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A NEW EXAMPLE OF A DETERMINISTIC CHAOS GAME

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Abstract

We give a new necessary and sufficient condition for an iterated function system to satisfy the deterministic chaos game. As a consequence, we give a new example of an iterated function system which satisfies the deterministic chaos game.

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1. Introduction

As is well known, fractals may appear as attractors for iterated function systems (IFSs). In this way, essentially contractive iterated function systems are simple tools to construct fractals. The main technique for studying essentially contractive iterated function systems is to apply the so-called coding map. However, this technique does not work for many interesting cases. The aim of the present work is to study iterated function systems without a coding map.

We begin by introducing definitions and notations for iterated function systems and then formulate our main result. Throughout this paper, X will stand for a compact metric space with metric d and $\mathcal{F}$ will stand for a finite family of continuous maps $f_1, \ldots, f_k$ on X.

We denote by $\langle \mathcal{F} \rangle^+$ the semigroup generated by these maps. We call the action of the semigroup $\langle \mathcal{F} \rangle^+$ on X the *iterated function system* associated to $\mathcal{F}$ and we denote it by $\mathrm{IFS}(X; \mathcal{F})$ or $\mathrm{IFS}(\mathcal{F})$.

Let us consider the spaces $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$ of sequences of symbols from the finite alphabet $\{1, \ldots, k\}$. For every $\ell \in \mathbb{N}$, set $\Sigma_k^\ell = \{1, \ldots, k\}^\ell$, whose elements are the finite words. Following [11, 17], we say that a sequence $(\omega_i)_{i=1}^\infty \in \Sigma_k^+$ is *disjunctive* if it contains all possible finite words as its subsequences. The sequence ω is disjunctive if and only if it has a dense orbit under the shift map, that is,

$$\forall \ell \ \forall \rho \in \mathbb{N}^\ell \ \exists n \in \mathbb{N} : \ (\omega_{n+1} \ldots \omega_{n+\ell}) = \rho.$$

We denote the set of all the disjunctive sequences by Ω.

For any sequence $\omega = (\omega_1\omega_2\ldots\omega_n\ldots) \in \Sigma_k^+$, set $f_\omega^0 := \mathrm{Id}$ and

$$f_\omega^n(x) = f_{\omega_1\ldots\omega_n}^n(x) = f_{\omega_n} \circ f_\omega^{n-1}(x), \quad \forall n \in \mathbb{N}.$$

If $\omega = (\omega_1\omega_2\ldots\omega_n\ldots)$ is a sequence in Σ_k^+, then the corresponding *chaos game orbit* [1] of a point $x_0 \in X$ is a sequence $\{x_n\}_{n=0}^\infty$ defined iteratively by $x_n = f_{\omega_n}(x_{n-1})$ for $n = 1, 2, \ldots$. A chaos game orbit is also called a *fibre-wise orbit* and denoted by $\mathcal{O}^+(x, \omega) = \{f_\omega^n(x)\}_{n=0}^\infty$.

Now consider the hyperspace $\mathcal{K}(X)$ consisting of all nonempty compact subsets of X endowed with the Hausdorff metric d_H. It is known that if X is compact, so is $\mathcal{K}(X)$ (see [15]). Associated with the set $\mathcal{F}$, we define the *Hutchinson operator* $F = F_{\mathcal{F}}$ as follows:

$$F : \mathcal{K}(X) \to \mathcal{K}(X), \quad F(\Theta) = \bigcup_{i=1}^{k} f_i(\Theta).$$

Simplify $F(\{x\})$ to $F(x)$ for $x \in X$. A nonempty closed subset A of X is said to be a *strict attractor* of IFS$(X; \mathcal{F})$ if there is an open neighbourhood U of A so that, in the Hausdorff metric space $(\mathcal{K}(X), d_H)$,

$$\lim_{n\to\infty} F^n(K) = A \quad \text{for every compact set } K \subset U.$$

We remark that it is usual to include in the definition of attractor that $F(A) = A$ (cf. [4, Definition 2.2]). Denote by $\mathcal{B}(A)$ the *basin* of the strict attractor. That is, the union of all open neighbourhoods U of A such that the above convergence holds.

The simple and main tool for approximation of attractors is the chaos game. We will focus on a definition of a kind of chaos game which is independent of start points. We say that a strict attractor A of IFS$(X; \mathcal{F})$ satisfies the *deterministic chaos game* if, for every $\omega \in \Omega$,

$$A \subset \overline{\mathcal{O}^+(x, \omega)} \quad \text{for all } x \in \mathcal{B}(A).$$

It is obvious that if a strict attractor A of IFS$(X; \mathcal{F})$ satisfies the deterministic chaos game, then for every $x \in \mathcal{B}(A)$, $i \in \mathbb{N}$ and every $\omega \in \Omega$, $A \subset \overline{\mathcal{O}^+(f_\omega^i(x), \sigma^i(\omega))}$, where $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is the shift map. Indeed, the assertion follows from the following facts: $F(\mathcal{B}(A)) \subseteq \mathcal{B}(A)$ (see [3]), $\sigma(\Omega) = \Omega$ and IFS$(X; \mathcal{F})$ satisfies the deterministic chaos game. Roughly speaking, a fibre-wise orbit simulates the attractor by drawing accumulation points.

Remark 1.1. From [2, 6–8, 14], the deterministic chaos algorithm holds in the following five cases:

(1) strict attractors of IFSs on Hausdorff topological spaces with certain contractions (such as weakly hyperbolic, strongly fibred, well-fibred, contractible, ...);
(2) strict attractors of nonexpansive or equicontinuous IFSs on metric spaces;
(3) forward and backward minimal IFSs of homeomorphisms of the circle;
(4) IFSs on a compact metric space having a minimal map; and
(5) strict attractors of symmetric IFSs.

Before adding another and different class of systems to this list, we note the following equivalency to the deterministic chaos game.

PROPOSITION 1.2 [8]. *A strict attractor A of* IFS$(X;\mathcal{F})$ *satisfies the deterministic chaos game if and only if for every open set I which has a nonempty intersection with the attractor A, there is a finite word $(\omega_1 \ldots \omega_t)$ such that*

$$\text{for each } x \in A \text{ there is } i \in \{0,\ldots,t\} \text{ so that } f_{\omega_i} \circ \cdots \circ f_{\omega_1}(x) \in I.$$

We say that a strict attractor A of IFS$(X;\mathcal{F})$ generated by a family of homeomorphisms $\mathcal{F} = \{f_1,\ldots,f_k\}$ of X satisfies the *vee-chaos game* if for every open set I which has a nonempty intersection with the attractor A, there are α,β in Σ_k^+ and $t,s \in \mathbb{N}$ and

$$\forall x \in \mathcal{B}(A), \quad \bigg[\bigg(\bigcup_{i=0}^{t}\{f_\alpha^i(x)\}\bigg) \cup \bigg(\bigcup_{j=0}^{s}\{g_\beta^j(x)\}\bigg)\bigg] \cap I \neq \emptyset, \tag{1.1}$$

where $g_i = f_i^{-1}$ for $1 \le i \le k$.

Now we can state the main results of this paper.

THEOREM 1.3. *A strict attractor of an IFS on a compact metric space satisfies the deterministic chaos game if and only if it satisfies the vee-chaos game.*

COROLLARY 1.4. *There exists an IFS which satisfies the deterministic chaos game and is not included in the cases* (1)–(5) *of Remark 1.1.*

2. Proof of the main theorem

Now we prove the main result of the paper.

PROOF OF THEOREM 1.3. Clearly, if A is a strict attractor which satisfies the deterministic chaos game, then A satisfies the vee-chaos game.

Conversely, consider a strict attractor A of an IFS$(X;\mathcal{F})$ generated by a family of homeomorphisms $\mathcal{F} = \{f_1,\ldots,f_k\}$ of X, where A satisfies the vee-chaos game. By Proposition 1.2, the attractor A satisfies the deterministic chaos game if for any nonempty open set $I \subset X$ with $A \cap I \neq \emptyset$, there exist $t \in \mathbb{N}$ and $\rho \in \Sigma_k^+$ such that

$$\forall x \in \mathcal{B}(A), \quad \bigg(\bigcup_{i=0}^{t}\{f_\rho^i(x)\}\bigg) \cap I \neq \emptyset.$$

Suppose that I is an open set which has a nonempty intersection with the attractor A. By the assumption, there are finite words α,β in Σ_k^+ and $t,s \in \mathbb{N}$ satisfying (1.1) with $g_r = f_r^{-1}$ for $1 \le r \le k$.

Take $z = f_{\beta_1} \circ \cdots \circ f_{\beta_s}(x)$. Since $x \in \mathcal{B}(A)$ and $F(\mathcal{B}(A)) \subset \mathcal{B}(A)$, $z \in \mathcal{B}(A)$. Thus,

$$\bigg[\bigg(\bigcup_{i=0}^{t}\{f_\alpha^i(z)\}\bigg) \cup \bigg(\bigcup_{j=0}^{s}\{g_\beta^j(z)\}\bigg)\bigg] \cap I \neq \emptyset.$$

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Now define ρ by

$$(\beta_s\beta_{s-1}\ldots\beta_1\alpha_1\alpha_2\alpha_3\ldots).$$

To complete the proof, we claim that

$$\forall x \in \mathcal{B}(A)\ \exists i \in \{0,\ldots,t+s\}, \quad f_\rho^i(z) \in I.$$

To see this, we consider the following two cases.

Case 1. $(\bigcup_{i=0}^t\{f_\alpha^i(z)\}) \cap I \neq \emptyset$.

Since $(\bigcup_{i=0}^t\{f_\alpha^i(z)\}) \cap I \neq \emptyset$, we have $f_\alpha^i(z) \in I$ for some $i \in \{1,\ldots,t\}$. On the other hand, $z = f_{\beta_1} \circ \cdots \circ f_{\beta_s}(x)$, which implies that

$$f_{\alpha_i} \circ \cdots \circ f_{\alpha_1}(z) = f_{\alpha_i} \circ \cdots \circ f_{\alpha_1} \circ f_{\beta_1} \circ \cdots \circ f_{\beta_s}(x) = f_\rho^{s+i}(x) \in I$$

for some i in $\{1,\ldots,t\}$.

Case 2. $(\bigcup_{j=0}^s\{g_\beta^j(z)\}) \cap I \neq \emptyset$.

Since $(\bigcup_{j=0}^s g_\beta^j(z)) \cap I \neq \emptyset$, there is $j \in \{1,\ldots,s\}$ so that $g_\beta^j(z) \in I$. Thus,

$$g_{\beta_j} \circ \cdots \circ g_{\beta_1}(z) = g_{\beta_j} \circ \cdots \circ g_{\beta_1} \circ f_{\beta_1} \circ \cdots \circ f_{\beta_s}(x) \in I.$$

However, $g_{\beta_r} = f_{\beta_r}^{-1}$ for $r = 1,\ldots,j$, which implies that

$$f_{\beta_{j+1}} \circ \cdots \circ f_{\beta_s}(x) = f_\rho^{s-j}(x) \in I.$$

This completes the proof of the theorem. □

3. The new example

We complete the paper with the proof of Corollary 1.4. To this end, we need to recall several definitions.

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a generalised north–south pole diffeomorphism on the torus $\mathbb{T}^2$ (see [16] for more details about the north–south pole diffeomorphism). By this we mean that the nonwandering set, $\Omega(f)$, of f consists of one fixed source, q, one fixed sink, p, and saddle-type periodic orbits. Let S be the set of all the saddle-type periodic points of f. For simplicity, we assume that S consists of two saddle points a and b so that $\mathcal{W} = W^s(S) \cup W^u(S) \cup \{p,q\}$ consists of four circles: two disjoint circles following the meridian direction and two other disjoint circles following the parallel directions. Figure 1 shows a 'typical' diagram of such a diffeomorphism. Notice that for every $x \in \mathbb{T}^2 \setminus \mathcal{W}$, we have $f^n(x) \to p$ and $f^{-n}(x) \to q$ as $n \to \infty$.

Furthermore, following [8], we remark that $A \in \mathcal{K}(X)$ is a *quasi-attractor* of the IFS generated by $\mathcal{F}$ if $F(A) = A$ and $A = \overline{\{h(x) | h \in \langle\mathcal{F}\rangle^+\}}$ for all $x \in A$. In particular, if X is a quasi-attractor of the IFS generated by $\mathcal{F}$, then we say that the IFS$(X;\mathcal{F})$ is minimal or $\langle\mathcal{F}\rangle^+$ acts minimally on X. Also, we say that a quasi-attractor A of the IFS is *well-fibred* if for every compact set K in A so that $K \neq A$ and for any open cover $\mathcal{U}$ of A, there exist $g \in \Gamma$ and $U \in \mathcal{U}$ such that $g(K) \subset U$.

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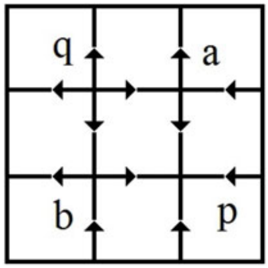

FIGURE 1. Diagram of the diffeomorphism f.

One of the tools which we need here is the 'blending region', which is the main tool to produce C^1-robust minimal actions, that is, actions whose minimality persists under small C^1-perturbations on the generators (see [9, 10, 12, 13]). An open subset Δ of a manifold M is called a *blending region* for a semigroup $\langle \mathcal{F} \rangle^+$ of diffeomorphisms on M if there exist $h_1, \ldots, h_k \in \langle \mathcal{F} \rangle^+$ and an open set $D \subset M$ such that $\overline{\Delta} \subset D$ and:

(1) $\overline{\Delta} \subset h_1(\Delta) \cup \cdots \cup h_k(\Delta)$;
(2) $h_i : \overline{D} \to D$ is a contracting map for $i = 1, \ldots, k$.

The construction of our example is based on the following facts: existence of a C^1-blending region for a semigroup $\langle \mathcal{F} \rangle^+$ of diffeomorphisms on the 2-torus and existence of an irrational translation in $\langle \mathcal{F} \rangle^+$. This implies existence of a blending region and the possibility of replacing an irrational translation by rational ones. All these phenomena occur sufficiently close to our original system.

Now we ready to prove Corollary 1.4.

PROOF OF COROLLARY 1.4. Consider a generalised north–south pole diffeomorphism f on the torus and two translations $R_{(\lambda,0)} : \mathbb{T}^2 \to \mathbb{T}^2$, $R_{(\lambda,0)}(x, y) = (x + \lambda, y)$ and $R_{(0,\kappa)} : \mathbb{T}^2 \to \mathbb{T}^2$, $R_{(0,\kappa)}(x, y) = (x, y + \kappa)$ with $\lambda, \kappa \in \mathbb{R} \setminus \mathbb{Q}$.

Since the IFS generated by $f, R_{(\lambda,0)}$ and $R_{(0,\kappa)}$ on $\mathbb{T}^2$ has minimal elements, according to [7, Proposition 1], it satisfies the deterministic chaos game. Moreover, it is not difficult to see that this IFS is C^1-robustly minimal. Indeed, one can construct a blending region around the attracting fixed point and then apply [5, Theorem 6.2].

Thus, there are rational numbers φ and ψ close to λ and κ, respectively, so that the IFS$(\mathbb{T}^2; f_1, f_2, f_3)$ is minimal, where $f_1 = f$, $f_2 = R_{(\varphi,0)}$ and $f_3 = R_{(0,\psi)}$. To apply Theorem 1.3, it is sufficient to show that, for every pair of open neighbourhoods I of the sink p and J of the source q, there exist α', β' in Σ_k^+ and $t, s \in \mathbb{N}$ so that, for all $x \in \mathbb{T}^2$,

$$f^t_{\alpha'}(x) \in I \quad \text{or} \quad g^s_{\beta'}(x) \in J, \tag{3.1}$$

where $g_i = f_i^{-1}$ for $1 \le i \le 3$.

Indeed, since IFS$(\mathbb{T}^2; f_1, f_2, f_3)$ and IFS$(\mathbb{T}^2; g_1, g_2, g_3)$ are minimal and I and J can be chosen sufficiently small, for every open set U there exist $h \in \langle f_1, f_2, f_3 \rangle^+$

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and $h' \in \langle g_1, g_2, g_3\rangle^+$ with

$$h(I) \subset U \quad \text{and} \quad h'(J) \subset U.$$

That is equivalent to the following: there exist sequences α and β with

$$\alpha_i = \alpha'_i \quad \text{and} \quad \beta_j = \beta'_j \quad \text{for } 0 \le i \le t,\ 0 \le j \le s,$$

and $\ell \in \mathbb{N}$ so that for every $x \in \mathbb{T}^2$,

$$f_\alpha^{t+\ell}(x) \in U \quad \text{or} \quad g_\beta^{s+\ell}(x) \in U.$$

Now suppose that I and J are sufficiently small open neighbourhoods around p and q, respectively. Take $c_1 = W^s(b) \cup \{b, q\}$ and $c_2 = W^s(a) \cup \{a, q\}$. Clearly, for every $x \in \mathbb{T}^2 \setminus (c_1 \cup c_2)$, there exists $n \in \mathbb{N}$ so that $f_1^n(x) \in I$. Hence, for strips C_1 and C_2 around c_1 and c_2, respectively, there exists $n_0 \in \mathbb{N}$ so that for every x outside of these strips and every $n > n_0$, $f_1^n(x) \in I$. Notice that $n_0 \to \infty$ as $d_H(c_1, C_1) \to 0$ and $d_H(c_2, C_2) \to 0$.

It is not difficult to see that the IFS$(c_2; g_1|_{c_2}, g_2|_{c_2})$ is forward and backward minimal. Thus, by [8], the circle c_2 satisfies the deterministic chaos game for the IFS$(c_2; g_1|_{c_2}, g_2|_{c_2})$. By shrinking C_1 and C_2, if necessary, $d_H(c_1, C_1)$ and $d_H(c_2, C_2)$ are sufficiently close to zero. Thus, there is a finite sequence $(\beta_1 \ldots \beta_{s'})$ so that for every $x \in C_2$, $g^j_{\beta_1..\beta_{s'}}(x) \in J$ for some $j = 0, 1, \ldots, s'$. Moreover, we can choose the eigenvalue of f at points p and a so that $g^{s'}_{\beta_1..\beta_{s'}}(C_1) \cap (W^u(a) \cup \{a, p\}) = \emptyset$. Take $(\beta_{s'+1} \ldots \beta_{s''}) = \underbrace{(1 \ldots 1)}_{(s''-s')\text{-times}}$ so that $d_H(g^{s''-s'}_{\beta_{s'+1}..\beta_{s''}}(g^{s'}_{\beta_1..\beta_{s'}}(C_1)), c_1)$ is sufficiently close to zero.

Similarly, the IFS$(c_1; g_1|_{c_1}, g_3|_{c_1})$ is also forward and backward minimal. Hence, there is a finite sequence $(\beta_{s''+1} \ldots \beta_s)$ so that for every $x \in C_1$, $g^j_{\beta_{s''+1}..\beta_s}(x) \in J$ for some $j = 0, 1, \ldots, s - s''$. In particular, since $g^{s''-s'}_{\beta_{s'+1}..\beta_{s''}}(g^{s'}_{\beta_1..\beta_{s'}}(C_1))$ is sufficiently close to c_1 for every $y \in g^{s''-s'}_{\beta_{s'+1}..\beta_{s''}}(g^{s'}_{\beta_1..\beta_{s'}}(C_1))$, $g^j_{\beta_{s''+1}..\beta_s}(y) \in J$ for some $j = 0, 1, \ldots, s - s''$. Therefore, $g^j_{\beta_1..\beta_s}(x) \in J$ for every $x \in C_1 \cup C_2$ and for some $j = 0, 1, \ldots, s$. This completes the proof of our claim at (3.1).

By Theorem 1.3, $\mathbb{T}^2$ satisfies the deterministic chaos game for the IFS$(\mathbb{T}^2; f_1, f_2, f_3)$. Clearly, IFS$(\mathbb{T}^2; f_1, f_2, f_3)$ does not contain any minimal element and it is not a nonexpansive and equicontinuous IFS (by [14]). Also, it is not well-fibred (indeed, it suffices to consider a compact neighbourhood of a circle that contains p and the unstable manifold of one saddle).

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