1

Introduction

1.1 Closed Time Contour

Consider a quantum many-body system governed by a time-dependent Hamiltonian $\hat{H}(t)$. Let us assume that in the distant past $t = -\infty$ the system was in a state specified by a many-body density matrix $\hat{\rho}(-\infty)$. The precise form of the latter is of no importance. It may be, for example, the equilibrium density matrix associated with the Hamiltonian $\hat{H}(-\infty)$. We shall also assume that the time-dependence of the Hamiltonian is such that at $t = -\infty$ the particles were noninteracting. The interactions are then adiabatically switched on to reach their actual physical strength sometime prior to the observation time. *In addition*, the Hamiltonian may contain *true* time dependence through, for example, external fields or boundary conditions. Due to such true time-dependent perturbations the density matrix is driven away from equilibrium.

The density matrix evolves according to the Von Neumann equation

$$\partial_t \hat{\rho}(t) = -\mathrm{i}\big[\hat{H}(t), \hat{\rho}(t)\big], \tag{1.1}$$

where we set $\hbar = 1$. It is formally solved with the help of the unitary evolution operator as $\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\big[\hat{\mathcal{U}}_{t,-\infty}\big]^\dagger = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\hat{\mathcal{U}}_{-\infty,t}$, where the † denotes Hermitian conjugation. The evolution operator obeys

$$\partial_t \hat{\mathcal{U}}_{t,t'} = -\mathrm{i}\hat{H}(t)\,\hat{\mathcal{U}}_{t,t'}\,; \qquad\qquad \partial_{t'}\hat{\mathcal{U}}_{t,t'} = \mathrm{i}\,\hat{\mathcal{U}}_{t,t'}\,\hat{H}(t').$$

Notice that the Hamiltonian operators taken at different moments of time, in general, do not commute with each other. As a result, $\hat{\mathcal{U}}_{t,t'}$ must be understood as an infinite product of incremental evolution operators with instantaneous locally constant Hamiltonians

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$$\hat{\mathcal{U}}_{t,t'} = \lim_{N\to\infty} \mathrm{e}^{-\mathrm{i}\hat{H}(t-\delta_t)\delta_t}\mathrm{e}^{-\mathrm{i}\hat{H}(t-2\delta_t)\delta_t}\ldots\mathrm{e}^{-\mathrm{i}\hat{H}(t-N\delta_t)\delta_t}\mathrm{e}^{-\mathrm{i}\hat{H}(t')\delta_t}$$
$$= \mathbb{T}\exp\left(-\mathrm{i}\int_{t'}^{t}\hat{H}(t)\,\mathrm{d}t\right), \tag{1.2}$$

where an infinitesimal time-step is $\delta_t = (t-t')/N$ and to shorten the notations the infinite product is abbreviated as the time-ordered, or $\mathbb{T}$-exponent.

One is usually interested to know an expectation value of some observable $\hat{\mathcal{O}}$ (say density or current operator) at a time t.[1] It is defined as

$$\langle\hat{\mathcal{O}}\rangle(t) \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{\rho}(t)\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}}\,\mathrm{Tr}\big\{\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\big\}, \tag{1.3}$$

where the trace is performed over many-body Hilbert space and in the last equality we cyclically permuted the $\hat{\mathcal{U}}_{-\infty,t}$ operator under the trace sign. The expression under the last trace describes (read from right to left) evolution from $t=-\infty$, where the initial density matrix is specified, toward t, where the observable is calculated, and then back to $t=-\infty$. Therefore, calculation of an observable implies evolving the initial state both *forward and backward*.

Such forward–backward evolution is avoided in the equilibrium quantum field theory with a special trick. Let us recall how it works, for example, in the zero-temperature equilibrium formalism [2]. The latter deals with the ground state expectation values of the type $\langle\mathrm{GS}|\hat{\mathcal{O}}|\mathrm{GS}\rangle$, where $|\mathrm{GS}\rangle$ is a ground state of an *interacting* many-body system. It is obtained from the known and simple ground state of the corresponding *noninteracting* system $|0\rangle$ by acting on the latter with the evolution operator $|\mathrm{GS}\rangle = \hat{\mathcal{U}}_{t,-\infty}|0\rangle$. Since we are dealing with the equilibrium situation, the only time dependence allowed for the Hamiltonian is an adiabatic switching of the interactions on and off in the distant past and distant future, respectively. The evolution operator therefore describes the evolution of a simple noninteracting ground state $|0\rangle$ toward $|\mathrm{GS}\rangle$ upon adiabatic switching of the interactions and thus $\langle\mathrm{GS}|\hat{\mathcal{O}}|\mathrm{GS}\rangle = \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle$.

Now comes the trick: one argues that

$$\hat{\mathcal{U}}_{+\infty,-\infty}|0\rangle = \mathrm{e}^{\mathrm{i}L}|0\rangle. \tag{1.4}$$

That is, evolution of the noninteracting ground state upon adiabatic switching of the interactions on and subsequent adiabatic switching them off brings the system back into the state $|0\rangle$, up to a phase factor $\mathrm{e}^{\mathrm{i}L}$. This statement is based on the belief that the adiabatic perturbation keeps the system in its (evolving) ground state at all times. If so, in view of normalization $\langle 0|0\rangle = 1$, the only

[1] We work in the Schrödinger picture, where observables are t-independent operators, while the wavefunctions and the density matrix evolve.

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possible change is the phase of the noninteracting ground state $e^{iL} = \langle 0|\hat{\mathcal{U}}_{+\infty,-\infty}|0\rangle$. Similarly $\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty} = \langle 0|e^{iL}$. Accepting this, one proceeds as follows:

$$\begin{aligned}\langle \mathrm{GS}|\hat{\mathcal{O}}|\mathrm{GS}\rangle &= \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle = e^{-iL}\langle 0|e^{iL}\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle \\ &= e^{-iL}\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty}\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle = \frac{\langle 0|\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle}{\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty}|0\rangle}\,, \qquad (1.5)\end{aligned}$$

where in the last equality we used $\hat{\mathcal{U}}_{+\infty,-\infty}\hat{\mathcal{U}}_{-\infty,t} = \hat{\mathcal{U}}_{+\infty,t}$, which is an immediate consequence of Eq. (1.2). The result of this procedure is that one needs to consider only the *forward* evolution. Indeed, the numerator in the last expression (being read from right to left) calls for evolving the noninteracting ground state $|0\rangle$ from the distant past to the observation time, where the observable operator acts, and then proceeding toward the distant future, where the overlap with the same known state $\langle 0|$ is evaluated.

The similar strategy works in the finite-temperature equilibrium formalism [11, 2, 4]. There, one treats the equilibrium density matrix $e^{-\beta\hat{H}}$, where $\beta = 1/T$ is the inverse temperature, as the evolution operator in the imaginary time τ. The latter is defined on a finite interval $0 \le \tau < \beta$. The observables (or correlation functions) are also evaluated at imaginary time points $\tau_1, \tau_2, \ldots$ and the result must be analytically continued back to the real-time axis. One may argue that, since the adiabatic switching of interactions does not drive the system out of equilibrium, a statement similar to Eq. (1.4) still holds. As a result one is again left to describe only the forward evolution, albeit along the finite time interval in the imaginary direction.

Let us mention that elimination of the backward evolution comes with a price: the normalization denominator in the last expression in Eq. (1.5). It offsets the phase accumulation e^{iL} of the noninteracting ground state $|0\rangle$. In diagrammatic language it amounts to subtracting the so-called disconnected or vacuum loop diagrams. This denominator is a serious liability in the theory of disordered systems. The reason is that the accumulated phase e^{iL} sensitively depends on a specific realization of the disorder (which may be thought of as being absent at $t \to \pm\infty$ and adiabatically switched on and off in the process of evolution). Therefore, the denominator absolutely must be included in any disorder averaging procedure, which complicates the treatment in a very substantial way.

The much more serious trouble with the outlined procedure is that Eq. (1.4) does *not* work in a non-equilibrium situation. If the Hamiltonian $\hat{H}(t)$ contains nonadiabatic time-dependent external fields, boundary conditions, and so on, the evolution drives the system away from equilibrium. Even if all such fields are eventually switched off in the distant future, there is no guarantee that the system returns to its ground (or equilibrium) state. Therefore, acting with the operator $\hat{\mathcal{U}}_{+\infty,-\infty}$ on

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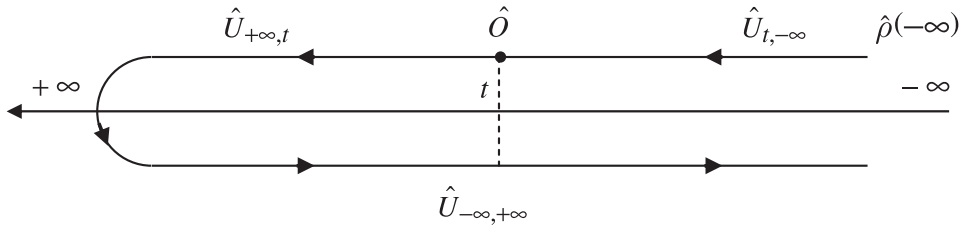

Figure 1.1 Closed time contour $\mathcal{C}$. Evolution along such a contour is described by Eq. (1.6).

the initial ground (or equilibrium) state results in an unknown superposition of excited states. As a result, the backward evolution, inherent to Eq. (1.3), can't be eliminated.

Nevertheless, it is still convenient to extend the evolution in Eq. (1.3) toward $t = +\infty$ and then back to t. This is achieved with the help of the trivial identity $\hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t} = \hat{1}$. Inserting it into Eq. (1.3) and using $\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{U}}_{t,+\infty} = \hat{\mathcal{U}}_{-\infty,+\infty}$, one finds

$$\langle\hat{\mathcal{O}}\rangle(t) = \frac{1}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}}\,\mathrm{Tr}\big\{\hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\big\}. \tag{1.6}$$

Here we also used the fact that, according to the Von Neumann equation (1.1), the trace of the initial density matrix is unchanged under the unitary evolution. Equation (1.6) describes evolution along the *closed time contour* $\mathcal{C}$ depicted in Fig. 1.1. The observable $\hat{\mathcal{O}}$ is inserted at time t, somewhere along the forward branch of the contour. Notice that, by inserting the operator $\hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t} = \hat{1}$ to the right of $\hat{\mathcal{O}}$ in Eq. (1.3), one could equally well arrange to have the observable on the backward branch of the contour. As we shall see, the most convenient choice is to take a half sum of these two equivalent representations.

Evolution along the closed time contour $\mathcal{C}$ is the central subject of this book. The fact that the field theory can be constructed with the time ordering along such a contour was first realized by Schwinger [12] and further developed in [25, 26]. About the same time Konstantinov and Perel [13] developed a diagrammatic technique, based on the time contour containing forward and backward branches in the real-time direction along with the imaginary time portion of length β. The formalism was significantly advanced, in particular its utility to derive the kinetic theory, in the seminal book of Kadanoff and Baym [1]. Independently Keldysh [14] (for some of the historic context, see [27]) suggested a formulation that does not rely on imaginary time (and thus on the equilibrium density matrix). He also introduced a convenient choice of variables (Keldysh rotation), which made derivation of the kinetic theory particularly transparent. The time contour without the imaginary time piece, along with the Keldysh variables (which we call "classical" and "quantum"), appear to be by far the most convenient choices for the functional

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formulation of the theory presented in this book. For this reason we occasionally refer to the construction as the Keldysh technique (this should by no way diminish the credit deserved by the other authors). Reformulation of the theory for the case of fermions, given later by Larkin and Ovchinnikov [28], became universally accepted. In fact, other theories developed about the same time, while not using the time contour explicitly, appear to be close relatives of the Schwinger–Kadanoff–Baym–Keldysh construction. Among them are Feynman and Vernon [29], Wyld's [30] diagrammatic technique for fluid dynamics, and Martin, Siggia, and Rose [31] and DeDominicis' [32] calculus for classical stochastic systems.

The central object of the theory is the evolution operator along the closed contour $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,-\infty}$. If the Hamiltonian is the same on the forward and backward branches, then the forward–backward evolution of *any* state brings it back exactly to the original state. (Not even a phase factor is accumulated; indeed, any phase gained on the forward branch is exactly "unwound" on the backward branch.) As a result $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{1}$ and the partition function, defined as $Z \equiv \mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}(-\infty)\}/\mathrm{Tr}\{\hat{\rho}(-\infty)\}$, is identically equal to unity, $Z = 1$. Nevertheless, the partition function is a convenient object to develop the functional representation, and the normalization identity $Z = 1$ is a useful check of its consistency. For this reason we shall use it widely in what follows.

To insert an observable somewhere along the forward (as prescribed by Eq. (1.6)) or backward branches it is convenient to modify the Hamiltonian $\hat{H}(t)$ by adding the source term $\hat{H}_V^{\pm}(t) \equiv \hat{H}(t) \pm \hat{\mathcal{O}}V(t)$, where the plus (minus) sign refers to the forward (backward) part of the contour. Now, since the Hamiltonian is different on the two branches, the evolution operator along the contour $\hat{\mathcal{U}}_{\mathcal{C}}[V] \neq \hat{1}$ becomes nontrivial and so does the *generating function*

$$Z[V] \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}[V]\,\hat{\rho}(-\infty)\}}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}}. \tag{1.7}$$

The expectation value of the observable $\hat{\mathcal{O}}$, given by Eq. (1.6) (or rather by a half sum of the observable inserted along the forward and backward branches) may be found as $\langle\hat{\mathcal{O}}\rangle(t) = (\mathrm{i}/2)\delta Z[V]/\delta V(t)\big|_{V=0}$. This expression should be compared with the equilibrium technique [2, 4], where the observables are given by variational derivatives of the *logarithm* of the generating (or partition) function. In our case, since $Z = Z[0] = 1$, the presence of the logarithm is optional.[2] Knowledge of

[2] It is worth mentioning that the denominators in Eqs. (1.5) and (1.7) have very different status. In the latter case $\mathrm{Tr}\{\hat{\rho}(-\infty)\}$ refers entirely to the distant past, when both interactions and disorder are switched off. It is therefore a simple constant, which may be easily evaluated. In the former case $\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty}|0\rangle$ involves evolution of the ground state upon switching on and off the interactions and disorder. It thus depends on both disorder and interactions and requires a separate calculation. The absence of a disorder-dependent denominator makes the closed time contour formalism especially suitable to deal with the averaging over the quenched disorder. The fact that observables do not require the *logarithm* is another manifestation of the absence of the nontrivial denominator.

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the generating function allows one thus to find observables of interest. Therefore, after developing the functional formalism for the partition function, we extend it to include the generating function as well.

1.2 Brief Outline of the Book

Part I is devoted to systems with one or few degrees of freedom. It is used as a pedagogical device to develop the functional integral treatment of the evolution along the closed time contour, introduced previously. There are several important steps that need to be dealt with within the time-discretized treatment, before the continuous notations turn into a practical theoretical device. We take extra care exposing those steps in Chapter 2 for Von Neumann unitary evolution and then again in Chapter 5 in Lindblad dissipative setting. To illustrate the utility of the developed framework we apply it to a number of quantum mechanical problems, which include treatment of systems in contact with classical or quantum bath. A particular focus is on non-perturbative phenomena, such as tunneling in a time-dependent potential, or quantum activation, where the functional integral treatment is truly indispensable.

Chapter 4 is used to demonstrate that the classical limit of the closed time contour theory yields the Martin–Siggia–Rose–DeDominicis framework for stochastic models. We use this observation as an opportunity to derive Langevin and Fokker–Planck equations, which are widely employed through the rest of the book. Once again, the developed machinery is deployed to treat a number of examples. Those include escape from a metastable state, full counting statistics, reaction models, fluctuation relations, and time-dependent perturbations in stochastic systems. These subjects resurface in subsequent chapters.

Part II extends the machinery to many-body quantum Bose systems as well as classical stochastic models. Chapter 6 serves to adopt the standard arsenal of the quantum field theory to the closed time contour setting. This includes the diagrammatic technique and the Dyson equation. Specifically we focus on the realization that a particular component of the Dyson equation yields the quantum kinetic equation. We present several pedagogical examples of how the kinetic equation is derived from the first principles and is used in practice. In Chapter 7, the classical collisionless plasma is used as a vehicle to expose some useful concepts: collective modes (plasmons), Landau damping, random phase approximation (RPA), Hubbard–Stratonovich transformation, and coupled kinetic equations for collective modes and quasiparticles degrees of freedom.

Chapter 8 employs the machinery to discuss kinetics of the Bose–Einstein condensation. Here we derive the mean-field Gross–Pitaevskii framework as the stationary point of the Keldysh action and Bogoliubov collective modes as its

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linearized excitations. We then develop the kinetic theory of coupled collective modes and quasiparticles. Chapter 9 is devoted to the dynamics of classical phase transitions. It includes nucleation of critical droplets in first-order transitions as well as dynamics of continuous phase transitions, starting from equilibrium transitions and their Hohenberg–Halperin classification. We then turn to essentially non-equilibrium phase transitions, which include absorbing state transitions in reaction-diffusion models and Kardar–Parisi–Zhang (KPZ) theory of the roughening transition on growing interfaces. The 1d KPZ equation is used to illustrate concepts of the macroscopic fluctuation theory (MFT), geared toward rare events statistics in stochastic models.

The rest of the book, Parts III and IV, is devoted to fermions. Chapter 10 extends treatment of Chapters 2 and 6 to fermionic systems, using Grassmann functional integrals on the closed time contour. Chapter 11 serves to derive the kinetic equation for interacting fermions and uses it to introduce a hydrodynamic description of the Fermi liquid. Chapter 12 deals with various aspects of the fermionic kinetic theory. Those include treatment of fermions in presence of a Berry curvature, kinetics of periodically driven Floquet systems, stochastic Boltzmann–Langevin terms in the kinetic equation. Finally we discuss the "butterfly effect" with the out-of-time-order correlation functions (OTOC), which requires four-branch time contour.

In Chapter 13 we use fermionic formalism to discuss non-equilibrium quantum transport. In particular we derive the Landauer formula for tunneling conductance, the Lesovik formula for shot-noise, the Levitov's result for the full counting statistics of transmitted charge, the Brouwer formula for adiabatic pumping of charge, and, following Nazarov and Tobiska, the exact fluctuation relation and its consequences. We also deal with the spin transport, deriving the Slonczewski–Berger spin-torque term in the Landau–Lifshitz equation along with the spin-torque noise and associated Gilbert damping.

Part IV of the book addresses fermions in the presence of a quenched disorder. The main technical tool of this part is Keldysh nonlinear sigma-model (NLSM), which is painstakingly developed in Chapter 14 and applied in various settings in this and subsequent chapters. Chapter 14 features weak-localization along with the scaling theory of Anderson localization. Chapter 15 deals with the mesoscopic phenomena ranging from spectral statistics and universal conductance fluctuations (UCF) to full counting statistics in tunneling, diffusive, and quantum chaotic systems.

We then include electron–electron interactions in disordered systems, which lead to singular behavior in the density of states (zero bias anomaly) and conductivity (Altshuler–Aronov corrections). Interactions also yield collision terms to the diffusive quantum kinetic equation. These phenomena are the subject of Chapter 16.

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Chapter 17 is devoted to disordered superconductors. We generalize the nonlinear sigma-model to include superconducting correlations. It yields the Usadel equation as its stationary point, which includes equations for the quasiparticle spectrum as well as the quasiparticle kinetic equation for the quasiparticles distribution function. Together with the self-consistency condition they provide a complete framework to study non-equilibrium superconductivity. As examples, we work out the spectrum of the collective (Carlson–Goldman) mode of the superconductor, and derive the time-dependent Ginzburg–Landau theory and fluctuation corrections to the conductivity above the critical temperature. Finally, Chapter 18 deals with electron–phonon interactions in disordered normal metals and superconductors. We derive coupled kinetic equations for electron and phonon degrees of freedom and use them to analyze electron relaxation as well as ultrasound attenuation.

All chapters are equipped with several problems. They are structured as compact guided research projects, which may be suggested to students for an independent exploration. Most of them carry useful supplemental information, which has not yet found its way to textbooks.

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