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Generating Some Symmetric Semi-classical Orthogonal Polynomials

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Abstract. We show that if v is a regular semi-classical form (linear functional), then the symmetric form u defined by the relation $x^2\sigma u = -\lambda v$, where $(\sigma f)(x) = f(x^2)$ and the odd moments of u are 0, is also regular and semi-classical form for every complex λ except for a discrete set of numbers depending on v. We give explicitly the three-term recurrence relation and the structure relation coefficients of the orthogonal polynomials sequence associated with u and the class of the form u knowing that of v. We conclude with an illustrative example.

1 Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$ and by $S(v)(z)$ the formal Stieltjes function of v defined by

$$S(v)(z) = -\sum_{n\geq 0} \frac{(v)_n}{z^{n+1}}, \tag{1.1}$$

where $(v)_n = \langle v, x^n\rangle$, $n \geq 0$, are the moments of v.

For any form v, any polynomial h and $c \in \mathbb{C}$ let $Dv = v'$, hv, δ_c and $(x-c)^{-1}v$ be the forms defined by:

$$\langle v', f\rangle := -\langle v, f'\rangle, \qquad \langle hv, f\rangle := \langle v, hf\rangle,$$
$$\langle \delta_c, f\rangle := f(c), \qquad \langle (x-c)^{-1}v, f\rangle := \langle v, \theta_c f\rangle,$$

where $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$, $f \in \mathcal{P}$.

Then it is straightforward to prove that for $v \in \mathcal{P}'$, we have [17]

$$x(x^{-1}v) = v, \tag{1.2}$$

$$x^{-1}(xv) = v - (v)_0\delta_0, \tag{1.3}$$

$$x^{-2}(x^2v) = v - (v)_0\delta_0 + (v)_1\delta_0'. \tag{1.4}$$

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Let us define the operator $\sigma\colon \mathcal{P} \to \mathcal{P}$ by $(\sigma f)(x) = f(x^2)$. Then we define the even part, σv, by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$. Therefore, we have [6, 16]

$$f(x)(\sigma v) = \sigma(f(x^2)v),$$
$$\sigma v' = 2(\sigma x v)'.$$

A form v is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ with $\deg S_n = n$, $n \geq 0$, such that

$$\langle v, S_n S_m\rangle = r_n \delta_{n,m}\ , \quad r_n \neq 0, \quad n \geq 0.$$

We can always assume that each S_n is monic, *i.e.*, $S_n(x) = x^n$+lower degree terms. Then the sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v (monic orthogonal polynomial sequence (MOPS) in short). It is a very well known fact (see, for instance, the monograph by Chihara [6]) that the sequence $\{S_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$\begin{gathered} S_{n+2}(x) = (x - \xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x), \quad n \geq 0, \\ S_1(x) = x - \xi_0, \qquad S_0(x) = 1. \end{gathered} \tag{1.5}$$

with $(\xi_n, \rho_{n+1}) \in \mathbb{C} \times (\mathbb{C} - \{0\})$, $n \geq 0$. By convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation

$$\begin{gathered} S_{n+2}^{(1)}(x) = (x - \xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x), \quad n \geq 0, \\ S_1^{(1)}(x) = x - \xi_1, \quad S_0^{(1)}(x) = 1, \quad \big(S_{-1}^{(1)}(x) = 0\big). \end{gathered} \tag{1.6}$$

Another important representation of $S_n^{(1)}(x)$ is, (see [7])

$$S_n^{(1)}(x) := \Big\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x - \zeta}\Big\rangle = (v\theta_0 S_{n+1})(x),$$

where the right-multiplication of a form by a polynomial is defined by

$$(vf)(x) := \Big\langle v, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta}\Big\rangle = \sum_{m=0}^{n}\Big(\sum_{j=m}^{n} a_j (v)_{j-m}\Big)x^m, \quad f(x) = \sum_{j=0}^{n} a_j x^j.$$

Also, let $\{S_n(\cdot\,, \mu)\}_{n\geq 0}$ be co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [6]

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)}, \quad n \geq 0. \tag{1.7}$$

We recall that a form v is called symmetric if $(v)_{2n+1} = 0, n \geq 0$. The conditions $(v)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding MOPS $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (1.5) with $\xi_n = 0, n \geq 0$ [6].

Let us recall that a form v is called semi-classical of class s when it is regular and there exist two polynomials Φ, a monic polynomial, and Ψ, $\deg(\Psi) \leq 1$, such that

$$\big(\Phi(x)v\big)' + \Psi(x)v = 0,$$

where

$$\prod_{c\in\mathcal{Z}} (|\Phi'(c) + \Psi(c)| + |\langle v, \theta_c^2\Phi + \theta_c\Psi\rangle|) \neq 0.$$

Here, $\mathcal{Z}$ denotes the set of zeros of Φ and $s = \max(\deg(\Phi) - 2, \deg(\Psi) - 1)$.

The corresponding MOPS $\{S_n\}_{n\geq 0}$ is said to be semi-classical of class s. When $s = 0$, v is a classical form (Hermite, Laguerre, Jacobi, or Bessel).

The semi-classical forms are a natural generalization of the classical forms. Since the system corresponding to the problem of determining all the semi-classical forms of class $s \geq 1$ becomes non-linear, the problem can only be solved when the class $s = 1$ and for some particular cases [1, 5, 12]. Thus, several authors use different processes in order to obtain semi-classical forms of class $s \geq 1$. For instance, we can mention the adjunction of a finite number of Dirac masses and their derivatives to classical forms [3, 9–11] and the product and the division of a form by a polynomial [2, 4, 8, 14, 18, 19]. So, some examples of semi-classical forms are given in terms of classical ones.

The whole idea of this paper is to build a new construction process of a semi-classical form, which has not yet been treated in the literature on semi-classical polynomials.

We study the form u related to a semi-classical form v by

$$x^2\sigma u = -\lambda v, \quad \lambda \neq 0, \quad \sigma(xu) = 0.$$

The structure of the paper is as follows. In Section 2, an explicit, necessary, and sufficient condition for the regularity of the new form is given. We will also give the coefficients of the three-term recurrence relation satisfied by the new family of orthogonal polynomials. In the third section, we compute the exact class of the semi-classical form obtained by the above modification, and the structure relation of the orthogonal polynomials sequence relative to the form u will follow. In the final section, we give a detailed study of an example. The regular form found in the example is semi-classical of class four at most.

2 Algebraic Properties

Let v be a regular, normalized form (*i.e.*, $(v)_0 = 1$) and let $\{S_n\}_{n\geq 0}$ be the corresponding MOPS. For a $\lambda \in \mathbb{C} - \{0\}$, we can define a new symmetric form u as follows:

$$x^2\sigma u = -\lambda v, \quad \sigma x u = 0, \quad (u)_2 = (u)_0 = 1. \tag{2.1}$$

From (1.2)–(1.4), we have

$$x\sigma u = -\lambda x^{-1} v + \delta_0, \tag{2.2}$$

$$\sigma u = -\lambda x^{-2} v + \delta_0 - \delta_0'. \tag{2.3}$$

Proposition 2.1 *The form u is regular if and only if $\Delta_n S_n(0,\lambda) \neq 0$, $n \geq 0$, where*

$$\Delta_n = \tau_n\Big(\lambda + \sum_{\nu=0}^{n} \frac{S_\nu^2(0,\lambda)}{\tau_\nu}\Big), \quad n \geq 0, \tag{2.4}$$

$$\tau_n = \langle v, S_n^2\rangle = \prod_{\nu=0}^{n} \rho_\nu, \quad n \geq 0. \tag{2.5}$$

For the proof, we need the following lemma.

Lemma 2.2 ([6, 16]) *If the form v is symmetric, then v is regular if and only if σv and $x\sigma v$ are both regular.*

Proof of Proposition 2.1 Let u be given by (2.1). Since u is a symmetric form, according to Lemma 2.2, u is regular if and only if $x\sigma u$ and σu are regular. But $x\sigma u = -\lambda x^{-1}v + \delta_0$ is regular if and only if $S_n(0,\lambda) \neq 0, n \geq 0$ (see [18]). So u is regular if and only if $S_n(0,\lambda) \neq 0$ and $\sigma u = -\lambda x^{-2}v + \delta_0 - \delta_0'$ is regular. Or, it was shown in [4] that the form $-\lambda x^{-2}v + \delta_0 - \delta_0'$ is regular if and only if $\Delta_n \neq 0$, $n \geq 0$. Then we deduce the desired result. ∎

If u is regular, let $\{Z_n\}_{n\geq 0}$ be its corresponding sequence of polynomials satisfying the three-term recurrence relation

$$Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1}Z_n(x), \quad n \geq 0, \qquad Z_1(x) = x, \quad Z_0(x) = 1. \tag{2.6}$$

Since $\{Z_n\}_{n\geq 0}$ is symmetric, let us consider its quadratic decomposition (see [16])

$$Z_{2n}(x) = P_n(x^2), \quad Z_{2n+1}(x) = xR_n(x^2). \tag{2.7}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to σu and $x\sigma u$.

We have, for instance,

$$P_{n+2}(x) = (x - \gamma_{2n+2} - \gamma_{2n+3})P_{n+1}(x) - \gamma_{2n+1}\gamma_{2n+2}P_n, \quad n \geq 0, \qquad P_1(x) = x - \gamma_1, \quad P_0(x) = 1,$$

and

$$R_{n+2}(x) = (x - \gamma_{2n+3} - \gamma_{2n+4})R_{n+1}(x) - \gamma_{2n+2}\gamma_{2n+3}R_n(x), \quad n \geq 0, \qquad R_1(x) = x - \gamma_1 - \gamma_2, \quad R_0(x) = 1.$$

Remarks (i) If w is the symmetrized form associated with the form v (*i.e.*, $(w)_{2n} = (v)_n$ and $(w)_{2n+1} = 0, n \geq 0$), then (2.1) is equivalent to $x^4u = -\lambda w$. Notice that w is not necessarily a regular form in the problem under study. In [13], the authors have solved it only when w is regular.

(ii) From (2.2)–(2.3), we have ([4, 18])

$$R_{n+1}(x) = S_{n+1}(x) + a_nS_n(x), \quad n \geq 0, \tag{2.8}$$

$$P_{n+2}(x) = S_{n+2}(x) + c_{n+1}S_{n+1}(x) + b_nS_n(x), \quad n \geq 0, \qquad P_1(x) = S_1(x) + c_0S_0(x), \tag{2.9}$$

where

$$a_n = -\frac{S_{n+1}(0,\lambda)}{S_n(0,\lambda)}, \quad n \geq 0, \tag{2.10}$$

$$b_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad n \geq 0, \tag{2.11}$$

$$c_0 = \xi_0 - 1, \quad c_{n+1} = \xi_{n+1} - \frac{S_{n+1}(0,\lambda)S_n(0,\lambda)}{\Delta_n}, \quad n \geq 0,$$

and also we have ([18])

$$\xi_{n+1} + a_n - a_{n+1} = \gamma_{2n+3} + \gamma_{2n+4}, a_n(\xi_n - a_n) = \gamma_{2n+2}\gamma_{2n+3}, \quad n \geq 0. \tag{2.12}$$

(iii) From (2.7) and (2.8)–(2.9), we get

$$\begin{aligned} Z_{2n+3}(x) &= xS_{n+1}(x^2) + a_n x S_n(x^2), \quad n \geq 0, \\ Z_{2n+4}(x) &= S_{n+2}(x^2) + c_{n+1}S_{n+1}(x^2) + b_n S_n(x^2), n \geq 0, \\ Z_2(x) &= S_1(x^2) + c_0 S_0(x^2). \end{aligned}$$

Proposition 2.3 *We may write*

$$\begin{cases} \gamma_1 = 1, \quad \gamma_3 = \lambda \dfrac{a_0}{\lambda+1}, \quad \gamma_{2n+5} = \rho_{n+1}\dfrac{a_{n+1}}{b_n}, \\ \qquad \gamma_2 = -\lambda - 1, \quad \gamma_{2n+4} = \dfrac{b_n}{a_n}, \quad n \geq 0. \end{cases} \tag{2.13}$$

For the proof, we use the following lemma.

Lemma 2.4

$$\langle u, xZ_1(x)\rangle = 1, \qquad \langle u, x^{2n+3}Z_{2n+3}(x)\rangle = -\lambda\tau_n a_n, \quad n \geq 0, \tag{2.14}$$

$$\langle u, x^2 Z_2(x)\rangle = -\lambda - 1, \quad \langle u, x^{2n+4}Z_{2n+4}(x)\rangle = -\lambda\tau_n b_n, \quad n \geq 0. \tag{2.15}$$

Proof From (2.7), we have

$$\langle u, x^{2n+3}Z_{2n+3}(x)\rangle = \langle x^2\sigma u, x^n R_{n+1}(x)\rangle, \quad n \geq 0.$$

Taking (2.1), (2.5), and relation (2.8) into account, we get (2.14).

By applying the same process as we did to obtain (2.14) and using (2.7) and (2.9), we can get (2.15). Finally, $\langle u, x^2 Z_2(x)\rangle = (u)_4 - (u)_2 = -\lambda - 1$. ∎

Proof of Proposition 2.3 By (2.6) and the orthogonality of $\{Z_n\}_{n\geq 0}$, we get

$$\gamma_{2n+2} = \frac{\langle u, x^{2n+2}Z_{2n+2}(x)\rangle}{\langle u, x^{2n+1}Z_{2n+1}(x)\rangle} \quad \text{and} \quad \gamma_{2n+1} = \frac{\langle u, x^{2n+1}Z_{2n+1}(x)\rangle}{\langle u, x^{2n}Z_{2n}(x)\rangle}, \quad n \geq 0.$$

Then, from (2.14)–(2.15) and the above relations, we get (2.13). ∎

Corollary 2.5 *When the form v is symmetric, then u is regular if and only if*

$$\lambda\Delta_{2n}\Delta_{2n+1} \neq 0, \quad n \geq 0,$$

with

$$\Delta_{2n} = \tau_{2n}(\lambda^2\varpi_{n-1} + \lambda + 1 + \omega_{n-1}), \quad n \geq 0, \tag{2.16}$$

$$\Delta_{2n+1} = \tau_{2n+1}(\lambda^2\varpi_n + \lambda + 1 + \omega_{n-1}), \quad n \geq 0, \tag{2.17}$$

$$\omega_n = \sum_{\nu=0}^{n} \frac{\tau_\nu^o}{\tau_{\nu+1}^e}, \quad \varpi_n = \sum_{\nu=0}^{n} \frac{\tau_\nu^e}{\tau_\nu^o}, n \geq 0, \quad \omega_{-1} = \varpi_{-1} = 0,$$

$$\tau_n^e = \prod_{\mu=0}^{n} \rho_{2\mu}, \quad \tau_n^o = \prod_{\mu=0}^{n} \rho_{2\mu+1}, \quad n \geq 0.$$

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Moreover, for $n \geq 0$,

$$(2.18)\quad \begin{cases} \gamma_1 = 1, \quad \gamma_2 = -\lambda - 1, \quad \gamma_3 = \dfrac{\lambda^2}{\lambda+1}, \quad \gamma_4 = \dfrac{\lambda^2 + \rho_1(\lambda+1)}{\lambda(\lambda+1)}, \\ \gamma_{4n+5} = -\lambda^{-1} \dfrac{\tau_n^o}{\tau_n^e} \dfrac{\varpi_{n-1}\lambda^2 + \lambda + 1 + \omega_{n-1}}{\varpi_n \lambda^2 + \lambda + 1 + \omega_{n-1}}, \\ \gamma_{4n+6} = -\lambda \dfrac{\tau_{n+1}^e}{\tau_n^o} \dfrac{\varpi_n \lambda^2 + \lambda + 1 + \omega_n}{\varpi_n \lambda^2 + \lambda + 1 + \omega_{n-1}}, \\ \gamma_{4n+7} = \lambda \dfrac{\tau_{n+1}^e}{\tau_n^o} \dfrac{\varpi_n \lambda^2 + \lambda + 1 + \omega_{n-1}}{\varpi_n \lambda^2 + \lambda + 1 + \omega_n}, \\ \gamma_{4n+8} = \lambda^{-1} \dfrac{\tau_{n+1}^o}{\tau_{n+1}^e} \dfrac{\varpi_{n+1}\lambda^2 + \lambda + 1 + \omega_n}{\varpi_n \lambda^2 + \lambda + 1 + \omega_n}. \end{cases}$$

Proof Taking into account (1.5)–(1.6), with $\xi_n = 0$, we get $S_{n+2}(0) = -\rho_{n+1} S_n(0)$ and $S^{(1)}_{n+2}(0) = -\rho_{n+2} S^{(1)}_n(0)$. Then

$$(2.19)\quad S_{2n+1}(0) = 0, \quad S_{2n+2}(0) = (-1)^{n+1} \prod_{\nu=0}^{n} \rho_{2\nu+1}, \quad n \geq 0,$$

$$(2.20)\quad S^{(1)}_{2n+1}(0) = 0, \quad S^{(1)}_{2n}(0) = (-1)^n \prod_{\nu=0}^{n} \rho_{2\nu}, \quad n \geq 0.$$

Therefore $S_{2n+1}(0,\lambda) = -\lambda S^{(1)}_{2n}(0) \neq 0$ and $S_{2n+2}(0,\lambda) = S_{2n+2}(0) \neq 0$. Hence, from (2.19)–(2.20) and (2.4) we get (2.16)–(2.17).

By virtue of (2.16)–(2.17), (2.13) becomes (2.18). ∎

3 The Semi-classical Case

Definition 3.1 ([17]) The form v is called *semi-classical* when it is regular and its formal Stieltjes function satisfies the Riccati equation

$$(3.1)\quad \Phi(z) S'(v)(z) = C_0(z) S(v)(z) + D_0(z),$$

where Φ monic, C_0, and D_0 are polynomials .

It was shown in [17] that equation (3.1) is equivalent to

$$(3.2)\quad \big(\Phi(x)v\big)' + \Psi(x)v = 0.$$

with $\Psi(x) = -\Phi'(x) - C_0(x)$.

Proposition 3.2 ([15]) *Define* $r = \deg(\Phi)$ *and* $p = \deg(\Psi)$. *The semi-classical form* v *satisfying* (3.2) *is of class* $s = \max(r-2, p-1)$ *if and only if*

$$(3.3)\quad \prod_{c \in \mathcal{Z}} \big(|\Phi'(c) + \Psi(c)| + |\langle v, \theta_c^2 \Phi + \theta_c \Psi \rangle| \big) \neq 0,$$

where $\mathcal{Z}$ *denotes the set of zeros of* Φ.

Corollary 3.3 ([1]) *The form v satisfying (3.1) is of class s if and only if*

$$\prod_{c \in \mathcal{Z}} \big(|C_0(c)| + |D_0(c)| \big) \neq 0. \tag{3.4}$$

In the sequel, the form v will be assumed to be semi-classical of class s satisfying (3.1) and (3.2).

Proposition 3.4 *If v is a semi-classical form and satisfies (3.1), then the form u defined by (2.1) is semi-classical. It satisfies*

$$\widetilde{\Phi}(z) S'(u)(z) = \widetilde{C}_0(z) S\big(u\big)(z) + \widetilde{D}_0(z), \tag{3.5}$$

where

$$\begin{cases} \widetilde{\Phi}(z) = z^2 \Phi(z^2), \\ \widetilde{C}_0(z) = 2z^3 C_0(z^2) - 3z\Phi(z^2), \\ \widetilde{D}_0(z) = 2(z^2+1) C_0(z^2) - 2\Phi(z^2) - 2\lambda D_0(z^2). \end{cases} \tag{3.6}$$

Moreover, the class of u depends only on the zero $x = 0$ of Φ.

Proof From (1.1) and (2.1), we obtain

$$S(v)(z^2) = -z^3 \lambda^{-1} S(u)(z) - \lambda^{-1} z^2 - \lambda^{-1}. \tag{3.7}$$

Make a change of variable $z \to z^2$ in (3.1); multiply by $-2\lambda z$ and substitute (3.7) in the obtained equation, we get (3.5)–(3.6).

From (3.6), the zeros of $\widetilde{\Phi}$ are 0 and those of Φ.

Let c^2 be any zero of Φ. Then, from (3.6), we get

$$\widetilde{C}_0(c) = 2c^3 C_0(c^2), \quad \widetilde{D}_0(c) = 2(c^2+1) C_0(c^2) + 2\lambda D_0(c^2). \tag{3.8}$$

Now, if $c \neq 0$ and $C_0(c^2) = 0$, from (3.8), we have $\widetilde{D}_0(c) \neq 0$, since v is semi-classical of class s and so satisfies (3.4), and if $c \neq 0$ and $C_0(c^2) \neq 0$, from (3.8), we have $\widetilde{C}_0(c) \neq 0$.

Thus, we cannot simplify the irreducible equation (3.5)–(3.6) by $z - c$ if $c \neq 0$. ■

The form u satisfies the distributional equation

$$\big(\widetilde{\Phi}(x) u\big)' + \widetilde{\Psi} u = 0, \tag{3.9}$$

where $\widetilde{\Phi}$ is the polynomial defined by (3.6) and

$$\widetilde{\Psi}(x) = -\widetilde{\Phi}'(x) - \widetilde{C}_0(x) = 2x^3 \Psi(x^2) + x\Phi(x^2). \tag{3.10}$$

Proposition 3.5 *Let $X(z) = C_0(z) - \Phi(z) - \lambda D_0(z)$ and $Y(z) = 2C_0(z) - 3\Phi'(z)$, where the polynomials Φ, C_0, and D_0 are defined by (3.1). For every $\lambda \in \mathbb{C} - \{0\}$ such that $\Delta_n S_n(0, \lambda) \neq 0$, $n \geq 0$, the linear functional u defined by (2.1) is regular and semi-classical of class $\widetilde{s}$ satisfying (3.5). Moreover,*

(i) *if $X(0) \neq 0$, then $\widetilde{s} = 2s + 4$;*
(ii) *if $X(0) = 0$ and $\Phi(0) \neq 0$, then $\widetilde{s} = 2s + 3$;*
(iii) *if $X(0) = \Phi(0) = 0$ and $X'(0) \neq 0$, then $\widetilde{s} = 2s + 2$;*
(iv) *if $X(0) = X'(0) = \Phi(0) = 0$ and $Y(0) \neq 0$, then $\widetilde{s} = 2s + 1$;*

(v) *if* $X(0) = X'(0) = \Phi(0) = Y(0) = 0$, *then* $\widetilde{s} = 2s$.

Proof (i) If $X(0) \neq 0$, then from (3.6) we obtain $\widetilde{D}_0(0) \neq 0$. Therefore, it is not possible to simplify (3.5)–(3.6). From (3.6) and (3.10), we have

$$\deg(\widetilde{\Phi}) = 2r + 2 \quad \text{and} \quad \deg(\widetilde{\Psi}) \leq \max(2p + 3, 2r + 1).$$

Thus, $\widetilde{s} = \max(\deg(\widetilde{\Phi}) - 2, \deg(\widetilde{\Psi}) - 1) = \max(2r, 2p + 2) = 2s + 4$.

(ii) If $X(0) = 0$, then from (3.6) we have $\widetilde{C}_0(0) = \widetilde{D}_0(0) = 0$; therefore, (3.5)–(3.6) is divisible by z. Thus, u fulfils (3.5) with

$$\left\{\begin{aligned} \widetilde{\Phi}(z) &= z\Phi(z^2), \\ \widetilde{C}_0(z) &= 2z^2 C_0(z^2) - 3\Phi(z^2), \\ \widetilde{D}_0(z) &= 2zC_0(z^2) + 2z\big(\theta_0(C_0 - \Phi - \lambda D_0)\big)(z^2). \end{aligned}\right. \tag{3.11}$$

If $\Phi(0) \neq 0$, it is not possible to simplify and thus the order of the class of u decreases by one unit.

(iii) If $\Phi(0) = X(0) = 0$, then it is possible to simplify (3.5)–(3.11), by z thus, u fulfils (3.5) with

$$\left\{\begin{aligned} \widetilde{\Phi}(z) &= \Phi(z^2), \\ \widetilde{C}_0(z) &= 2zC_0(z^2) - 3z(\theta_0\Phi)(z^2), \\ \widetilde{D}_0(z) &= 2C_0(z^2) + 2\big(\theta_0(C_0 - \Phi - \lambda D_0)\big)(z^2). \end{aligned}\right. \tag{3.12}$$

Therefore, if $C_0(0) + X'(0) \neq 0$, it is not possible to simplify, which means that the class of u is $\widetilde{s} = 2s + 2$.

(iv) If $\Phi(0) = X(0) = C_0(0) + X'(0) = 0$, then it is possible to simplify (3.5)–(3.12) by z. Thus u fulfils (3.5) with

$$\left\{\begin{aligned} \widetilde{\Phi}(z) &= z(\theta_0\Phi)(z^2), \\ \widetilde{C}_0(z) &= 2C_0(z^2) - 3(\theta_0\Phi)(z^2), \\ \widetilde{D}_0(z) &= 2z(\theta_0 C_0)(z^2) + 2z\big(\theta_0^2(C_0 - \Phi - \lambda D_0)\big)(z^2). \end{aligned}\right. \tag{3.13}$$

If $Y(0) \neq 0$, it is not possible to simplify, which means that the class of u is $\widetilde{s} = 2s + 1$.

(v) If $\Phi(0) = X(0) = C_0(0) + X'(0) = Y(0) = 0$, then it is possible to simplify (3.5)–(3.13) by z. Thus, u fulfils (3.5) with

$$\left\{\begin{aligned} \widetilde{\Phi}(z) &= (\theta_0\Phi)(z^2), \\ \widetilde{C}_0(z) &= 2z(\theta_0 C_0)(z^2) - 3z(\theta_0^2\Phi)(z^2), \\ \widetilde{D}_0(z) &= 2(\theta_0 C_0)(z^2) + 2\big(\theta_0^2(C_0 - \Phi - \lambda D_0)\big)(z^2). \end{aligned}\right. \tag{3.14}$$

Assuming that $\widetilde{\Phi}(0) = \Phi'(0) = 0$, from the condition $Y(0) = 0$, we obtain $C_0(0) = 0$. Thus, from the last result and the condition $X(0) = \Phi(0) = 0$, we get $D_0(0) = 0$, which is a contradiction with (3.3). Then it is not possible to simplify (3.5)–(3.14), which means the class of u is $\widetilde{s} = 2s$. ∎

Note that the sequence of orthogonal polynomials relative to a semi-classical form has a structure relation [17]. Then if we consider that the form ν is semi-classical, its orthogonal polynomial sequence $\{S_n\}_{n\geq 0}$ fulfils the following structure relation (written in a compact form):

$$\Phi(x)S'_{n+1}(x) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)\, S_{n+1}(x) - \rho_{n+1}D_{n+1}(x)S_n(x), \quad n \geq 0, \tag{3.15}$$

where for $n \geq 0$,

$$\begin{cases} C_{n+1}(x) = -C_n(x) + 2(x-\xi_n)D_n(x), \\ \rho_{n+1}D_{n+1}(x) = -\Phi(x) + \rho_n D_{n-1}(x) - (x-\xi_n)C_n(x) \\ \qquad + (x-\xi_n)^2 D_n(x). \end{cases} \tag{3.16}$$

Here, Φ, $C_0(x)$, and $D_0(x)$ are the same polynomials introduced in (3.1); ξ_n, ρ_n are the coefficients of the three term recurrence relation (1.5). Notice that $D_{-1}(x) = 0, \deg(C_n) \leq s+1$ and $\deg(D_n) \leq s, n \geq 0$ [17].

According to Proposition 3.4, the form u is also semi-classical and its orthogonal polynomials sequence $\{Z_n\}_{n\geq 0}$ satisfies a structure relation. In general, $\{Z_n\}_{n\geq 0}$ fulfils

$$\widetilde{\Phi}(x)Z'_{n+1}(x) = \frac{1}{2}\big(\widetilde{C}_{n+1}(x) - \widetilde{C}_0(x)\big)\, Z_{n+1}(x) - \gamma_{n+1}\widetilde{D}_{n+1}(x)Z_n(x), \quad n \geq 0, \tag{3.17}$$

with

$$\begin{cases} \gamma_{n+1}\widetilde{D}_{n+1}(x) = -\widetilde{\Phi}(x) + \gamma_n\widetilde{D}_{n-1}(x) - x\widetilde{C}_n(x) + x^2\widetilde{D}_n(x), \\ \widetilde{C}_{n+1}(x) = -\widetilde{C}_n(x) + 2x\widetilde{D}_n(x), \quad , n \geq 0, \end{cases} \tag{3.18}$$

where $\widetilde{C}_0(x)$, $\widetilde{D}_0(x)$ are given by (3.6) and $\widetilde{D}_{-1}(x) = 0$.

We are going to establish the expression of $\widetilde{C}_n$ and $\widetilde{D}_n$, $n \geq 0$ in terms of those of the sequence $\{S_n\}_{n\geq 0}$.

Proposition 3.6 *The sequence $\{Z_n\}_{n\geq 0}$ fulfills (3.17) with*

$$\begin{cases} \widetilde{C}_{2n+3}(x) = -x\Phi(x^2) + 2x(x^2+\gamma_{2n+3}-a_n)(C_{n+1}(x^2) + 2a_nD_n(x^2)) \\ \qquad + 2x(\gamma_{2n+3}-a_n)(C_{n+1}(x^2) + 2\dfrac{\rho_{n+1}}{a_n}D_{n+1}(x^2)), \quad n \geq 0, \\ \widetilde{D}_{2n+3}(x) = 2x^2(C_{n+1}(x^2) + a_nD_n(x^2) + \dfrac{\rho_{n+1}}{a_n}D_{n+1}(x^2)), \quad n \geq 0; \end{cases} \tag{3.19}$$

$$\begin{cases} \widetilde{C}_{2n+4}(x) = x\Phi(x^2) + 2x(x^2+a_n-\gamma_{2n+3})(C_{n+1}(x^2) + 2\dfrac{\rho_{n+1}}{a_n}D_{n+1}(x^2)) \\ \qquad + 2x(a_n-\gamma_{2n+3})(C_{n+1}(x^2) + 2a_nD_n(x^2)), \quad n \geq 0, \\ \widetilde{D}_{2n+4}(x) = (x^2+a_n-\gamma_{2n+3})(C_{n+1}(x^2) + 2\dfrac{\rho_{n+1}}{a_n}D_{n+1}(x^2)) \\ \qquad + (a_n-\gamma_{2n+3})(C_{n+1}(x^2) + 2a_nD_n(x^2)) \\ \qquad + (x^2+\gamma_{2n+5}-a_{n+1})(C_{n+2}(x^2) + 2a_{n+1}D_{n+1}(x^2)) \\ \qquad + (\gamma_{2n+5}-a_{n+1})(C_{n+2}(x^2) + 2\dfrac{\rho_{n+2}}{a_{n+1}}D_{n+2}(x^2)), \quad n \geq 0; \end{cases} \tag{3.20}$$

$$
(3.21)\begin{cases}
\widetilde{C}_1(x) = -x\Phi(x^2) + 2x(x^2+2)C_0(x^2) - 4\lambda x D_0(x^2),\\
\widetilde{C}_2(x) = x\Phi(x^2) + 2x(x^2-2)C_0(x^2) - 4\lambda x(x^2-1)D_0(x^2),\\
\widetilde{D}_1(x) = 2x^2(C_0(x^2) - \lambda D_0(x^2)),\\
\widetilde{D}_2(x) = -\dfrac{2}{\lambda+1}(\Phi(x^2) + (x^2-1)C_0(x^2) + \lambda(x^2-1)^2 D_0(x^2));
\end{cases}
$$

where $\widetilde{C}_0(x)$ and $\widetilde{D}_0(x)$ are given by (3.6) and γ_{n+1} by (2.13).

To prove the above proposition, we need the following lemmas.

Lemma 3.7 ([18]) *We have*

$$
(3.22)\qquad xS_n(x) = R_{n+1}(x) + (\xi_n - a_n)R_n(x), \quad n \geq 0.
$$

Lemma 3.8 *We have*

$$
(3.23)\qquad x^3 S_n(x^2) = \left(1 - \frac{\gamma_{2n+3}}{a_n}\right) Z_{2n+3}(x) + \frac{\gamma_{2n+3}}{a_n} Z_{2n+2}(x), \quad n \geq 0.
$$

Proof Change $x \to x^2$ in (3.22) and multiply by x; we obtain (3.23) by taking (2.6)–(2.7) and (2.12) into account. ∎

Proof of Proposition 3.6 From (1.5), (2.7), and (2.8), we have

$$
(3.24)\quad Z_{2n+3}(x) = -\frac{a_n}{\rho_{n+1}} x S_{n+2}(x^2) + \left\{1 + \frac{a_n}{\rho_{n+1}}(x^2 - \xi_{n+1})\right\} x S_{n+1}(x^2), \quad n \geq 0.
$$

After a derivation of (3.24), multiplying by $x^3\Phi(x^2)$, and using (3.15)–(3.16), we get

$$
\begin{aligned}
x^3\Phi(x^2)Z'_{2n+3}(x) &= \left\{\Phi(x^2) + x^2\left(2a_nD_n(x^2) + C_{n+1}(x^2)\right) - C_0(x^2))\right\} x^3 S_{n+1}(x^2)\\
&+ \{a_n\Phi(x^2) - a_n x^2\left(C_{n+1}(x^2) + C_0(x^2)\right) - 2\rho_{n+1}x^2 D_{n+1}(x^2)\} x^3 S_n(x^2), \quad n \geq 0.
\end{aligned}
$$

From (2.6), (2.12), and (3.23), we can write

$$
x^3 S_{n+1}(x^2) = \left(x^2 + \gamma_{2n+3} - a_n\right) Z_{2n+3}(x) - \gamma_{2n+3} x Z_{2n+2}(x), \quad n \geq 0.
$$

Then

$$
(3.25)\qquad \widetilde{\Phi}(x) Z'_{2n+3}(x) = X_n(x) Z_{2n+3}(x) - \gamma_{2n+3} Y_n(x) Z_{2n+2}(x), \quad n \geq 0,
$$

with for $n \geq 0$,

$$
\begin{aligned}
X_n(x) &= x\Phi(x^2) + x(x^2 + \gamma_{2n+3} - a_n)\left(C_{n+1}(x^2) + 2a_nD_n(x^2)\right)\\
&\quad + x(\gamma_{2n+3} - a_n)(C_{n+1}(x^2) + 2\frac{\rho_{n+1}}{a_n} D_{n+1}(x^2)
\end{aligned}
$$

and

$$
Y_n(x) = 2x^2(C_{n+1}(x^2) + a_n D_n(x^2) + \frac{\rho_{n+1}}{a_n} D_{n+1}(x^2).
$$

From (3.17) and (3.25), we have

$$
\begin{aligned}
&\left\{X_n(x) - \frac{1}{2}\left(\widetilde{C}_{2n+3}(x) - \widetilde{C}_0(x)\right)\right\} Z_{2n+3}(x) =\\
&\qquad\qquad \gamma_{2n+3}\left\{Y_n(x) - \widetilde{D}_{2n+3}(x)\right\} Z_{2n+2}(x), \quad n \geq 0.
\end{aligned}
$$

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Since $Z_{2n+3}(x)$ and $Z_{2n+2}(x)$ have no common roots, $Z_{2n+3}(x)$ divides $Y_n(x) - \widetilde{D}_{2n+3}(x)$, which is a polynomial of degree at most equal to $2s+4$. Therefore, we have necessarily $Y_n(x) - \widetilde{D}_{2n+3}(x) = 0$ for $n \geq s+1$, and also $X_n(x) = \frac{1}{2}(\widetilde{C}_{2n+3}(x) - \widetilde{C}_0(x))$, $n \geq s+1$.

Then, by (3.6), we get (3.19) for $n \geq s+1$.

By virtue of the recurrence relation (3.18) and (3.6), we can easily prove by induction that the system (3.19) is valid for $0 \leq n \leq s$. Hence, (3.19) is valid for $n \geq 0$.

Finally, using (3.6) and (3.18)–(3.19) we give (3.20)–(3.21). ■

4 Illustrative Example

We study the problem (2.1), with $v = \mathcal{J}(-\frac{1}{2}, \frac{1}{2})$, where $\mathcal{J}$ is the Jacobi form. The form v is classical, and it satisfies (3.2) with ([16])

$$\Phi(x) = x^2 - 1, \quad \Psi(x) = -2x - 1. \tag{4.1}$$

The sequence $\{S_n\}_{n\geq 0}$ fulfils (1.5) with ([16])

$$\xi_0 = -\frac{1}{2}, \quad \xi_{n+1} = 0, \quad \rho_{n+1} = \frac{1}{4}, \quad n \geq 0. \tag{4.2}$$

It also fulfils (3.15) with ([16])

$$C_0(x) = 1, \quad C_{n+1}(x) = 2(n+1)x, \quad D_n(x) = 2n+1, \quad n \geq 0. \tag{4.3}$$

First, we study the regularity of the form u.

From (1.5)–(1.6), and (4.2) , we can obtain the following by induction:

$$S_{2n}(0) = \frac{(-1)^n}{4^n}, \qquad S_{2n+1}(0) = \frac{(-1)^n}{2^{2n+1}}, \quad n \geq 0, \tag{4.4}$$

$$S^{(1)}_{2n}(0) = \frac{(-1)^n}{4^n}, \qquad S^{(1)}_{2n+1}(0) = 0, \quad n \geq 0. \tag{4.5}$$

Then, from (1.7), we get for $n \geq 0$,

$$S_{2n}(0,\lambda) = \frac{(-1)^n}{4^n}, \quad S_{2n+1}(0,\lambda) = (1-2\lambda)\frac{(-1)^n}{2^{2n+1}}. \tag{4.6}$$

Therefore, from (2.4) and (4.4), we have, for $n \geq 0$,

$$\Delta_{2n} S_{2n}(0,\lambda) = \frac{(-1)^n}{2^{6n}}(\lambda + 1 + n\theta),$$
$$\Delta_{2n+1} S_{2n+1}(0,\lambda) = \frac{(-1)^n}{2^{6n+3}}(1-2\lambda)(\lambda + (n+1)\theta),$$

where $\theta = 1 + (2\lambda - 1)^2$. Then we distinguish the two following cases.

- $\theta = 0$: The regularity means that we must have $\lambda = \frac{1\pm i}{2}$.
- $\theta \neq 0$: $\Delta_n S_n(0,\lambda) \neq 0$, $n \geq 0$, means that we must have

  $$\lambda + 1 + n\theta \neq 0 \quad \text{and} \quad (1-2\lambda)(\lambda + (n+1)\theta) \neq 0, \quad n \geq 0,$$

  which gives

  $$\lambda \neq \frac{1}{2}, \quad \frac{\lambda+1}{\theta} \neq -n, \quad \frac{\lambda+\theta}{\theta} \neq -n, \quad n \geq 0.$$

Now, we give the coefficients of the second-order recurrence relation satisfied by $\{Z_n\}_{n\geq 0}$. For this, first we calculate the coefficients a_n and b_n, $n \geq 0$, given by (2.10)–(2.11) and (4.4)–(4.5):

$$a_{2n} = \frac{2\lambda - 1}{2}, \qquad a_{2n+1} = \frac{1}{2(1-2\lambda)}, \quad n \geq 0,$$
$$b_{2n} = \frac{\lambda + (n+1)\theta}{4(\lambda + 1 + n\theta)}, \qquad b_{2n+1} = \frac{\lambda + 1 + (n+1)\theta}{4(\lambda + (n+1)\theta)}, \quad n \geq 0.$$

Using the above results and (2.13), we obtain, for $n \geq 0$,

$$\gamma_1 = 1, \ \gamma_2 = -\lambda - 1, \ \gamma_3 = \frac{\lambda(2\lambda-1)}{2(\lambda+1)},$$
$$\gamma_{4n+4} = \frac{\lambda + (n+1)\theta}{2(2\lambda-1)(\lambda+1+n\theta)},$$
$$\gamma_{4n+5} = \frac{\lambda + 1 + n\theta}{2(1-2\lambda)(\lambda+(n+1)\theta)},$$
$$\gamma_{4n+6} = \frac{(1-2\lambda)(\lambda+1+(n+1)\theta)}{2(\lambda+(n+1)\theta)},$$
$$\gamma_{4n+7} = \frac{(2\lambda-1)(\lambda+(n+1)\theta)}{2(\lambda+1+(n+1)\theta)}.$$

Since v is classical, according to Proposition 3.4, the form u is semi-classical. It satisfies (3.5) and (3.9) with

$$\widetilde{\Phi}(x) = x^2(x^4 - 1), \qquad \widetilde{\Psi}(x) = -x(3x^4 + x^2 + 1),$$
$$\widetilde{C}_0(x) = x(-3x^4 + 2x^2 + 3), \quad \widetilde{D}_0(x) = 2(-x^4 + x^2 + 2 - \lambda).$$

From (4.1) and (4.3), we have $\Phi(0) = -1$, $\quad X(0) = 2 - \lambda$. Now we can use Proposition 3.5 to obtain the following:

- If $\lambda \neq 2$, then the class of u is $\widetilde{s} = 4$.
- If $\lambda = 2$, then the class of u is $\widetilde{s} = 3$.

Finally, according to Proposition 3.6 and using (4.3), (4.6), we give the elements of the structure relation of the sequence $\{Z_n\}_{n\geq 0}$.

$$\widetilde{C}_0 = x(-3x^4 + 2x^2 + 3),$$
$$\widetilde{C}_1 = x(-x^4 + 2x^2 + 5 - 4\lambda),$$
$$\widetilde{C}_2 = x\big(x^4 + 2(1-2\lambda)x^2 + 4\lambda - 5\big),$$
$$\widetilde{D}_0 = -2x^4 + 2x^2 + 2(2-\lambda),$$
$$\widetilde{D}_1 = 2(1-\lambda)x^2,$$
$$\widetilde{D}_2 = 2x^4 + \frac{2}{\lambda+1}\big((1-2\lambda)x^2 + \lambda - 2\big),$$

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$$
\begin{aligned}
\widetilde{C}_{2n+3} &= (4n+3)x^5 + 4\Big((2n+1)a_n + 2(n+1)(\gamma_{2n+3} - a_n)\Big)x^3 \\
&\quad + \Big(1 + 4(2n+1)(\gamma_{2n+3} - a_n)a_n + \frac{2n+3}{a_n}\Big)x, \\
\widetilde{C}_{2n+4} &= (4n+5)x^5 + \Big(8(n+1)(a_n - \gamma_{2n+3}) + \frac{2n+3}{a_n}\Big)x^3 \\
&\quad + \Big((a_n - \gamma_{2n+3})\Big(4(2n+1)a_n + \frac{2n+3}{a_n}\Big) - 1\Big)x, \\
\widetilde{D}_{2n+3} &= 2x^2\Big(2(n+1)x^2 + (2n+1)a_n + \frac{2n+3}{a_n}\Big), \\
\widetilde{D}_{2n+4} &= 2(2n+3)x^4 + \Big(\Big(2n+3)(2a_{n+1} + \frac{1}{2a_n}\Big) + 4(n+2)(\gamma_{2n+5} - a_{n+1}) \\
&\quad + 4(a_n - \gamma_{2n+3})\Big)x^2 + (\gamma_{2n+5} - a_{n+1})\Big(2(2n+3)a_{n+1} + \frac{2n+5}{2a_{n+1}}\Big) \\
&\quad + (a_n - \gamma_{2n+3})\Big(2(2n+1)a_n + \frac{2n+3}{2a_n}\Big).
\end{aligned}
$$

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References

[1] J. Alaya and P. Maroni, *Symmetric Laguerre-Hahn forms of class $s = 1$.* Integral Transform. Spec. Funct. **4**(1996), no. 4, 301–320.
http://dx.doi.org/10.1080/10652469608819117

[2] ———, *Semi-classical and Laguerre-Hahn forms defined by pseudo-functions.* Methods Appl. Anal. **3**(1996), no. 1, 12–30.

[3] R. Álvarez-Nodarse, J. Arvesú, and F. Marcellán, *Modifications of quasi-definite linear functionals via addition of delta and derivatives of delta Dirac functions.* Indag. Math. (N.S.) **15**(2004), no. 1, 1–20.
http://dx.doi.org/10.1016/S0019-3577(04)90001-8

[4] D. Beghdadi and P. Maroni, *On the inverse problem of the product of a form by a polynomial.* J. Comput. Appl. Math. **88**(1998), no. 2, 377–399.
http://dx.doi.org/10.1016/S0377-0427(97)00227-6

[5] B. Bouras and A. Alaya, *A large family of semi-classical polynomials of class one.* Integral Transforms Spec. Funct. **18**(2007), no. 11–12, 913–931.
http://dx.doi.org/10.1080/10652460701511269

[6] T. S. Chihara, *An introduction to orthogonal polynomials.* Mathematics and its Applications, 13, Gordon and Breach, New York, 1978.

[7] T. S. Chihara, *On co-recursive orthogonal polynomials.* Proc. Amer. Math. Soc. **8**(1957), 899–905.
http://dx.doi.org/10.1090/S0002-9939-1957-0092015-5

[8] J. Dini and P. Maroni, *Sur la multiplication d'une forme semi-classique par un polynôme.* Publ. Sem. Math. Univ. d'Antananarivo **3**(1989), 76–89.

[9] D. H. Kim, K. H. Kwon, and S. B. Park, *Delta perturbation of a moment functional.* Appl. Anal., **74**(2000), no. 3–4, 463–477.
http://dx.doi.org/10.1080/00036810008840828

[10] K. H. Kwon and S. B. Park, *Two point masses perturbation of quasi-define moment functionals.* Indag. Math. (N.S.) **8**(1997), no. 1, 79–93.
http://dx.doi.org/10.1016/S0019-3577(97)83352-6

[11] F. Marcellán and P. Maroni, *Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique.* Ann. Mat. Pura Appl. (4) **162**(1992), 1–22.
http://dx.doi.org/10.1007/BF01759996

[12] P. Maroni and M. Mejri, *Some semi-classical orthogonal polynomials of class one.* Eurasian Math. J. **2**(2011), 108–128.

[13] P. Maroni and I. Nicolau, *On the inverse problem of product of a form by a monomial: the case $n = 4$. I.* Integral Transforms Spec. Funct. **21**(2010), no. 1–2, 35–56.
http://dx.doi.org/10.1080/10652460903016117

[14] ———, *On the inverse problem of the product of a form by a polynomial: The cubic case.* Appl. Numer. Math. **45**(2003), no. 4, 419–451.
http://dx.doi.org/10.1016/S0168-9274(02)00250-7

[15] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems.* In: Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA) (Granada, 1991), J. Comput. Appl. Math. **48**(1993), no. 1–2, 133–155.
http://dx.doi.org/10.1016/0377-0427(93)90319-7

[16] ———, *Sur la décomposition quadratique d'une suite de polynômes orthogonaux. I.* Riv. Mat. Pura Appl. **6**(1990), 19–53.

[17] ———, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques.* In: Orthogonal Polynomials and their applications (Erice, 1990), IMACS Ann. Comput. Appl. Math., 9, Baltzer, Basel. 1991, pp. 95–130.

[18] ———, *Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x-c)^{-1}L$.* Period. Math. Hungar. **21**(1990), no. 3, 223–248.
http://dx.doi.org/10.1007/BF02651091

[19] M. Sghaier and J. Alaya, *Orthogonal polynomials associated with some modifications of a linear form.* Methods Appl. Anal. **11**(2004), no. 2, 267–293.

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