad \Theta \in \Omega^{0,2}(M,\mathbb{C}) \oplus \Omega^{0,1}(M, T_{1,0}) \oplus \Gamma(\wedge^2 T_{1,0})$$

satisfy $dH = 0$ and $-\Theta$ defines a deformation of J_I to a $-H$-twisted generalized complex structure.

Remark 2.23. (1) The graded vector space $\mathfrak{C}$ is concentrated in degrees $\{-3, \ldots, \dim_{\mathbb{R}}(M) - 2\}$, and its degree i component is $\Omega^{i+3}(M,\mathbb{R}) \oplus \bigoplus \Omega^{0,r}(M, \wedge^s T_{1,0})$ for $r + s = i + 2$.

(2) We make precise the meaning of '$-\Theta$ defines a deformation of J_I to a $-H$-twisted generalized complex structure': it means that $\mathrm{graph}(-\Theta) \subset A \oplus \bar A = (TM \oplus T^*M) \otimes \mathbb{C}$ is the $+i$-eigenbundle of a generalized complex structure in the Courant algebroid $(TM \oplus T^*M, [\![\cdot,\cdot]\!]_{-H})$ (the Courant bracket twisted by $-H$ was defined in § 2.3). For instance, if $\Theta = B \in \Omega^{0,2}(M,\mathbb{C})$, then $\mathrm{graph}(-\Theta) = \{X + \xi - \iota_X B : X \in T_{0,1}, \xi \in T^*_{1,0}\}$.

We make more explicit the Maurer–Cartan condition for the $L_\infty[1]$-algebra of Corollary 2.22. The pair (H, Θ) is a Maurer–Cartan element if $dH = 0$ and the following equation of order four is satisfied:

$$\bar\partial\Theta + H^{0,3} \pm \tfrac{1}{2}[\Theta, \Theta] + \Theta^\sharp H^{1,2} \pm \tfrac{1}{2}(\Theta^\sharp \wedge \Theta^\sharp) H^{2,1} \pm \tfrac{1}{6}(\Theta^\sharp \wedge \Theta^\sharp \wedge \Theta^\sharp) H^{3,0} = 0, \tag{31}$$

where the signs $\pm$ depend on Θ.

We spell out three special cases. When $\Theta = B \in \Omega^{0,2}(M,\mathbb{C})$, (31) is equivalent to

$$\bar\partial B + H^{0,3} = 0 \in \Omega^{0,3}(M,\mathbb{C}).$$

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When $\Theta = \varphi \in \Omega^{0,1}(M, T_{1,0})$, decomposing the left-hand side of (31) according to bi-degrees, we see that (31) is equivalent to

$$\bar\partial\varphi + \tfrac{1}{2}[\varphi,\varphi] = 0 \in \Omega^{0,2}(M, T_{1,0}),$$
$$H^{0,3} + \varphi^\sharp H^{1,2} - \tfrac{1}{2}(\varphi^\sharp \wedge \varphi^\sharp)H^{2,1} - \tfrac{1}{6}(\varphi^\sharp \wedge \varphi^\sharp \wedge \varphi^\sharp)H^{3,0} = 0 \in \Omega^{0,3}(M, \mathbb{C}).$$

The first equation states that $-\varphi$ defines a deformation of I to an (integrable) complex structure $I_{-\varphi}$. The second condition is equivalent to H being of type $(2,1)+(1,2)$ with respect to $I_{-\varphi}$. This is not surprising, since, for any closed $H' \in \Omega^3(M, \mathbb{R})$, a complex structure defines an H'-twisted generalized complex structure if and only if H' is of type $(2,1)+(1,2)$ [Gua11, Example 2.14]. (To see the second condition, first verify that the evaluation of H on three vectors of the form $X - \varphi(X)$ vanishes for $X \in T_{0,1}$. Hence, the component of H of type $(0,3)$ with respect to $I_{-\varphi}$ vanishes. Then use that H is real, to conclude that H is of type $(2,1)+(1,2)$ with respect to $I_{-\varphi}$.)

The most interesting case is when $\Theta = \beta \in \Gamma(\wedge^2 T_{1,0})$. In that case, (31) is equivalent to

$$[\beta,\beta] = 0 \in \Gamma(\wedge^3 T_{1,0}),$$
$$\bar\partial\beta + \tfrac{1}{2}(\beta^\sharp \wedge \beta^\sharp)H^{2,1} = 0 \in \Omega^{0,1}(M, \wedge^2 T_{1,0}),$$
$$\beta^\sharp H^{1,2} = 0 \in \Omega^{0,2}(M, T_{1,0}),$$
$$H^{0,3} = 0 \in \Omega^{0,3}(M, \mathbb{C}).$$

(Here we used $H^{3,0} = \overline{H^{0,3}} = 0$.) By the first equation, β is a Poisson bivector field; however, it is not holomorphic in general due to the second equation. The form H is of type $(2,1)+(1,2)$ by the fourth equation.

We do not discuss here the equivalences on the set of Maurer–Cartan elements of $\mathfrak{C}$. We just point out that they are induced by elements of $\Omega^2(M,\mathbb{R})[3] \oplus \Gamma(T_{1,0})[2] \oplus \Omega^{0,1}(M,\mathbb{C})[2]$.

Proof of Corollary 2.22. Apply Corollary 2.20 to the standard Courant algebroid $TM \oplus T^*M$ and to the generalized complex structure J_I (i.e. to $A = T_{0,1} \oplus T^*_{1,0}$). It delivers an $L_\infty[1]$-algebra structure on $(C(\mathcal{M})[2])[1] \oplus \mathfrak{a}$ governing deformations of the Courant algebroid and of generalized complex structures. Recall that, given $H \in \Omega^3(M)$, the degree 3 function $\Delta + H$ on $\mathcal{M}$ defines a Courant algebroid structure on $TM \oplus T^*M$ if and only if H is closed, and in this case it induces the $(-H)$-twisted Courant bracket [Roy02a, § 4], [Zam12, § 8].

To conclude the proof, we just need to show that $\Omega(M,\mathbb{R})[3] \oplus \mathfrak{a}$ is an $L_\infty[1]$-subalgebra of $(C(\mathcal{M})[2])[1] \oplus \mathfrak{a}$, and that the restricted multibrackets are those given in the statement.

We use the following notation for the canonical local coordinates on $\mathcal{N} = T^*[2]A[1] = T^*[2](T_{0,1} \oplus T^*_{1,0})[1]$ (i.e. for local generators of $C_{\mathbb{C}}(\mathcal{N}) = C(T^*[2]T[1]M) \otimes \mathbb{C}$). For $j = 1, \ldots, \dim_{\mathbb{C}}(M)$, we denote by z_j complex local coordinates on M, by $\bar z_j$ the conjugate coordinates, by p_j the canonical coordinates on the fibers of $T^*_{1,0}$, and by $\bar v_j$ those on the fibers of $T_{0,1}$ (so the degrees are $|z_j| = |\bar z_j| = 0, |p_j| = |\bar v_j| = 1$). By $P_j, \bar P_j, v_j, \bar p_j$, we denote the coordinates on the fibers of $T^*[2]A[1] \to A[1]$ conjugate to $z_j, \bar z_j, p_j, \bar v_j$, respectively (their degrees are $|P_j| = |\bar P_j| = 2, |v_j| = |\bar p_j| = 1$).

The quadruple listed in Lemma 2.19 reads:

- $L = C_{\mathbb{C}}(T^*[2]A[1])[2]$, whose Lie bracket we denote by $\{\cdot\,,\cdot\}$;
- $\mathfrak{a} = C_{\mathbb{C}}(A[1])[2] \cong \Gamma(\wedge A^*)[2]$;
- the natural projection $P : L \to \mathfrak{a}$ given by evaluation on the base $A[1]$;
- $\Delta = \sum_i P_i v_i + \sum_i \bar P_i \bar v_i$.

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Notice that Δ is given essentially by the de Rham differential $d = \partial + \bar{\partial}$. The multibrackets of the $L_\infty[1]$-algebra $(L[1] \oplus \mathfrak{a})^P_\Delta$ are given in Theorem 2. Clearly, $\Omega(M, \mathbb{R})[2]$ is a Lie subalgebra of L (for it is abelian), and further it is closed under $\{\Delta, \cdot\}$ since the latter acts as the de Rham differential. By Remark 1.10, it follows that $\mathfrak{C} = \Omega(M, \mathbb{R})[3] \oplus \mathfrak{a}$ is an $L_\infty[1]$-subalgebra of $(C(\mathcal{M})[2])[1] \oplus \mathfrak{a} \subset (L[1] \oplus \mathfrak{a})^P_\Delta$.

The restriction of the multibrackets to $\mathfrak{C}$ is the one described in the statement of this corollary, as one computes in coordinates: (a) is obtained from (5), (b) from (8), and (c) from (7). □

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