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EXACT SOLUTION OF A DIFFERENCE APPROXIMATION TO DUFFING'S EQUATION

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Abstract

Duffing's equation, in its simplest form, can be approximated by various non-linear difference equations. It is shown that a particular choice can be solved in closed form giving periodic solutions.

1. Introduction

In contrast to the extensive research which has been undertaken on non-linear differential equations, little work has been done on non-linear difference equations. In particular, the few such difference equations which have been solved in closed form tend to be somewhat artificial and unrelated to applications.

With the recent interest in representation of non-linear phenomena [2], the question arises whether any of the classical non-linear differential equations can be approximated by appropriate non-linear difference equations for which solutions in closed form are possible.

The classical non-linear differential equation considered in this paper is the simple Duffing's equation [3],

$$\ddot{x}(t) + ax(t) + bx^3(t) = 0, \tag{1.1}$$

which describes, for example, the undamped unforced vibrations of an anharmonic oscillator, of a "hard" or "soft" spring, or of a simple pendulum. The well-known analytic solutions of (1.1) in terms of the Jacobian functions are summarized in the next section.

In approximating (1.1) by a difference equation using a discrete time interval $h > 0$,

$$t \text{ is replaced by } nh, \text{ with } n \text{ an integer,} \tag{1.2}$$

$$x(t) \text{ is replaced by } x_n \tag{1.3}$$

and

$$\ddot{x}(t) \text{ is replaced by } h^{-2}(x_{n+1} - 2x_n + x_{n-1}). \tag{1.4}$$

But how should the non-linear term $x^3(t)$ be replaced? Possible choices are the obvious one x_n^3, or perhaps the more symmetrical product form $x_{n-1}x_nx_{n+1}$. However it is the purpose of this paper to show that if

$$x^3(t) \text{ is replaced by } \tfrac{1}{2}x_n^2(x_{n+1} + x_{n-1}), \tag{1.5}$$

then the resulting difference equation

$$h^{-2}(x_{n+1} - 2x_n + x_{n-1}) + ax_n + \tfrac{1}{2}bx_n^2(x_{n+1} + x_{n-1}) = 0, \tag{1.6}$$

or, equivalently,

$$\tfrac{1}{2}(x_{n+1} + x_{n-1})(2 + bx_n^2h^2) - (2 - ah^2)x_n = 0, \quad \text{for } n \text{ an integer,} \tag{1.7}$$

is an appropriate approximation of Duffing's equation which can be solved in closed form. The other suggested replacements for $x^3(t)$ would also give satisfactory approximations to the differential equation; the advantage of (1.7) is that it can be solved completely.

2. Solutions of Duffing's equation

The classical analysis [3] of Duffing's equation classifies the periodic solution for three cases governed by the boundary conditions and the constraints on the constants:

Case	I	II	III
boundary conditions	$\begin{cases} x(0) = A \\ \dot{x}(0) = 0 \end{cases}$	$\begin{cases} x(0) = A \\ \dot{x}(0) = 0 \end{cases}$	$\begin{cases} x(0) = 0 \\ x = A, \dot{x} = 0 \end{cases}$
constant constraints	$\begin{cases} b > 0 \\ a > -\frac{1}{2}bA^2 \end{cases}$	$\begin{cases} b > 0 \\ -bA^2 < a < -\frac{1}{2}bA^2 \end{cases}$	$\begin{cases} b < 0 \\ a > -bA^2 \end{cases}$
solution $x(t)$	$A \operatorname{cn}[(a + bA^2)^{1/2}t]$	$A \operatorname{dn}[A(\frac{1}{2}b)^{1/2}t]$	$A \operatorname{sn}[(a + \frac{1}{2}bA^2)^{1/2}t]$
parameter m	$\frac{1}{2}bA^2/(a + bA^2)$	$2[1 + a/(bA^2)]$	$-\frac{1}{2}bA^2/(a + \frac{1}{2}bA^2)$
period T	$4K/(a + bA^2)^{1/2}$	$2K/A(\frac{1}{2}b)^{1/2}$	$4K/(a + \frac{1}{2}bA^2)^{1/2}$

The notation of [1] has been used for the Jacobian elliptic functions cn, dn, sn and the complete elliptic integral of the first kind K, with m as the parameter, $0 \leqslant m < 1$, on which these functions and integral depend.

3. Periodic solutions of the difference equation (1.7)

It is easy to establish sufficient conditions for periodic solutions of (1.7) appropriate to the boundary conditions for Cases I, II and III respectively.

For Cases I and II the boundary conditions are taken as

$$x_0 = A \tag{3.1}$$

and

$$x_{-1} = x_1. \tag{3.2}$$

For $n = 0$, (1.7) with (3.1) and (3.2) gives

$$x_1 = \frac{2 - ah^2}{2 + bA^2h^2} A \tag{3.3}$$

and repeated use of (1.7) for $n = 1, 2, \ldots,$ and noting that $x_{-n} = x_n$, gives x_n for all n in terms of the constants a, b and A and the arbitrary time interval h. The possibility of periodic solutions of the difference equation (1.7) for Cases I and II is decided by the following:

THEOREM 1. *If, for given constants a, b and A and any positive integer p, the time interval h can be chosen so that $x_{p+1} = x_{p-1}$, then the solution x_n of* (1.7) *satisfying* (3.1) *and* (3.2) *is periodic with period*

$$T = 2ph. \tag{3.4}$$

PROOF. Equation (1.7) for $n = p - 1$ and $n = p + 1$ gives

$$\tfrac{1}{2}(x_p + x_{p-2})\left(2 + bx_{p-1}^2h^2\right) - (2 - ah^2)x_{p-1} = 0 \tag{3.5}$$

and

$$\tfrac{1}{2}(x_{p+2} + x_p)\left(2 + bx_{p+1}^2h^2\right) - (2 - ah^2)x_{p+1} = 0; \tag{3.6}$$

the condition $x_{p+1} = x_{p-1}$ therefore forces

$$x_{p+2} = x_{p-2}. \tag{3.7}$$

Similarly (1.7) for $n = p - 2$ and $n = p + 2$ forces

$$x_{p+3} = x_{p-3}, \tag{3.8}$$

and so on, until $n = 1$ and $n = 2p - 1$ forces

$$x_{2p} = x_0 = A. \tag{3.9}$$

Finally, $n = 0$ and $n = 2p$ forces

$$x_{2p+1} = x_{-1} = x_1. \tag{3.10}$$

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With $x_{2p} = x_0$ and $x_{2p+1} = x_1$, (1.7) gives

$$x_{2p+n} = x_n \quad \text{for all } n, \tag{3.11}$$

completing the proof of the theorem.

The situation for Case III is somewhat different; the boundary conditions are taken as

$$x_0 = 0 \tag{3.12}$$

and

$$x_p = A \quad \text{when } x_{p-1} = x_{p+1}. \tag{3.13}$$

A theorem similar to Theorem 1 can now be proved showing that if, for any positive integer p, the time interval h can be chosen so that (3.12) and (3.13) are satisfied, then the solution is an odd function which is periodic with period

$$T = 4ph. \tag{3.14}$$

4. Solutions of the difference equation (1.7): Case I

To solve (1.7) we try, for an integer $p > 2$, the periodic solution

$$x_n = A\,\mathrm{cn}(2nK/p), \tag{4.1}$$

which correctly satisfies the boundary conditions (3.1) and (3.2) as well as $x_{p+1} = x_{p-1}$. This trial solution will be valid if, for the given constraints $b > 0$ and $a > -\frac{1}{2}bA^2$, the associated parameter m lies in the allowed interval (0, 1) and h is real.

The parameter m (and hence the time interval h and the period T) are determined by first substituting (4.1) into (3.3), giving

$$\mathrm{cn}(2K/p) = \frac{2 - ah^2}{2 + bA^2h^2}, \tag{4.2}$$

and then (4.1) into (1.7), giving

$$\frac{A\,\mathrm{cn}(2nK/p)\,\mathrm{cn}(2K/p)}{1 - m\,\mathrm{sn}^2(2nK/p)\mathrm{sn}^2(2K/p)}\left[2 + bA^2h^2\,\mathrm{cn}^2(2nK/p)\right] = (2 - ah^2)A\,\mathrm{cn}(2nK/p), \tag{4.3}$$

where use has been made of the formula [1]

$$\mathrm{cn}(u+v) + \mathrm{cn}(u-v) = \frac{2\,\mathrm{cn}\,u\,\mathrm{cn}\,v}{1 - m\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}. \tag{4.4}$$

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With the assistance of (4.2), equation (4.3) simplifies to

$$\left[(2 + bA^2h^2)m\,\mathrm{sn}^2(2K/p) - bA^2h^2\right]\mathrm{sn}^2(2nK/p) = 0,$$

which is satisfied for all n provided

$$m\,\mathrm{sn}^2(2K/p) = bA^2h^2/\,(2 + bA^2h^2). \tag{4.5}$$

Eliminating h^2 from (4.2) and (4.5) gives

$$m = \frac{bA^2}{a + bA^2}\left[1 + \mathrm{cn}(2K/p)\right]^{-1}. \tag{4.6}$$

This is a transcendental equation for m (noting that K depends on m). The constraints $b > 0$ and $a > -\frac{1}{2}bA^2$ ensure that the parameter m, depending on the ratio $bA^2/(a + bA^2)$ as well as on p (assumed > 2), falls in the required interval $0 < m < 1$.

The time interval h can now be calculated from (4.2):

$$h^2 = \frac{2\left[1 - \mathrm{cn}(2K/p)\right]}{a + bA^2\,\mathrm{cn}(2K/p)}, \tag{4.7}$$

giving real h for $p > 2$. Alternatively, (4.6) gives

$$h^2 = 2\frac{2m(a + bA^2) - bA^2}{m(a^2 - b^2A^4) + b^2A^4}. \tag{4.8}$$

The period is finally deduced from $T = 2ph$, completing the details of a valid periodic solution.

Essentially the analysis shows that if the time interval h is determined from (4.8) using the solution m of (4.6), then the iterative solution of (1.7) gives, apart from round-off errors, an exactly periodic solution.

It is easy to verify that the solution to Duffing's equation is obtained in the limit as $p \to \infty$, for then $K/p \to 0$, $h \to 0$, and

$$m \to \tfrac{1}{2}bA^2/\,(a + bA^2). \tag{4.9}$$

Since $\mathrm{cn}\,u = 1 - \frac{1}{2}u^2 + O(u^4)$, equation (4.7) gives, as $p \to \infty$,

$$T = 2ph \to 4K/\,(a + bA^2)^{1/2}. \tag{4.10}$$

Finally

$$\begin{aligned} x_n &= A\,\mathrm{cn}(2nK/p) = A\,\mathrm{cn}(2nhK/ph) \\ &\to A\,\mathrm{cn}\left[(a + bA^2)^{1/2}t\right] = x(t), \end{aligned} \tag{4.11}$$

so that in the limit all the results for Duffing's equation are realized.

5. Solutions of the difference equation (1.7): Case II

The boundary conditions (3.1) and (3.2) are the same as for Case I but now we try, for an integer $p > 2$, the periodic solution

$$x_n = A\ \mathrm{dn}(nK/p), \tag{5.1}$$

which satisfies $x_{p+1} = x_{p-1}$. This trial solution is valid if, for the given constraints $b > 0$ and $-bA^2 < a < -\frac{1}{2}bA^2$, the associated parameter m lies in the allowed interval (0, 1) and the corresponding h is real.

The parameter m (and hence the time interval h and the period T) is determined as for Case I by substituting (5.1) into (3.3), giving

$$\mathrm{dn}(K/p) = \frac{2 - ah^2}{2 + bA^2h^2}, \tag{5.2}$$

and then (5.1) into (1.7), giving

$$\frac{A\ \mathrm{dn}(nK/p)\ \mathrm{dn}(K/p)}{1 - m\ \mathrm{sn}^2(nK/p)\ \mathrm{sn}^2(K/p)}\left[2 + bA^2h^2\ \mathrm{dn}^2(nK/p)\right] = (2 - ah^2)A\ \mathrm{dn}(nK/p), \tag{5.3}$$

where use has been made of the formula [1]

$$\mathrm{dn}(u + v) + \mathrm{dn}(u - v) = \frac{2\ \mathrm{dn}\ u\ \mathrm{dn}\ v}{1 - m\ \mathrm{sn}^2\ u\ \mathrm{sn}^2\ v}. \tag{5.4}$$

With the assistance of (5.2), (5.3) simplifies to

$$\left[(2 + bA^2h^2)\mathrm{sn}^2(K/p) - bA^2h^2\right]\mathrm{sn}^2(nK/p) = 0,$$

which is satisfied for all n provided

$$\mathrm{sn}^2(K/p) = bA^2h^2/\,(2 + bA^2h^2). \tag{5.5}$$

Eliminating h^2 from (5.2) and (5.5) gives

$$m = \frac{a + bA^2}{bA^2}\left[1 + \mathrm{dn}(K/p)\right]. \tag{5.6}$$

For $b > 0$ and $-bA^2 < a < -\frac{1}{2}bA^2$ and for $p > 2$, the parameter m falls in the required interval $0 < m < 1$ so that the solution is valid.

The time interval h can now be calculated from (5.2):

$$h^2 = \frac{2\left[1 - \mathrm{dn}(K/p)\right]}{a + bA^2\ \mathrm{dn}(K/p)}, \tag{5.7}$$

or, using (5.6), from

$$h^2 = 2\frac{2(a + bA^2) - mbA^2}{(a^2 - b^2A^4) + mb^2A^4}. \tag{5.8}$$

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For $p > 2$, h is real. The period is finally deduced from $T = 2ph$, completing the solution.

Again it is easy to verify that the solution to the differential equation is obtained in the limit as $p \to \infty$, for then $K/p \to 0$, $h \to 0$ and

$$m \to 2\left[1 + a/(bA^2)\right]. \tag{5.9}$$

Since $\operatorname{dn} u = 1 - \frac{1}{2}mu^2 + O(u^4)$, equation (5.7) gives, as $p \to \infty$,

$$T = 2ph \to 2K/A\left(\tfrac{1}{2}b\right)^{1/2}. \tag{5.10}$$

Finally,

$$x_n = A \operatorname{dn}(nK/p) = A \operatorname{dn}(nhK/ph) \to A \operatorname{dn}\left[a\left(\tfrac{1}{2}b\right)^{1/2} t\right] = x(t), \tag{5.11}$$

in agreement with the results for Duffing's equation.

6. Solutions of the difference equation (1.7): Case III

For Case III we try for any positive integer p,

$$x_n = A \operatorname{sn}(nK/p) \tag{6.1}$$

which satisfies $x_0 = 0$, $x_p = A$ and $x_{p+1} = x_{p-1}$. For $n = p$, (1.7) gives

$$x_{p-1} = \frac{2 - ah^2}{2 + bA^2h^2} A, \tag{6.2}$$

and (6.1) with $n = p - 1$ gives

$$x_{p-1} = A \operatorname{sn}\left[K - (K/p)\right] = A \frac{\operatorname{cn}(K/p)}{\operatorname{dn}(K/p)}. \tag{6.3}$$

Using the formula [1]

$$\operatorname{sn}(u + v) + \operatorname{sn}(u - v) = 2\frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v}{1 - m \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{6.4}$$

and substituting (6.1) in (1.7) gives

$$\begin{aligned} \frac{A \operatorname{sn}(nK/p)\operatorname{cn}(K/p)\operatorname{dn}(K/p)}{1 - m \operatorname{sn}^2(nK/p) \operatorname{sn}^2(K/p)} &\left[2 + bA^2h^2 \operatorname{sn}^2(nK/p)\right] \\ &= (2 - ah^2)A \operatorname{sn}(nK/p), \end{aligned} \tag{6.5}$$

which, with (6.2) and (6.3), simplifies to

$$\left[2m \operatorname{sn}^2(K/p) + bA^2h^2\right]\operatorname{cn}^2(nK/p) = 0. \tag{6.6}$$

This is satisfied for all n provided

$$m \operatorname{sn}^2(K/p) = -\tfrac{1}{2}bA^2h^2. \tag{6.7}$$

Eliminating h^2 from (6.2), (6.3) and (6.7) gives the transcendental equation for determining m:

$$m = -\frac{bA^2}{a}\frac{1 - \mathrm{cn}(K/p)\mathrm{dn}(K/p)}{1 - \mathrm{cn}^2(K/p)}. \tag{6.8}$$

With the constraints $b < 0$ and $a > -bA^2$, we have $0 < m < 1$ as required for a valid solution.

Equation (6.8) is not convenient for calculation purposes for large p; an alternative formula is

$$\begin{aligned} -bA^2m^{-1} = {} & a\left[\mathrm{dn}^2(K/p) + \mathrm{dn}(K/p)\mathrm{cn}(K/p)\right] \\ & + bA^2\left[\mathrm{dn}(K/p)\mathrm{cn}(K/p) - \mathrm{sn}^2(K/p)\right]. \end{aligned} \tag{6.9}$$

The time interval h can be calculated from (6.7):

$$h^2 = -\frac{2m\,\mathrm{sn}^2(K/p)}{bA^2} \tag{6.10}$$

or, using (6.2) and (6.3),

$$h^2 = 2\frac{m(2a + bA^2) + bA^2}{ma^2 - b^2A^4}. \tag{6.11}$$

The value of h is real and the period is $T = 4ph$, completing the solution.

In the limit as $p \to \infty$, $K/p \to 0$, $h \to 0$ and

$$m \to -bA^2/(2a + bA^2), \tag{6.12}$$

$$T = 4ph \to 4K\left(a + \tfrac{1}{2}bA^2\right)^{1/2}, \tag{6.13}$$

and

$$x_n \to A\,\mathrm{sn}\left[\left(a + \tfrac{1}{2}bA^2\right)^{1/2}t\right] = x(t), \tag{6.14}$$

giving the results for Duffing's equation.

7. The linear case $b = 0$

For the linear problem $b = 0$ and $a > 0$, Duffing's equation simplifies to

$$\ddot{x}(t) + ax(t) = 0 \tag{7.1}$$

and the approximating difference equation to

$$h^{-2}(x_{n+1} - 2x_n + x_{n-1}) + ax_n = 0. \tag{7.2}$$

For $b = 0$ it follows for Case I and Case III that $m = 0$, $K = \pi/2$, cn = cos and sn = sin. Case II does not arise.

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For the boundary conditions

$$x(0) = A \quad \text{and} \quad x_{-1} = x_1, \tag{7.3}$$

the Case I solutions of (1.7) become

$$x_n = A\cos(n\pi/p), \tag{7.4}$$

$$h = 2a^{-1/2}\sin(\pi/2p), \tag{7.5}$$

and

$$T = 4pa^{-1/2}\sin(\pi/2p), \tag{7.6}$$

which are well-known periodic solutions of (7.2).

For the boundary conditions

$$x(0) = 0, \tag{7.7}$$

$$x_p = A \quad \text{and} \quad x_{p+1} = x_{p-1}, \tag{7.8}$$

the Case III solutions become other known solutions of (7.2):

$$x_n = A\sin(n\pi/2p), \tag{7.9}$$

$$h = (2/a)^{1/2}\big[(1 - \cos(\pi/2p))\big]^{1/2}, \tag{7.10}$$

and

$$T = 4p(2/a)^{1/2}\big[1 - \cos(\pi/2p)\big]^{1/2}. \tag{7.11}$$

8. Numerical results: Case I

To illustrate the analysis given above, numerical results are presented first for $a = 10$, $b = 90$ and $A = 1$.

Duffing's equation is then

$$\ddot{x}(t) + 10x(t) + 90x^3(t) = 0, \tag{8.1}$$

with

$$x(0) = 1 \quad \text{and} \quad \dot{x}(0) = 0 \tag{8.2}$$

and

$$m = 0.45, \tag{8.3}$$

$$T = 0.4K(0.45) = 0.72555, \tag{8.4}$$

and

$$x(t) = \mathrm{cn}(10t). \tag{8.5}$$

The solution is exhibited graphically in Fig. 1.

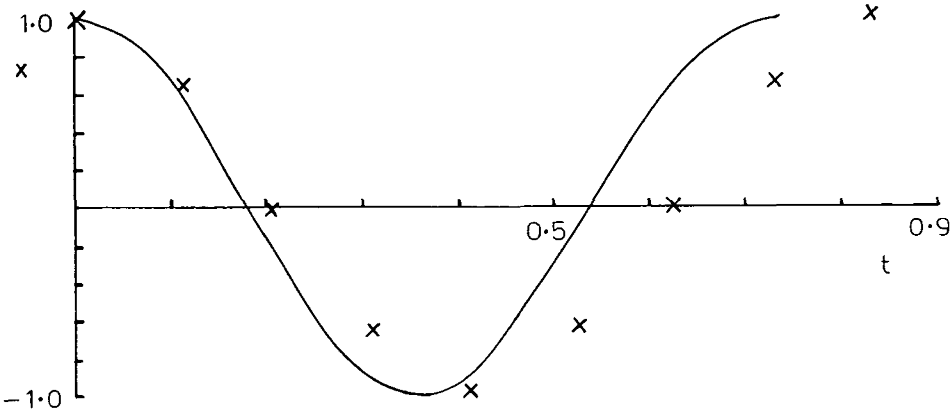

Fig. 1. The solid curve represents the solution for one period of Duffing's equation $\ddot{x} + 10x + 90x^3 = 0$. Crosses represent the solution for one period of the difference equation (1.7) with $p = 4$, that is, with $2p = 8$ intervals a period.

Equation (1.7) becomes

$$\tfrac{1}{2}(x_{n+1} + x_{n-1})(2 + 90x_n^2h^2) - (2 - 10h^2)x_n = 0, \tag{8.6}$$

with

$$x_0 = 1 \quad \text{and} \quad x_{-1} = x_1. \tag{8.7}$$

From (4.6) the transcendental equation for m is

$$m = 0.9[1 + \mathrm{cn}(2K/p)]^{-1}, \tag{8.8}$$

and from (4.8)

$$h^2 = 0.2(20m - 9)/(81 - 80m) \tag{8.9}$$

and

$$T = 2ph. \tag{8.10}$$

TABLE 1

Numerical results for (1.7) with $a = 10$, $b = 90$, $A = 1$

integer p	parameter m	time interval h	period T
2	0.900	0.447	1.788
3	0.640	0.160	0.959
4	0.551	0.105	0.836
12	0.460	0.031	0.736
20	0.454	0.018	0.728
∞	0.450		0.726

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Numerical results for various values of p are given in Table 1; results for $p = \infty$ are (8.3) and (8.4). The solution for $p = 4$ is shown in Fig. 1.

As p increases the approach to the solution of Duffing's equation is evident. It has to be emphasized that for the calculated values of h the solutions of the difference equation are, apart from round-off errors, periodic.

9. Numerical results: Case II

For $a = -24$, $b = 32$ and $A = 1$, Duffing's equation is

$$\ddot{x}(t) - 24x(t) + 32x^3(t) = 0 \tag{9.1}$$

with

$$x(0) = 1 \quad \text{and} \quad \dot{x}(0) = 0, \tag{9.2}$$

and

$$m = 0.5, \tag{9.3}$$

$$T = 0.5K(0.5) = 0.92704, \tag{9.4}$$

and

$$x(t) = \mathrm{dn}(4t). \tag{9.5}$$

The solution is shown in Fig. 2.

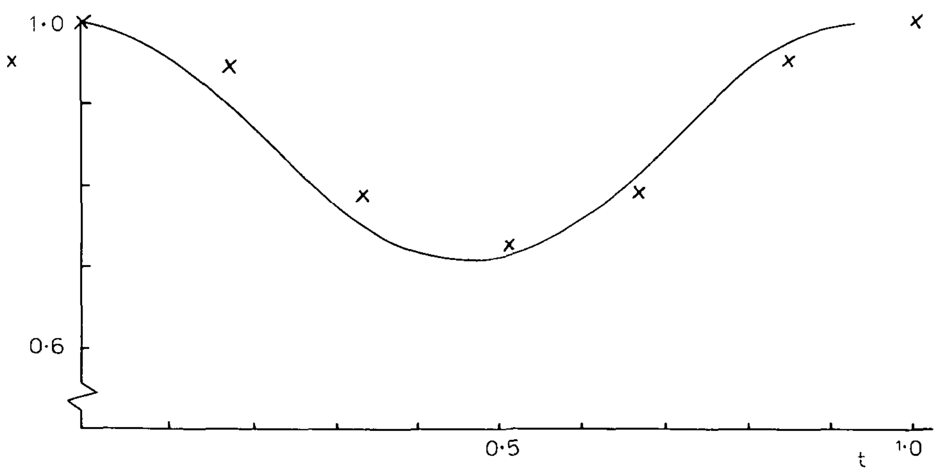

Fig. 2. The solid curve represents the solution for one period of Duffing's equation $\ddot{x} - 24x + 32x^3 = 0$. Crosses represent the solution for one period of the difference equation (1.7) with $p = 3$, that is, with $2p = 6$ intervals a period.

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The corresponding difference equation is

$$\tfrac{1}{2}(x_{n+1} + x_{n-1})(2 + 32x_n^2h^2) - (2 + 32h^2)x_n = 0 \tag{9.6}$$

with

$$x_0 = 1 \quad \text{and} \quad x_{-1} = x_1. \tag{9.7}$$

From (5.6) the transcendental equation for m is

$$m = 0.25[1 + \mathrm{dn}(K/p)] \tag{9.8}$$

and from (5.8)

$$h^2 = (1 - 2m)/(32m - 14) \tag{9.9}$$

and

$$T = 2ph. \tag{9.10}$$

Numerical results for various values of p are given in Table 2; results for $p = \infty$ are (9.3) and (9.4). The solution for $p = 3$ is shown in Fig. 2.

TABLE 2

Numerical results for (1.7) with $a = -24$, $b = 32$, $A = 1$

integer p	parameter m	time interval h	period T
2	0.464	0.288	1.154
3	0.480	0.169	1.016
4	0.488	0.122	0.978
5	0.492	0.095	0.953
∞	0.500		0.927

10. Numerical results: Case III

Finally, for $a = 6$, $b = -4$ and $A = 1$, we consider the Duffing's equation

$$\ddot{x}(t) + 6x(t) - 4x^3(t) = 0 \tag{10.1}$$

with

$$x(0) = 0 \quad \text{and} \quad x(t) = 1 \tag{10.2}$$

for the least $t > 0$ for which $\dot{x}(t) = 0$. Then

$$m = 0.5, \tag{10.3}$$

$$T = 2K(0.5) = 3.70815, \tag{10.4}$$

and

$$x(t) = \mathrm{sn}(2t). \tag{10.5}$$

The solution is shown in Fig. 3.

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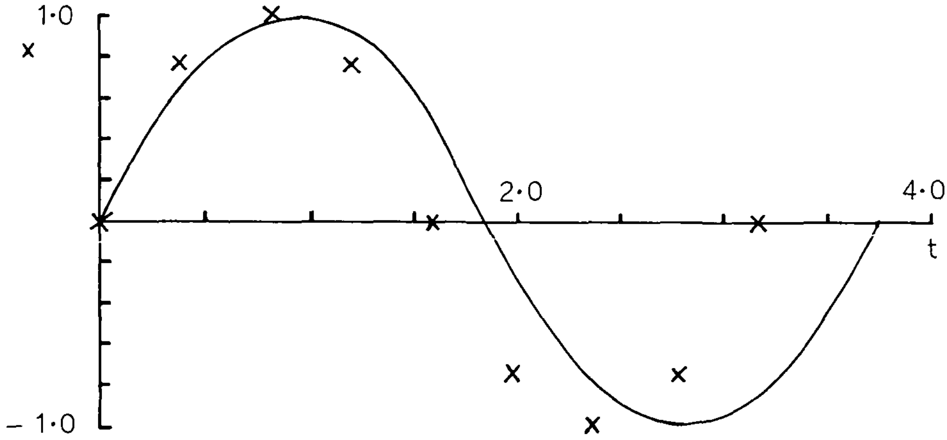

Fig. 3. The solid curve represents the solution for one period of Duffing's equation $\ddot{x} + 6x - 4x^3 = 0$. Crosses represent the solution for one period of the difference equation (1.7) with $p = 2$, that is, with $4p = 8$ intervals a period.

The difference equation is

$$\tfrac{1}{2}(x_{n+1} + x_{n-1})(2 - 4x_n^2h^2) - (2 - 6h^2)x_n = 0 \tag{10.6}$$

with

$$x_0 = 0, \quad x_p = 1 \quad \text{and} \quad x_{p+1} = x_{p-1}. \tag{10.7}$$

From (6.9)

$$m^{-1} = 1.5\,\mathrm{dn}^2(K/p) + 0.5\,\mathrm{dn}(K/p)\,\mathrm{cn}(K/p) + \mathrm{sn}^2(K/p) \tag{10.8}$$

and from (6.11)

$$h^2 = 2(2m - 1)/(9m - 4) \tag{10.9}$$

and

$$T = 4ph.$$

Numerical results for various values of p are given in Table 3; the solution for $p = 2$ is shown in Fig. 3.

TABLE 3

Numerical results for (1.7) with $a = 6$, $b = -4$, $A = 1$

integer p	parameter m	time interval h	period T
2	0.530	0.396	3.17
3	0.512	0.283	3.40
4	0.507	0.226	3.62
∞	0.500		3.71

11. Discussion

This paper has been concerned with an investigation of a non-linear difference equation approximating Duffing's differential equation in its simplest form. Various choices of the difference equation are possible and it is shown that a particular choice, given by (1.7), has the desirable property that it can be solved analytically in closed form, the solutions being strictly periodic. With calculated values of the time interval h, the difference equation (1.7) when solved iteratively with given boundary conditions gives, apart from round-off errors, strictly periodic solutions. With other approximating difference equations periodic solutions are possible but the determination of the appropriate time interval is too difficult.

It is intended to extend the approach used in this paper to other non-linear problems.

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