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On Lagrangian Catenoids

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Abstract. Recently I. Castro and F. Urbano introduced the Lagrangian catenoid. Topologically, it is $\mathbb{R} \times S^{n-1}$ and its induced metric is conformally flat, but not cylindrical. Their result is that if a Lagrangian minimal submanifold in $\mathbb{C}^n$ is foliated by round $(n-1)$-spheres, it is congruent to a Lagrangian catenoid. Here we study the question of conformally flat, minimal, Lagrangian submanifolds in $\mathbb{C}^n$. The general problem is formidable, but we first show that such a submanifold resembles a Lagrangian catenoid in that its Schouten tensor has an eigenvalue of multiplicity one. Then, restricting to the case of at most two eigenvalues, we show that the submanifold is either flat and totally geodesic or is homothetic to (a piece of) the Lagrangian catenoid.

1 Introduction

In 1744 Euler showed that a catenoid is a minimal surface of revolution and in 1785 Meusnier proved the converse that the catenoid is the only surface of revolution that is minimal [3, 11].

For hypersurfaces of dimension $n \geq 4$ in Euclidean space, Cartan [2] showed that a conformally flat hypersurface is quasi-umbilical, *i.e.,* the Weingarten map has an eigenvalue of multiplicity $\geq n-1$. Common examples, also due to Cartan, are the canal hypersurfaces, *i.e.,* envelopes of one-parameter families of hyperspheres. Thus conformal flatness can be viewed as a natural generalization of a surface of revolution.

In [1] the author showed that a conformally flat minimal hypersurface M^n, $n \geq 4$ of Euclidean space $\mathbb{R}^{n+1}$ is either totally geodesic or a hypersurface of revolution $S^{n-1} \times \gamma(s)$, where the profile curve γ is determined by its curvature as a function of arc length by $\kappa = (1-n)/u^n$ and

$$s = \int \frac{u^{n-1}\, du}{\sqrt{Cu^{2n-2} - 1}},$$

where C is a constant. For $n = 3$, replacing conformal flatness by quasi-umbilicity gives the same result with the same proof. For $n = 2$, the profile curve is a catenary and hence these hypersurfaces are called *generalized catenoids.* Jagy gave an independent study of this question by assuming that the minimal hypersurface is foliated by spheres from the outset [10].

Recently I. Castro and F. Urbano [4] (see also Castro [3]) introduced the Lagrangian catenoid. The manifold itself was introduced by Harvey and Lawson [9] as an example of a minimal Lagrangian submanifold and is defined by

$$M_0 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{C}^n : |x|y = |y|x,\ \Im(|x| + i|y|)^n = 1,\ |y| < |x| \tan(\pi/n)\}.$$

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Topologically M_0 is $\mathbb{R} \times S^{n-1}$. To describe it precisely, let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and view a point $p \in S^{n-1}$ as an n-tuple in $\mathbb{R}^n$ giving its coordinates. Define a map $\phi_0 \colon \mathbb{R} \times S^{n-1} \to \mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$ by

$$\phi_0(u, p) = \cosh^{1/n}(nu)e^{i\beta(u)}p,$$

where $\beta(u) = \frac{\pi}{2n} - \frac{2}{n}\arctan(\tanh\frac{nu}{2}) \in (0, \frac{\pi}{n})$ and the multiplication $e^{i\beta}p$ multiplies each coordinate of p by $e^{i\beta}$ and lists the real and imaginary parts as a $2n$-tuple in $\mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$. Let g_0 be the standard metric of constant curvature 1 on S^{n-1}; then the metric induced on $\mathbb{R} \times S^{n-1}$ by ϕ_0 is $ds^2 = \cosh^{2/n}(nu)(du^2 + g_0)$, which is clearly conformally flat. This Lagrangian submanifold defined by the mapping $\phi_0 \colon \mathbb{R} \times S^{n-1} \to \mathbb{C}^n$, together with its induced metric, is known as the *Lagrangian catenoid*. The second fundamental form is given in terms of the Weingarten maps as follows. Let $\{e_1, \dots, e_n\}$ be a local orthonormal basis with e_1 in the u-direction and let A_i denote the Weingarten map corresponding to the normal Je_i. Then the matrices of the Weingarten maps with respect to this basis are the following:

$$A_1 = \begin{pmatrix} -(n-1)a & & & \bigcirc \\ & a & & \\ & & \ddots & \\ \bigcirc & & & a, \end{pmatrix} \quad A_i = \begin{pmatrix} 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & & & & & \\ a & & & \bigcirc & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix},$$

where $a = \cosh^{-(1+\frac{1}{n})}(nu)$.

The main result of Castro and Urbano [4] is the following.

Theorem 1 *Let $\phi \colon M^n \to \mathbb{C}^n$ be a minimal (non-flat) Lagrangian immersion. Then M^n is foliated by pieces of round $(n-1)$-spheres in $\mathbb{C}^n$ if and only if ϕ is congruent (up to dilations) to an open subset of the Lagrangian catenoid.*

In view of the conformal flatness of the Lagrangian catenoid and the author's result on conformally flat minimal hypersurfaces in $\mathbb{R}^{n+1}$, it is natural to ask if, aside from the flat totally geodesic case, the Lagrangian catenoids are the only conformally flat minimal Lagrangian submanifolds in $\mathbb{C}^n$. We first show in Proposition 4 that the Schouten tensor of such a submanifold has an eigenvalue of multiplicity 1. Then, noting that the Schouten tensor of the Lagrangian catenoid has only two eigenvalues, we restrict ourselves to the case that Schouten tensor, equivalently the Ricci tensor, has at most two eigenvalues. In particular we prove the following theorem.

Theorem 2 *Let M^n, $n \geq 4$, be a conformally flat, minimal, Lagrangian submanifold of $\mathbb{C}^n$. If the Schouten tensor has at most two eigenvalues, then either M^n is flat and totally geodesic or is homothetic to (a piece of) the Lagrangian catenoid.*

We comment that our proof is not a reduction to the result of Castro and Urbano. Even when one is at the stage of having essentially a warped product of a 1-dimensional manifold and a space of constant curvature, it does not follow immediately that the spaces of constant curvature which now have codimension $n+1$ are round spheres in $\mathbb{C}^n$. Instead we establish both the first and second fundamental forms of M^n.

We close the paper with a remark showing that the Lagrangian catenoid is not a quasi-umbilical submanifold.

2 Preliminaries

Let $(x_1 + iy_1, \dots, x_n + iy_n)$ be the coordinates on $\mathbb{C}^n$ and J the almost complex structure. An n-dimensional submanifold M^n of $\mathbb{C}^n$ is said to be *Lagrangian* if the restriction of the canonical symplectic form $\Omega = \sum_{i=1}^n dx_i \wedge dy_i$ to M^n vanishes. Note also that if a vector X is tangent to M^n, JX is normal.

For an isometrically immersed submanifold (M, g) of $(\mathbb{C}^n, \langle\, ,\rangle)$ the Levi–Civita connection ∇ of g and the second fundamental form σ are related to the ambient Levi–Civita connection $\overline{\nabla}$ by $\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$. For a normal vector field ζ, let A_ζ denote the corresponding Weingarten map and let D denote the connection in the normal bundle; in particular A_ζ and D are defined by $\overline{\nabla}_X \zeta = -A_\zeta X + D_X \zeta$. The Gauss equation is

$$R(X, Y, Z, W) = \langle \sigma(Y, Z), \sigma(X, W)\rangle - \langle \sigma(X, Z), \sigma(Y, W)\rangle.$$

Defining the covariant derivative of σ by $(\nabla'\sigma)(X, Y, Z) = D_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$, the Codazzi equation is

$$(R_{XY}Z)^\perp = (\nabla'\sigma)(X, Y, Z) - (\nabla'\sigma)(Y, X, Z).$$

Now for a Lagrangian submanifold, the almost complex structure J of the Kähler manifold $\mathbb{C}^n$ is an isometry between the tangent bundle and the normal bundle, and hence the equation of Ricci–Kühne, $R^\perp(X, Y, \xi, \zeta) = g([A_\xi, A_\zeta]X, Y)$, gives no new information; in particular we can find the commutators of the Weingarten maps directly.

Also for a Lagrangian submanifold of a Kähler manifold with local orthonormal basis $\{e_1, \dots, e_n\}$, let A_i denote the Weingarten map corresponding to the normal Je_i. Then we have from

$$-A_i e_j + D_{e_j} Je_i = \bar{\nabla}_{e_j} Je_i = J\bar{\nabla}_{e_j} e_i = J\nabla_{e_j} e_i + J\sigma(e_i, e_j)$$

and the symmetry of the second fundamental form that

$$A_i e_j = -J\sigma(e_i, e_j) = A_j e_i,$$

an important relation that we will use frequently.

In the course of our work we will use several properties of Codazzi tensors as developed by Derdziński [7]. A symmetric tensor field L of type $(1, 1)$ is a Codazzi tensor if it satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$. Derdziński proves the following Lemma.

Lemma 3 *If a Codazzi tensor has more than one eigenvalue, then the eigenspaces for each eigenvalue form an integrable subbundle on open sets of constant multiplicity. If an eigenvalue has multiplicity greater than* 1*, then the eigenvalue is constant on its integral submanifolds. Moreover the integral submanifolds are umbilical submanifolds, and if the eigenvalue is constant on the manifold then the integral submanifolds are totally geodesic.*

As a matter of notation, for a Riemannian metric g, let Q denote its Ricci operator and τ its scalar curvature.

Recall that if the Weyl conformal curvature tensor of a Riemannian manifold M^n vanishes, the curvature tensor is given in terms of the Schouten tensor

$$L = -\frac{Q}{n-2} + \frac{\tau}{2(n-1)(n-2)} I$$

by

$$\begin{aligned} g(R_{XY}Z, V) = -g(LX, V)g(Y, Z) + g(LX, Z)g(Y, V) \\ - g(LY, Z)g(X, V) + g(LY, V)g(X, Z). \end{aligned}$$

It is well known that for $n \geq 4$, M^n is conformally flat if and only if the Weyl conformal curvature tensor vanishes, and that this implies that L is a Codazzi tensor. For $n = 3$, the Weyl conformal curvature tensor vanishes identically, and the manifold is conformally flat if and only if the Schouten tensor is a Codazzi tensor.

3 Conformally Flat, Minimal, Lagrangian Submanifolds

In this section we develop to some extent the theory of conformally flat, minimal, Lagrangian submanifolds in general and specifically prove our theorem on the case of the Schouten tensor having two eigenvalues.

Let $\{e_1, \dots, e_n\}$ be a local orthonormal eigenvector basis of L and let A_i denote the Weingarten map corresponding to the normal Je_i. Contracting the Gauss equation and using the minimality, we see that the Ricci operator and the scalar curvature are given by

$$Q = -\sum_i A_i^2, \quad \tau = -|\sigma|^2, \tag{3.1}$$

and hence the Schouten tensor becomes

$$L = \frac{1}{n-2} \sum_i A_i^2 - \frac{|\sigma|^2}{2(n-1)(n-2)} I. \tag{3.2}$$

We also note that

$$\operatorname{tr} L = \frac{-\tau}{2(n-1)} = \frac{|\sigma|^2}{2(n-1)}.$$

We will approach our work by studying the eigenvalues of L. We will assume that not all the eigenvalues are equal on any neighborhood, for if they were, the submanifold would be of constant curvature. Ejiri proved that a minimal Lagrangian submanifold of constant curvature in a complex space form must be totally geodesic or flat [8]. Thus with the ambient space being $\mathbb{C}^n$ we have both these properties, giving the trivial case of the theorem.

Let ν_j denote the j-th eigenvalue of L and recall that for a Lagrangian submanifold $A_i e_j = A_j e_i$. Then computing $g(Le_j, e_m)$, we see that

$$\nu_j \delta_{jm} = \frac{1}{n-2} \operatorname{tr} A_j A_m - \frac{|\sigma|^2}{2(n-1)(n-2)} \delta_{jm}.$$

Consequently we have that $\operatorname{tr} A_j A_m = 0$ for $j \neq m$ and that the eigenvalues of L are given by

$$\nu_j = \frac{1}{n-2} \operatorname{tr} A_j^2 - \frac{|\sigma|^2}{2(n-1)(n-2)}.$$

Denote by $\mathcal{V}_{\nu_j}$ the eigenspace of ν_j. If $X, V \in \mathcal{V}_{\nu_j}$ and $Y, Z \in \mathcal{V}_{\nu_k}$, direct computation of the curvature gives $g(R_{XY}Z, V) = -(\nu_j + \nu_k) g(X, V) g(Y, Z)$. On the other hand, by the Gauss equation,

$$\begin{aligned} g(R_{e_j e_k} e_l, e_m) &= \sum_i g(A_k e_l, e_i) g(A_j e_m, e_i) - \sum_i g(A_j e_l, e_i) g(A_k e_m, e_i) \\ &= g(A_k e_l, A_j e_m) - g(A_j e_l, A_k e_m) \\ &= g([A_j, A_k] e_l, e_m). \end{aligned}$$

From the form of the curvature tensor of a conformally flat space, this is zero for j, k, l, m distinct. Similarly, for j, k, m distinct and $k = l$,

$$\begin{aligned} g([A_j, A_k] e_k, e_m) &= -\frac{1}{n-2} \operatorname{tr} A_j A_m = 0, \\ g([A_j, A_k] e_k, e_j) &= g(R_{e_j e_k} e_k, e_j) = -(\nu_j + \nu_k). \end{aligned}$$

Note that $g([A_j, A_k] e_k, e_j)$ is the (j, k)-component of the matrix of $[A_j, A_k]$. Thus, the only non-zero components of $[A_j, A_k]$ are the (j, k) and (k, j) components, and if $\{\omega^1, \dots, \omega^n\}$ is the dual basis of $\{e_1, \dots, e_n\}$,

$$[A_j, A_k] = (\nu_j + \nu_k)(\omega^j \otimes e_k - \omega^k \otimes e_j). \tag{3.3}$$

We next make the simple observation that from the minimality $g(\sum_k A_k e_k, e_l) = \sum_k g(e_k, A_k e_l) = \sum_k g(e_k, A_l e_k) = \operatorname{tr} A_l = 0$, and hence

$$\sum_k A_k e_k = 0. \tag{3.4}$$

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We now compute the commutator of L and A_j. From equation (3.2) we have

$$LA_j - A_jL = \frac{1}{n-2}\sum_i (A_i^2 A_j - A_j A_i^2) = \frac{1}{n-2}\sum_i \big(A_i[A_i, A_j] + [A_i, A_j]A_i\big).$$

Using (3.3), this gives

$$(LA_j - A_jL)e_k = \frac{-1}{n-2}\sum_i \Big(A_i(\nu_i + \nu_j)(\omega^j \otimes e_i - \omega^i \otimes e_j)e_k + (\nu_i + \nu_j)(\omega^j \otimes e_i - \omega^i \otimes e_j)A_i e_k\Big).$$

Noting that $\sum_i g(A_i e_k, e_j)e_i = \sum_i g(A_k e_i, e_j)e_i = \sum_i g(A_k e_j, e_i)e_i = A_k e_j$ and using (3.4),

$$(LA_j - A_jL)e_k = \frac{-1}{n-2}\Big(\delta_{jk}\sum_i \nu_i A_i e_i - \nu_k A_k e_j + \sum_i \nu_i g(A_i e_k, e_j)e_i - \sum_i \nu_i g(A_i e_k, e_i)e_j\Big).$$

Taking the inner product of both sides with e_m and evaluating, we have

$$\begin{aligned} g((LA_j - A_jL)e_k, e_m) &= (\nu_m - \nu_k)g(A_j e_k, e_m) \\ &= \frac{-1}{n-2}\Big(\delta_{jk}\sum_i \nu_i g(A_i e_i, e_m) \\ &\qquad - \nu_k g(A_k e_j, e_m) + \nu_m g(A_m e_k, e_j) - \delta_{jm}\sum_i \nu_i g(A_i e_k, e_i)\Big) \\ &= -\frac{\nu_m - \nu_k}{n-2} g(A_k e_j, e_m) \\ &\qquad - \frac{1}{n-2}\Big(\delta_{jk}\sum_i \nu_i g(A_i e_i, e_m) - \delta_{jm}\sum_i \nu_i g(A_i e_i, e_k)\Big). \end{aligned}$$

In particular,

$$(n-1)(\nu_m - \nu_k)g(A_j e_k, e_m) = \delta_{jm}\sum_i \nu_i g(A_i e_i, e_k) - \delta_{jk}\sum_i \nu_i g(A_i e_i, e_m).$$

Thus for j, k, m distinct,

$$(\nu_m - \nu_k)g(A_j e_k, e_m) = 0, \tag{3.5}$$

and for $j = k \neq m$,

$$(n-1)(\nu_m - \nu_j)g(A_j e_j, e_m) = -\sum_i \nu_i g(A_i e_i, e_m) = (n-1)(\nu_m - \nu_l)g(A_l e_l, e_m) \tag{3.6}$$

for $l \neq m$.

We now prove the following proposition, which does not use the hypothesis on the number of eigenvalues of the Schouten tensor.

Proposition 4 *Let M^n be a conformally flat, minimal, Lagrangian submanifold of $\mathbb{C}^n$. Then L has an eigenvalue of multiplicity 1.*

Proof From equation (3.6) if $\nu_m = \nu_j$, then $g\big(\sum_i \nu_i A_i e_i, e_m\big) = 0$, and hence $\sum_i \nu_i A_i e_i$ is orthogonal to each eigenspace corresponding to an eigenvalue of multiplicity > 1. Therefore either $\sum_i \nu_i A_i e_i = 0$ or there exists an eigenvalue of multiplicity 1.

Suppose now that L has no eigenvalue of multiplicity 1. Order the $e_1, \dots, e_n$ corresponding to the multiplicity of eigenvalues of L, *i.e.*, $e_1, \dots, e_p$ correspond to ν_1 having multiplicity p, etc., and consider the corresponding block decomposition of a Weingarten map. Using $\sum_i \nu_i A_i e_i = 0$ and (3.6) with ν_m and ν_j corresponding to different eigenspaces, we see that $g(A_j e_j, e_m) = 0$. This together with equation (3.5) shows that the off-diagonal blocks of the block decomposition are zero. Also from $g(A_j e_j, e_m) = 0$ we have $g(A_m e_j, e_j) = 0$ and hence in the matrix of any A_m, the diagonal blocks not corresponding to ν_m have zeros along their diagonals. Now from the property (3.3) of $[A_j, A_k]$, we see that the sum of any two distinct eigenvalues is zero and hence there exist at most two distinct eigenvalues, one being the negative of the other, say ν and $-\nu$. From Lemma 3, we see that ν is constant on M and hence the leaves of both foliations are totally geodesic submanifolds with curvatures of opposite sign. Now from $\sum_i \nu_i A_i e_i = 0$, we see that $\sum_{i=1}^{p} A_i e_i = \sum_{i=p+1}^{n} A_i e_i$, but the sum on the left is tangent to one foliation and the sum on the right is tangent to the other, and hence both sides vanish. Therefore the leaves of positive curvature are minimal submanifolds in $\mathbb{C}^n$, a contradiction. ∎

For the theorem, since we now know that L has an eigenvalue of multiplicity 1, say ν_1, the other eigenvalue, ν_2, has multiplicity $n-1$. We show next that relative to the basis $\{e_i\}$, the Weingarten map A_1 is given by a diagonal matrix.

Lemma 5 *The matrix of A_1 is diagonal. Moreover, setting $\Phi = \frac{-1}{n-1} g\big(\sum_i \nu_i A_i e_i, e_1\big)$, the j-th diagonal entry for $j > 1$ is $\frac{\Phi}{\nu_1 - \nu_2}$ and the first entry is $-\frac{(n-1)\Phi}{\nu_1 - \nu_2}$.*

Proof We first show that the lower right-hand $(n-1) \times (n-1)$ block of A_1 is diagonal. From equation (3.5) and the general property $A_i e_j = A_j e_i$, we have

$$(\nu_1 - \nu_k) g(A_1 e_j, e_k) = 0$$

for $j \neq k$ and $j, k \neq 1$, giving the diagonal form of the lower right-hand block. From equation (3.6)

$$(n-1)(\nu_1 - \nu_j) g(A_1 e_j, e_j) = -g(\sum_i \nu_i A_i e_i, e_1),$$

giving the diagonal entry. In the proof of Proposition 4 we saw that $\sum_i \nu_i A_i e_i$ is orthogonal to each eigenspace corresponding to an eigenvalue of multiplicity > 1. Therefore from equation (3.6) we see that $(\nu_m - \nu_1)g(A_1 e_1, e_m) = 0$. Therefore the matrix of A_1 is diagonal. Finally by the minimality,

$$g(A_1 e_1, e_1) = -\frac{(n-1)\Phi}{\nu_1 - \nu_2}. \qquad \blacksquare$$

Proof of Theorem 2 With A_1 diagonal and $A_1 e_2 = A_2 e_1$ we can easily compute the $(1,2)$ component of $[A_1, A_2]$; comparing with (3.3) we have two cases.

Case I $\qquad \nu_1 + \nu_2 = \dfrac{n\Phi^2}{(\nu_1 - \nu_2)^2} > 0,$

Case II $\qquad \Phi = 0(\nu_1 + \nu_2 = 0).$

If for a conformally flat manifold, the Schouten tensor has an eigenvalue ν_1 of multiplicity 1 and a second eigenvalue ν_2 of multiplicity $n-1$, then by Lemma 3, the eigenspaces of ν_2 form an integrable subbundle and ν_2 is constant along the integral submanifolds. Moreover the integral submanifolds are umbilical in M^n. This and the fact that $n - 1 \geq 3$ imply that the integral submanifolds are of constant curvature, and hence we can write the metric in the form

$$ds^2 = e^{2f(u_1)}\left(du_1^2 + \frac{du_2^2 + \cdots + du_n^2}{\left(1 + \frac{\varepsilon}{4}\sum_{i=2}^n u_i^2\right)^2}\right), \tag{3.7}$$

where $\varepsilon = 1, -1, 0$ according as the submanifolds $u_1 = \text{const}$ have positive, negative or zero constant curvature. With respect to the orthonormal basis

$$e_1 = e^{-f}\frac{\partial}{\partial u_1} \quad \text{and} \quad e_j = e^{-f}\left(1 + \frac{\varepsilon}{4}\sum_{i=2}^n u_i^2\right)\frac{\partial}{\partial u_j}, \; j > 1,$$

the Levi–Civita connection is given as follows, where $i, j > 1$, $i \neq j$:

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_j = 0, \quad \nabla_{e_i} e_1 = (e_1 f)e_i,$$
$$\nabla_{e_i} e_i = -(e_1 f)e_1 + e^{-f}\frac{\varepsilon}{2}\sum_{l \neq 1, i} u_l e_l, \quad \nabla_{e_i} e_j = -e^{-f}\frac{\varepsilon}{2}u_j e_i.$$

The computation of the curvature is now straightforward, and $\{e_1, \ldots, e_n\}$ is an eigenvector basis of L.

Now using the fact that L is a Codazzi tensor, we have $(\nabla_{e_1} L)e_i - (\nabla_{e_i} L)e_1 = 0$; expanding and taking inner products with $e_1, e_i, i \neq 1$, we have

$$(\nu_2 - \nu_1)g(\nabla_{e_1} e_i, e_1) - e_i \nu_1 = 0, \quad e_1 \nu_2 + (\nu_2 - \nu_1)g(\nabla_{e_i} e_1, e_i) = 0.$$

Continuing by direct use of the Levi–Civita connection of the metric (3.7), we see first that since the integral curves of e_1 are geodesics, $e_i \nu_1 = 0$, and hence that ν_1

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is a function of u_1 alone. Secondly, since $\nabla_{e_i} e_1 = (e_1 f) e_i$, $i > 1$, we have $e_1 \nu_2 = (e_1 f)(\nu_1 - \nu_2)$, or

$$\nu_2' = (\nu_1 - \nu_2) f', \tag{3.8}$$

where the prime denotes differentiation with respect to u_1. Computing the curvature, we see that $\nu_1 + \nu_2 = -g(R_{e_1 e_2} e_2, e_1) = e_1 e_1 f + (e_1 f)^2$, giving

$$\nu_1 + \nu_2 = e^{-2f} f''. \tag{3.9}$$

We now use the Codazzi equation

$$g((\nabla_{e_j} A_i) e_k, e_m) - g(A_k e_m, \nabla_{e_j} e_i) = g((\nabla_{e_k} A_i) e_j, e_m) - g(A_j e_m, \nabla_{e_k} e_i). \tag{3.10}$$

Taking $i = j = 1$, $k = m = 2$ and expanding, we have

$$g\Big(\nabla_{e_1} \frac{\Phi}{\nu_1 - \nu_2} e_2, e_2\Big) = -(n-1) \frac{\Phi}{\nu_1 - \nu_2} g((e_1 f) e_2, e_2) - 2g(A_1 (e_1 f) e_2, e_2),$$

and simplifying,

$$e_1 \frac{\Phi}{\nu_1 - \nu_2} = -(n+1) \frac{\Phi}{\nu_1 - \nu_2} e_1 f. \tag{3.11}$$

From the statement of Case I,

$$\frac{\Phi}{\nu_1 - \nu_2} = \sqrt{\frac{\nu_1 + \nu_2}{n}},$$

from which follows

$$e_1 \frac{\Phi}{\nu_1 - \nu_2} = \frac{1}{2n} \Big(\frac{\Phi}{\nu_1 - \nu_2}\Big)^{-1} (e_1 \nu_1 + (\nu_1 - \nu_2) e_1 f).$$

Comparing with (3.11), we have $-(n+1)(\nu_1 + \nu_2) e_1 f = \frac{1}{2}(e_1 \nu_1 + (\nu_1 - \nu_2) e_1 f)$, and simplifying,

$$\nu_1' = -((2n+3)\nu_1 + (2n+1)\nu_2) f'. \tag{3.12}$$

Our task will be to solve equations (3.8), (3.9), and (3.12). Differentiating (3.9) and comparing with the sum of (3.8) and (3.12), we obtain $f''' = -2n f'' f'$. Let $v = f'$ and $w = v'$. Then $w \frac{dw}{dv} = -2nvw$, from which follows either

Case Ia $$w = -nv^2 + C_1,$$

where C_1 is the constant of integration, or

Case Ib $$w = 0.$$

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Integrating the first case:

$$u_1 = \int \frac{dv}{C_1 - nv^2}. \tag{3.13}$$

A second constant of integration is unnecessary as it represents a change of position of the origin in the u_1 variable.

We remark at this stage that if $C_1 = n$, then $v = \tanh(nu_1)$ and therefore

$$f = \frac{1}{n} \ln \cosh(nu_1)$$

giving the Lagrangian catenoid; note again that another constant of integration is unnecessary as it would give a homothetic change of metric and the problem is homothetic invariant.

Of course we must deal with a general C_1 which we now do in cases by sign: $C_1 > 0, C_1 < 0, C_1 = 0$.

If $C_1 > 0$,

$$u_1 = \frac{1}{C_1} \int \frac{dv}{1 - \frac{n}{C_1} v^2} = \begin{cases} \frac{1}{\sqrt{C_1 n}} \tanh^{-1} \sqrt{\frac{n}{C_1}} v, & |v| < \sqrt{\frac{n}{C_1}}, \\ \frac{1}{\sqrt{C_1 n}} \coth^{-1} \sqrt{\frac{n}{C_1}} v, & |v| > \sqrt{\frac{n}{C_1}}, \end{cases}$$

where again we may take the constant of integration equal to zero. From these we see that

$$f = \ln \cosh^{\frac{1}{n}} \left(\sqrt{C_1 n}\, u_1 \right) \quad \text{or} \quad f = \ln \sinh^{\frac{1}{n}} \left(\sqrt{C_1 n}\, u_1 \right).$$

For the first of these we have from $-2\nu_2 = g(R_{e_i e_j} e_j, e_i) = e^{-2f} - (e_1 f)^2$, $i, j > 1$, that

$$\nu_2 = -\frac{C_1}{2n} \cosh^{-2-\frac{2}{n}} \left(\sqrt{C_1 n}\, u_1 \right) + \frac{C_1 - n}{2n} \cosh^{\frac{-2}{n}} \left(\sqrt{C_1 n}\, u_1 \right),$$

and from (3.8),

$$\nu_1 = \nu_2 + \frac{\nu_2}{f'} = \frac{(2n+1)C_1}{2n} \cosh^{-2-\frac{2}{n}} \left(\sqrt{C_1 n}\, u_1 \right) - \frac{C_1 - n}{2n} \cosh^{\frac{-2}{n}} \left(\sqrt{C_1 n}\, u_1 \right).$$

We noted the Ricci operator (3.1) at the outset and therefore

$$\begin{aligned} g(Qe_2, e_2) &= -g\Big(\sum_{i=1}^{n} A_i^2 e_2, e_2 \Big) = -\sum_{i=1}^{n} g(A_2^2 e_i, e_i) = -\operatorname{tr} A_2^2 = -2\frac{\Phi^2}{(\nu_1 - \nu_2)^2} \\ &= -2\frac{\nu_1 + \nu_2}{n} = -\frac{2C_1}{n} \cosh^{-2-\frac{2}{n}} \left(\sqrt{C_1 n}\, u_1 \right). \end{aligned}$$

On the other hand, we can compute $g(Qe_2, e_2)$ intrinsically:

$$\begin{aligned} g(Qe_2, e_2) &= (n-2)e^{ef} - e_1 e_1 f - (n-1)(e_1 f)^2 \\ &= -\frac{2C_1}{n} \cosh^{-2-\frac{2}{n}} \left(\sqrt{C_1 n}\, u_1 \right) + \frac{(n-2)(n-C_1)}{n} \cosh^{-\frac{2}{n}} \left(\sqrt{C_1 n}\, u_1 \right). \end{aligned}$$

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Comparing, we have that $C_1 = n$, the case of the Lagrangian catenoid.

To complete the analysis of the case $C_1 > 0$, we treat $f = \ln \sinh^{\frac{1}{n}}\left(\sqrt{C_1 n}\, u_1\right)$. Here we have from equation (3.9),

$$\nu_1 + \nu_2 = e^{-2f} f'' = -C_1 \sinh^{-2-\frac{2}{n}}(\sqrt{C_1 n}\, u_1) < 0,$$

contradicting the fact that in Case I $\nu_1 + \nu_2 > 0$.

We now turn to the case $C_1 < 0$; let $C = -C_1$. Equation (3.13) now takes the form

$$u_1 = -\int \frac{dv}{C + nv^2} = -\frac{1}{\sqrt{Cn}} \tan^{-1} \sqrt{\frac{n}{c}}\, v.$$

From this we obtain $f = \ln \cos^{1/n}\left(\sqrt{Cn}\, u_1\right)$, and $\nu_1 + \nu_2$ becomes

$$\nu_1 + \nu_2 = e^{-2f} f'' = -C \cos^{-2-\frac{2}{n}}\left(\sqrt{Cn}\, u_1\right) < 0.$$

Again this contradicts $\nu_1 + \nu_2 > 0$.

Finally for $C_1 = 0$, $v' = -nv^2$ from which $f(u_1) = \frac{1}{n}\ln(nu_1)$. From this

$$\nu_1 + \nu_2 = -\frac{1}{n}(u_1)^{-2-\frac{2}{n}} < 0,$$

a contradiction.

Thus we have shown that in Case Ia, M^n is (to within homothety) locally isometric to the Lagrangian catenoid. To show that the second fundamental form agrees with that of the Lagrangian catenoid, it remains to show that for the Weingarten maps, $A_k, k \geq 2$, the lower right-hand $(n-1)\times(n-1)$ blocks vanish. This now is immediate from $\tau = -|\sigma|^2$, equation (3.1). The left side can be computed intrinsically from the curvature of (3.7) with the now known function f. The terms in the right-hand side coming from A_1 and the first columns and rows of the other A_k contribute the same as τ on the left. Thus the lower right-hand $(n-1) \times (n-1)$ blocks vanish.

Turning to Case Ib and Case II, Case Ib leads immediately to $f'' = 0$, and therefore by equation (3.9) to $\nu_1 + \nu_2 = 0$, and hence these two cases are the same. Thus we have $\Phi = 0$, and hence $A_1 = 0$. Since $f'' = 0$, f is linear, say $f = au_1$, a second constant of integration being unnecessary as before. An intrinsic computation yields $\nu_2 = \frac{a^2+1}{2} e^{-2au_1}$. Using the Codazzi equation (3.10) with index $k = 1$, we get $-ag(A_i e_j, e_m) = \frac{\partial}{\partial u_1} g(A_i e_j, e_m)$, and therefore each $g(A_i e_j, e_m)$ is e^{-au_1} times a function of $u_2, \dots, u_n$.

Now since $A_1 = 0$, the first row and column of A_2 vanish and hence L and A_2 commute. We may therefore choose $e_2, \dots, e_n$ such that A_2 is diagonal, say e^{-au_1} times a diagonal matrix with entries b_{kk}, b_{11} being zero. The matrix of any other A_k is e^{-au_1} times a matrix whose first row and column vanish and whose second row and column have b_{kk} as the only possible non-zero entry. Now compute the $(2, k)$ entry of $[A_2, A_k]$ and use (3.3) to obtain $b_{22}b_{kk} - b_{kk}^2 = a^2 + 1$. This is a quadratic in b_{kk} and we obtain $b_{kk} = (b_{22} \pm D)/2$, where $D = \sqrt{b_{22}^2 - 4(a^2+1)}$. However the submanifold is minimal and hence the trace of A_2 vanishes. Therefore

$$b_{22} + \frac{(n-2)}{2} b_{22} + \frac{1}{2}(\text{a sum, difference of } (n-2) \text{ copies of } D) = 0.$$

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Rearranging we have

$$nb_{22} = \text{a sum, difference of } (n-2) \text{ copies of } D,$$

which, since $D < |b_{22}|$, is impossible. ∎

4 The Lagrangian Catenoid Is Not Quasi-Umbilical

We close with a remark showing that the Lagrangian catenoid is not a quasi-umbilical submanifold. It is known [6] that a quasi-umbilical submanifold of dimension ≥ 4 of a conformally flat space is conformally flat, but in general not conversely. Generalizing the result of Cartan mentioned in the introduction, Moore and Morvan [12] showed that a conformally flat submanifold M^n of Euclidean space E^{n+p} is quasi-umbilical if $p \leq \min\{4, n-3\}$. Chen and Verstraelen [5] showed that an n-dimensional submanifold of a conformally flat space of dimension $(n+p)$ with flat normal connection is quasi-umbilical for $p \leq n-3$. However here in the case of Lagrangian submanifolds, the codimension is n. We begin by recalling the definition of quasi-umbilicity.

Consider an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold. A (local) normal vector field is a *quasi-umbilical section* of the normal bundle if the corresponding Weingarten map has at least $n-1$ eigenvalues equal. The submanifold is said to be *quasi-umbilical* if there exist p mutually orthogonal quasi-umbilical normal sections.

We have already noted the Weingarten maps of the Lagrangian catenoid with respect to the normal fields Je_i, namely

$$A_1 = \begin{pmatrix} -(n-1)a & & & \bigcirc \\ & a & & \\ & & \ddots & \\ \bigcirc & & & a \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & & & & & \\ a & & & \bigcirc & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix},$$

where $a = \cosh^{-(1+\frac{1}{n})}(nu)$.

Suppose now that $\zeta_i = \sum_j \alpha_{ij} Je_j$ is an orthonormal basis of normal vectors with respect to which the Weingarten maps B_i have $n-1$ eigenvalues equal. Then from the Weingarten equation $\overline{\nabla}_X \zeta_i = -B_i X + D_X \zeta_i$ we have

$$-B_i X + D_X \zeta_i = \sum_j (X\alpha_{ij}) Je_j + \sum_j \alpha_{ij}(-A_j X + D_X Je_j).$$

Thus the matrix of B_i is

$$\begin{pmatrix} -(n-1)a\alpha_{i1} & a\alpha_{i2} & \cdots & a\alpha_{in} \\ a\alpha_{i2} & a\alpha_{i1} & & \bigcirc \\ \vdots & & \ddots & \\ a\alpha_{in} & \bigcirc & & a\alpha_{i1} \end{pmatrix}.$$

We now compute the characteristic polynomial $P(\lambda)$ of this matrix. Since (α_{ij}) is an orthognal matrix, $(a\alpha_{i1})^2 + \cdots + (a\alpha_{in})^2 = a^2$, and we find that

$$P(\lambda) = (a\alpha_{i1} - \lambda)^{n-2}\big[\lambda^2 + (n-2)a\alpha_{i1}\lambda - (n-2)a^2\alpha_{i1}^2 - a^2\big].$$

The zeros of the quadratic factor are

$$\frac{-(n-2)a\alpha_{i1} \pm \sqrt{(n^2-4)a^2\alpha_{i1}^2 + 4a^2}}{2}.$$

If now the eigenvalue $a\alpha_{i1}$ has multiplicity $n-1$, we have

$$na\alpha_{i1} = \pm\sqrt{(n^2-4)a^2\alpha_{i1}^2 + 4a^2}.$$

Squaring gives $4a^2\alpha_{i1}^2 = 4a^2$ or $\alpha_{i1}^2 = 1$, thus each entry in the first column of (α_{ij}) is ± 1, contradicting its orthogonality.

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