CONGRUENCES FOR THE COEFFICIENTS OF MODULAR FORMS AND SOME NEW CONGRUENCES FOR THE PARTITION FUNCTION

MORRIS NEWMAN

If n is a non-negative integer, define $p_r(n)$ as the coefficient of x^n in

$$\prod_{n=1}^{\infty} (1 - x^n)^r;$$

otherwise define $p_r(n)$ as 0. In a recent paper **(2)** the author established the following congruence:

Let $r = 4, 6, 8, 10, 14, 26$. Let p be a prime greater than 3 such that $r(p + 1)/24$ is an integer, and set $\Delta = r(p^2 - 1)/24$. Then if $R \equiv r \pmod p$ and $n \equiv \Delta \pmod p$,

$$p_R(n) \equiv 0 \pmod p. \tag{1}$$

The choices $r = 4$, $p = 5$; $r = 6$, $p = 7$; and $r = 10$, $p = 11$ (all with $R = -1$) give the Ramanujan congruences

$$p(5n+4) \equiv 0 \pmod 5 \tag{2}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{3}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{4}$$

It is also possible to determine from (1) the Ramanujan congruence modulo 25.

In this paper we establish a congruence similar to (1) (Theorem 1). By appropriate specialization we obtain congruences of the Ramanujan type for $p(n)$ with modulus 13 (formulas (11), (12) and (14)). A significant difference emerges, however. The congruences (2), (3), (4) are statements concerning arithmetic progressions. The congruence (14) is valid for sequences which are essentially geometric progressions. Thus divisibility of $p(n)$ by 13 seems a rarer phenomenon than divisibility by 5, 7 or 11.

The author has shown **(3; 4)** that the following identity is valid:

Suppose that r is even, $0 < r \leqslant 24$. Let p be a prime greater than 3 such that $\delta = r(p - 1)/24$ is an integer. Then for all integral n

$$p_r(np + \delta) = p_r(n)p_r(\delta) - p^{\frac{1}{2}r-1}p_r\left(\frac{n - \delta}{p}\right). \tag{5}$$

Thus for $r \geqslant 4$,

$$p_r(np + \delta) \equiv p_r(n)p_r(\delta) \pmod p. \tag{6}$$

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We use this congruence to prove the following theorem:

THEOREM 1. Suppose that r is even, $4 \leqslant r \leqslant 24$. Let p be a prime greater than 3 such that $\delta = r(p-1)/24$ is an integer. Then if Q,n are integers and $R = Qp + r$,

$$p_R(np + \delta) \equiv p_r(\delta) p_{Q+r}(n) \pmod{p}. \tag{7}$$

Proof. We have

$$\sum_{n=0}^{\infty} p_R(n) x^n = \prod_{n=1}^{\infty} (1 - x^n)^R = \prod_{n=1}^{\infty} (1 - x^n)^{Qp+r}$$
$$\equiv \prod_{n=1}^{\infty} (1 - x^{np})^Q \prod_{n=1}^{\infty} (1 - x^n)^r \pmod{p}.$$

Thus comparing coefficients,

$$p_R(n) \equiv \sum_{0 \leqslant j \leqslant n/p} p_Q(j) p_r(n - pj) \qquad \pmod{p}.$$

Replace n by $np + \delta$. Since $\delta/p < 1$, j runs from 0 to n inclusive, and making use of (6) we obtain

$$p_R(np + \delta) \equiv p_r(\delta) \sum_{j=0}^{n} p_Q(j) p_r(n - j) \equiv p_r(\delta) p_{Q+r}(n) \quad \pmod{p},$$

which proves the theorem.

We now choose $p = 13$, $r = 12$, $R = -1$, $Q = -1$. Then (7) becomes

$$p(13n + 6) \equiv p_{12}(6) p_{11}(n) \equiv 11 p_{11}(n) \quad \pmod{13}. \tag{8}$$

Formula (8) requires a knowledge of $p_{12}(6)$, which may be found in **(5)**. This congruence seems first to have been given by Zuckerman **(7)**.

Similarly the choices $p = 13$, $r = 24$, $R = 11$, $Q = -1$; and $p = 13$, $r = 10$, $R = 23$, $Q = 1$ yield

$$p_{11}(13n + 12) \equiv p_{24}(12) p_{23}(n) \equiv 8 p_{23}(n) \qquad \pmod{13} \tag{9}$$

$$p_{23}(13n + 5) \equiv p_{10}(5) p_{11}(n) \equiv 4 p_{11}(n) \qquad \pmod{13}. \tag{10}$$

The number $p_{10}(5)$ is also given in **(5)**, and $p_{24}(12) = \tau(13)$ may be found, for example, in Watson's table of the τ-function **(6)**.

Congruences (8), (9), (10) may now be combined in obvious fashion to give a congruence involving $p(n)$ only.

THEOREM 2. For $n \equiv 6 \pmod{13}$,

$$p(13^2 n - 7) \equiv 6 p(n) \qquad \pmod{13}. \tag{11}$$

It is plain that this theorem implies that $p(n)$ is divisible by 13 infinitely often, since for example $p(84)$ is divisible by 13 and $84 = 13.6 + 6$. More precisely, define the sequence

$$t_0 = 13t + 6,$$
$$t_n = 13^2 t_{n-1} - 7, \quad n \geqslant 1.$$

Here t is an arbitrary integer. Replacing n by t_{n-1} in (11) we find

$$p(t_n) \equiv 6p(t_{n-1}) \pmod{13}$$

which upon iteration becomes

$$p(t_n) \equiv 6^n p(t_0) \pmod{13}.$$

We thus obtain

COROLLARY 1. Let t be an arbitrary integer. Put $a = 24t + 11$, $b = 13t + 6$. Let n be a non-negative integer and put

$$\Delta_n = \frac{13}{24}(13^{2n} - 1)$$

(Δ_n is an integer).

Then

$$p(a\Delta_n + b) \equiv 6^n p(b) \pmod{13}. \tag{12}$$

We are interested in determining when $p(b) \equiv 0 \pmod{13}$. Since $b \equiv 6 \pmod{13}$ and $p_{11}(n)$ is tabulated in **(5)**, congruence (8) may be used to determine the first few b's such that $p(b)$ is divisible by 13. We find in fact that this occurs for the following values:

(13)

t	a	b	t	a	b
6	155	84	57	1379	747
10	251	136	68	1643	890
17	419	227	69	1667	903
18	443	240	74	1787	968
24	587	318	90	2171	1176
27	659	357	95	2291	1241

We obtain therefore

COROLLARY 2. Let a, b have the values given in table (13). Then

$$p(a\Delta_n + b) \equiv 0 \pmod{13}. \tag{14}$$

Formula (14) is a new congruence of the Ramanujan type.

The size of the numbers Δ_n prevents checking (11), (12) or (14) by any existing table of $p(n)$ (Gupta's table of the partition function extends only to $n = 600$). Since the numbers t_n are all congruent to 6 modulo 13, formula (8) may be applied to the table of $p_{11}(n)$ for $1 \leqslant n \leqslant 800$ given in **(5)**. It was found that (11), (12) and (14) were verified for all values obtainable from this table.

The divisibility of $p(b)$ by 13 for the first 6 values of b given in table (13) may be checked by Gupta's table **(1)** which gives the residues of $p(n)$ modulo 13 and modulo 19 for $0 \leqslant n \leqslant 721$.

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