

## LETTER TO THE EDITOR

# THE HIGHER MOMENTS OF THE NUMBER OF RETURNS OF A SIMPLE RANDOM WALK

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### Abstract

We consider a simple random walk starting at 0 and leading to 0 after  $2n$  steps. By a generating functions approach we achieve closed formulae for the moments of the random variables ‘number of visits to the origin’.

GENERATING FUNCTIONS; CATALAN NUMBERS

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60J15

Let  $X_k$ ,  $k = 1, 2, \dots$  be independent and identically distributed random variables with  $\mathbb{P}\{X_k = 1\} = \mathbb{P}\{X_k = -1\} = \frac{1}{2}$ . Consider the simple random walk

$$S_m = \sum_{k=1}^m X_k \quad \text{with } S_0 = 0 \quad \text{and} \quad S_{2n} = 0,$$

i.e. a simple random walk starting at 0 and leading to 0 after  $2n$  steps. Let the variable  $T$  be the number of visits to the origin.

In [2] the higher moments of this random variable were expressed as sums where the number of terms increases with  $n$ . The authors also gave asymptotic formulae by means of a Mellin transform approximation of the sums. In a following paper [4] the higher moments are described by certain recurrence relations with ‘full history’, i.e. using all moments of smaller order.

The aim of this note is, motivated by a comment in [2], to give closed-form expressions (i.e. the number of terms is independent of  $n$ ) for the moments in question. Our generating functions approach would also allow one to get the asymptotics in an elementary way. We mention two other problems where this kind of approach can be used.

In order to get a suitable expression for the generating function we decompose the family  $\mathcal{W}$  of random walks in question according to their returns. Noting that between any two consecutive returns a walk is either positive ( $\mathcal{W}_+$ ) or negative ( $\mathcal{W}_-$ ) we have

$$(1) \quad \mathcal{W} = (\mathcal{W}_+ + \mathcal{W}_-)^*,$$

where the asterisk denotes the combinatorial construction of forming finite sequences of elements of the concerned set of objects.

It is well known that the generating function of  $\mathcal{W}_+$  (or  $\mathcal{W}_-$ ) involves the Catalan

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Received 21 April 1993; revision received 7 July 1993.

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numbers. Counting the upward steps by the variable  $z$  and the returns by  $u$  we find the bivariate generating function

$$(2) \quad W(z, u) = \frac{1}{1 - 2uC(z)},$$

where

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2}.$$

The reader should observe that (not as in [2] and [4]) we do not count the starting position (at the origin) as a return to the origin.

The generating function of the  $s$ th factorial moments multiplied by  $\binom{2n}{n}$  is given by

$$(3) \quad M_s(z) = \frac{\partial^s}{\partial u^s} W(z, u)|_{u=1} = s! \frac{(1 - \sqrt{1 - 4z})^s}{(\sqrt{1 - 4z})^{s+1}}.$$

Expanding the numerator by the binomial theorem we rewrite (3) as

$$(4) \quad M_s(z) = s! \sum_{i=0}^s \binom{s}{i} (-1)^i (1 - 4z)^{\frac{1}{2}(i - s - 1)},$$

and hence the factorial moments  $m_s(n)$  satisfy

$$\binom{2n}{n} m_s(n) = [z^n] M_s(z) = s! \sum_{i=0}^s \binom{s}{i} (-1)^i (-4)^n \binom{\frac{1}{2}(i - s - 1)}{n}.$$

By substituting  $s - i$  for  $i$  we get

$$(5) \quad m_s(n) = \left\{ s! (-1)^s 4^n \sum_{i=0}^s \binom{s}{i} (-1)^i \binom{\frac{1}{2}(i - 1) + n}{n} \right\} / \binom{2n}{n}.$$

We note that for even values  $i = 2j$  Legendre’s duplication formula enables us to restate the binomial coefficients in (5) as

$$\binom{\frac{1}{2}(i - 1) + n}{n} = \frac{(2n + 2j)! j!}{4^n n! (2j)! (n + j)!},$$

and so (5) can be rewritten as

$$(6) \quad m_s(n) = \left\{ s! (-1)^{s+1} 4^n \sum_{j=0}^s \binom{s}{2j+1} \binom{n+j}{n} + s! (-1)^s \sum_{j=0}^s \binom{s}{2j} \frac{(2n+2j)! j!}{n! (2j)! (n+j)!} \right\} / \binom{2n}{n}.$$

The main advantage of formulae (5) and (6) is that the number of terms only depends on  $s$  (despite  $n$  in [2]).

Furthermore, a full asymptotic expansion of the moments follows immediately from (4) by Darboux’s method (or ‘singularity analysis’); compare [1].

By means of the Stirling numbers, these formulae can easily be rewritten in terms of the usual moments, still comprising only about  $s$  terms.

Finally, we note that this kind of approach could be used to investigate the problem of the more general random walk that finishes an arbitrary number of steps away from the

origin, and also the problem of the number of times where a simple random walk reaches its maximum; the latter problem was studied recently in [3] by different methods. (We note that the corresponding maximum problem for non-negative paths is considerably harder, and was studied in [5].)

## References

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