K. Sakai
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DIFFEOMORPHISMS WITH PSEUDO ORBIT TRACING PROPERTY

KAZUHIRO SAKAI

We shall discuss a differentiable invariant that arises when we consider a class of diffeomorphisms having the pseudo orbit tracing property (abbrev. POTP).

Let M be a closed C^∞ manifold and $\mathrm{Diff}^1(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. We denote $\mathscr{P}^1(M)$ the C^1 interior of the set of all diffeomorphisms having POTP belonging to $\mathrm{Diff}^1(M)$. Recently Aoki [1] proved that the C^1 interior of the set of all diffeomorphisms whose periodic points are hyperboric, $\mathscr{F}^1(M)$, is characterized as Axiom A diffeomorphisms with no-cycle. After this Moriyasu [8] showed that $\mathscr{P}^1(M) \subset \mathscr{F}^1(M)$ and if $\dim M = 2$ then every $f \in \mathscr{P}^1(M)$ satisfies strong transversality.

In this paper the following two theorems will be proved.

THEOREM A. *There exists a closed C^∞ 3-manifold M such that set of all diffeomorphisms having POTP is not dense in* $\mathrm{Diff}^1(M)$.

The Theorem answers to a problem stated in Morimoto [7].

THEOREM B. *If M is a closed C^∞ 3-manifold, then $\mathscr{P}^1(M)$ is characterized as* Axiom A *diffeomorphisms satisfying strong transversality.*

A diffeomorphism f of M is *quasi-Anosov* if the fact that $\|Df^n(v)\|$ is bounded for all $n \in \mathbf{Z}$ implies that $v = 0$. Theorem A is easily obtained in combining with Franks and Robinson [2] and Sakai [12]. The set of all quasi-Anosov diffeomorphisms belonging to $\mathrm{Diff}^1(M)$, $\mathrm{QA}^1(M)$, is open and $\mathrm{QA}^1(M) \subset \mathscr{F}^1(M)$. It is easy to see that when $\dim M = 2$, every $f \in \mathrm{QA}^1(M)$ is Anosov (see [5]). However an example of a diffeomorphism f' on the connected sum M' of two 3-tori that is quasi-Anosov but not Anosov was given in Franks and Robinson [2]. Since f' is Ω-stable, there is C^1 neighborhood $\mathscr{U}$ of f' in $\mathrm{Diff}^1(M')$ such that every $g \in \mathscr{U}$ is quasi-Anosov but not Anosov. Thus, by [12] every $g \in \mathscr{U}$ cannot have POTP,

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and so Theorem A is proved.

Before beginning the proof of Theorem B we give some notations and definitions.

Let (X, d) be a compact metric space, and let $f : X \to X$ be a homeomorphism. A sequence of points $\{x_i\}_{i=a}^{b-1}$ $(-\infty \leq a < b \leq \infty)$ in X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $a \leq i \leq b-1$. Given $\varepsilon > 0$ a sequence of points $\{x_i\}_{i=a}^{b}$ is said to be *f-ε-traced* by a point $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for $a \leq i \leq b$. We say that f has the *pseudo orbit tracing property* (abbrev. POTP) if for $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit for f can be f-ε-traced by some point of X. For compact spaces the notions stated above are independent of compatible metrics used. It is easy to see that if f has POTP then the *non-wandering set* $\Omega(f)$ coincides with the *chain recurrent set* $R(f)$ for f, where $R(f)$ is the set of $x \in X$ such that for every $\delta > 0$, there is a *δ-pseudo* orbit of f from x to x (see [11]). For $x \in X$ and $\varepsilon > 0$ the *local stable* and *unstable sets* are defined by

$$W_\varepsilon^s(y, f) = \{x \in X : d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0\},$$

$$W_\varepsilon^u(x, f) = \{y \in X : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \text{ for all } n \geq 0\}.$$

Suppose that f has POTP. Then it is checked that for every $\varepsilon > 0$, there is $0 < \delta < \varepsilon/2$ such that if $d(x, y) < \delta$ $(x, y \in X)$ then

$$W_\varepsilon^s(x, f) \cap W_\varepsilon^u(y, f) \neq \phi. \tag{1}$$

Let M be as before and denote by d a Riemannian metric on M. Then for a hyperbolic set Λ of $f \in \mathrm{Diff}^1(M)$ and for $x \in \Lambda$ the *stable* and *unstable manifolds* are defined by

$$W^s(x, f) = \{y \in M : d(f^n(y), f^n(x)) \to 0 \text{ as } n \to \infty\}.$$

$$W^u(x, f) = \{y \in M : d(f^{-n}(y), f^{-n}(x)) \to 0 \text{ as } n \to \infty\}.$$

When Λ can be written as the finite disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ of closed invariant sets Λ_i such that each of $f_{|\Lambda_i}$ is topologically transitive. Such a set Λ_1 is called a *basic set* with respect to Λ. The *stable set* $W^s(\Lambda_i. f)$ and *unstable set* $W^u(\Lambda_i, f)$ are defined by

$$W^s(\Lambda_i. f) = \{y \in M : d(f^n(y), \Lambda_i) \to 0 \text{ as } n \to \infty\}.$$

$$W^u(\Lambda_i. f) = \{y \in M : d(f^{-n}(y), \Lambda_i) \to 0 \text{ as } n \to \infty\}.$$

Then $W^\sigma(\Lambda_i, f) = \bigcup \{W^\sigma(x, f) : x \in \Lambda_i\}$ for $\sigma = s, u$. If $\varepsilon > 0$ is small enough, then for $x \in \Lambda$ the local stable and unstable sets, $W_\varepsilon^\sigma(x, f)$ $(\sigma = s, u)$, are C^1 disks tangent to certain subspaces $E^s(x)$ and $E^u(x)$, respectively, such that x $T_xM = E^s(x) \oplus E^u(x)$. Moreover there exists $0 < \lambda < 1$ such that

$$(2) \quad \begin{cases} d(f^n(y), f^n(z)) \le \lambda^n d(y, z) \text{ for } y, z \in W_\varepsilon^s(x, f) \text{ and } n \ge 0. \\ d(f^{-n}(y), f^{-n}(z)) \le \lambda^n d(y, z) \text{ for } y, z \in W_\varepsilon^u(x, f) \text{ and } n \ge 0 \end{cases}$$

(see Hirsch and Pugh [3]). Thus $W_\varepsilon^\sigma(x, f) \subset W^\sigma(x, f)$ for $x \in \Lambda$ $(\sigma = s, u)$ and

$$W^s(x, f) = \bigcup_{n \ge 0} f^{-n}(W_\varepsilon^s(f^n(x), f)),$$
$$W^u(x, f) = \bigcup_{n \ge 0} f^n(W_\varepsilon^u(f^{-n}(x), f)).$$

We denote $W^\sigma(x, f)$ by $W^\sigma(x)$ $(\sigma = s, u)$ if there is no confusion.

If f is Axiom A diffeomorphism then we have $M = \bigcup(W^\sigma(x) : x \in \Omega(f)\}$ for $\sigma = s, u$ and $\Omega(f)$ is expressed as the union $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ of disjoint basic sets for f. Such union is called the *spectral decomposition* for f. We say that f has a *cycle* if there is a subsequence $\{\Lambda_{i_j}\}_{j=1}^{s+1}$ $(2 \le s \le \ell)$ of $\{\Lambda_i\}_{i=1}^{\ell}$ such that $W^u(\Lambda_{i_j}) \cap W^s(\Lambda_{i_{j+1}}) \ne \phi$ $(1 \le j \le s)$ and $\Lambda_{i_{s+1}} = \Lambda_{i_1}$. We say that f satisfies the *strong transversality condition* if for all $x, y \in \Omega(f)$, $W^s(x)$ and $W^u(y)$ meet transversely. Remark that $W^s(\Lambda_i) \cap W^u(\Lambda_i) = \Lambda_i$ for $1 \le i \le \ell$.

Now to obtain the conclusion of Theorem B it is enough to see that every $f \in \mathscr{P}^1(M)$ satisfies strong transversality because f satisfies Axiom A as stated above.

Let $f \in \mathscr{P}^1(M)$ and $x \in M$. Then it was proved in [8] that if $0 < \dim W^\sigma(x) < \dim M$ for $\sigma = s, u$ then $T_xW^s(x) \not\subset T_xW^u(x)$ and $T_xW^u(x) \not\subset T_xW^s(x)$. This tells us that if

$$(3) \qquad \dim W^s(x) + \dim W^u(x) \ge \dim M,$$

then $W^s(x)$ and $W^u(x)$ meet transversely.

Therefore, to complete the proof of Theorem B it only remains to show (3) .

Since $f \in \mathscr{P}^1(M)$ satisfies Axiom A, $\Omega(f)$ is decomposed as $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_\ell$, where each Λ_i is a basic set. Then by [4] for each i there exists a compact neighborhood $B(\Lambda_i)$ satisfying the following (4), (5) and (6).

(4) There exists a continuous extension $T_{B(\Lambda_i)}M = \tilde{E}_i^s \oplus \tilde{E}_i^u$ of $T_{\Lambda_i}M = E_i^s \oplus E_i^u$ such that for $x \in B(\Lambda_i) \cap f^{-1}(B(\Lambda_i))$,

$$D_xf(\tilde{E}_i^s(x)) = \tilde{E}_i^s(f(x)) \text{ and } \| D_xf_{|\tilde{E}_i^s(x)}\| < \lambda,$$

and for $x \in B(\Lambda_i) \cap f(B(\Lambda_i))$.

$$D_xf^{-1}(\tilde{E}_i^u(x)) = \tilde{E}_i^u(f^{-1}(x)) \text{ and } \| D_xf^{-1}_{|\tilde{E}_i^u(x)}\| < \lambda.$$

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(5) There exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$ there exist submanifolds $\widetilde{W}^{\sigma}_{\varepsilon}(x)$ $(x \in B(\Lambda_i),\ \sigma = s, u)$ satisfying

$$\begin{cases} \text{(i)}\ f(\widetilde{W}^s_\varepsilon(x)) \subset \widetilde{W}^s_\varepsilon(f(x)) \text{ and } d(f(x), f(y)) < \lambda d(x, y) \\ \qquad \text{for evey } y \in \widetilde{W}^s_\varepsilon(x) \text{ if } x \in B(\Lambda_i) \cap f^{-1}(B(\Lambda_i)), \\ \text{(ii)}\ f^{-1}(\widetilde{W}^u_\varepsilon(x)) \subset \widetilde{W}^u_\varepsilon(f^{-1}(x)) \text{ and } d(f^{-1}(x), f^{-1}(y)) < \lambda d(x, y) \\ \qquad \text{for evey } y \in \widetilde{W}^u_\varepsilon(x) \text{ if } x \in B(\Lambda_i) \cap f(B(\Lambda_i)). \end{cases}$$

(6) There exists $\delta > 0$ such that if $d(x, y) < \delta$ $(x, y \in B(\Lambda_i))$ then $\widetilde{W}^s_\varepsilon(x)$ *and* $\widetilde{W}^u_\varepsilon(y)$ meet transversely.

For E and F subspaces of $T_{\Lambda_i}M$ define

$$\tan \sphericalangle (F, E) = \sup\left\{\frac{\|w_2\|}{\|w_1\|} : w_1 \in E,\ w_2 \in E^{\perp}, \text{ and } w_1 + w_2 \in F - \{0\}\right\}.$$

Then we find $\theta_{1,i} > 0$ satisfying $\tan \sphericalangle (E^s_i, E^{u\perp}_i) < \theta_{1,i}$ (See [10]) . The continuity of $\widetilde{E}^{\sigma}_i$ $(\sigma = s, u)$ ensures the existence of $\theta_{2,i} > 0$ satisfying $\tan \sphericalangle (\widetilde{E}^s_i, \widetilde{E}^{u\perp}_i) < \theta_{2,i}$.

CLAIM 1. *Define* $\theta_2 = \max\{\theta_{2,i} : 1 \leq i \leq \ell\}$. *For* $0 < \theta < \theta_2^{-1} \cdot (2 + \theta_2)^{-1}$, *there exists* $K(\theta) > 0$ *such that* $K(\theta) \to 0$ as $\theta \to 0$ *and for every* $v \in T_xM$ $(x \in B(\Lambda_i))$ *if* $\tan \sphericalangle (v, \widetilde{E}^u_i(x)) < \theta$ *and* $\{x, f(x), \cdots, f^N(x)\} \subset B(\Lambda_i)$ *for some* $N > 0$ *then* $\tan \sphericalangle (D_xf^N(v), \widetilde{E}^u_i(f^N(x))) \leq K(\theta) \cdot \lambda^{2N}$.

Proof. *Let* $x \in B(\Lambda_i)$ be fixed. For $v \in T_xM - \{0\}$, let $v = v^s + v^u = (v)_1 + (v)_2$, where $v^s \in \widetilde{E}^s_i$, $v^u \in \widetilde{E}^u_i$, $(v)_1 \in \widetilde{E}^u_i$, and $(v)_2 \in \widetilde{E}^{u\perp}_i$. Clearly $(v^s)_2 = (v)_2$, $(v^s)_1 + v^u = (v)_1$, $(v^u)_1 = v^u$ and $(v^u)_2 = 0$. Since $f^j(x) \in B(\Lambda_i)$ for $0 \leq j \leq N$ and $\tan \sphericalangle (\widetilde{E}^s_i, \widetilde{E}^{u\perp}_i) < \theta_2$,

$$\frac{\|(D_xf^N(v^s))_1\|}{\|(D_xf^N(v))_2\|} < \theta_2$$

and since $f^j(x) \in B(\Lambda_i)$ for $0 \leq j \leq N$,

$$\frac{\|D_xf^N(v^u)\|}{\|D_xf^N(v^s)\|} \geq \lambda^{-2N}\frac{\|v^u\|}{\|v^s\|}.$$

It is checked that

$$\frac{\|v^s\|}{\|v^u\|} \leq \frac{(1+\theta_2)\|(v)_2\|}{\|(v)_1\| - \theta_2\|(v)_2\|} = \frac{1+\theta_2}{\dfrac{\|(v)_1\|}{\|(v)_2\|} - \theta_2} \leq$$

$$\leq \frac{1+\theta_2}{1/\theta + \theta_2} = \frac{\theta(1+\theta_2)}{1 - \theta\,\theta_2} \text{ and}$$

$$\frac{\|(D_x f^N(v))_2\|}{\|(D_x f^N(v))_1\|} \leq \left| \frac{\|(D_x f^N(v^s))_1\|}{\|(D_x f^N(v^s))_2\|} - \frac{\|(D_x f^N(v^u))_1\|}{\|(D_x f^N(v^s))_2\|} \right|^{-1}.$$

From these inequalities we have

$$\left| \frac{\|(D_x f^N(v^u))_1\|}{\|(D_x f^N(v^s))_2\|} - \frac{\|(D_x f^N(v^s))_1\|}{\|(D_x f^N(v^s))_2\|} \right| \geq \lambda^{-2N} \frac{\|v^u\|}{\|v^s\|} - \theta_2 \geq$$

$$\geq \lambda^{-2N} \frac{1-\theta\,\theta_2}{\theta(2+\theta_2)} - \theta_2 = \lambda^{-2N} \left(\frac{1-\theta\,\theta_2}{\theta(1+\theta_2)} - \lambda^{2N} \cdot \theta_2 \right),$$

and so

$$\frac{\|(D_x f^N(v))_2\|}{\|(D_x f^N(v))_1\|} \leq \lambda^{2N} \cdot K(\theta),$$

where $K(\theta) = \left(\frac{1-\theta\,\theta_2}{\theta(1+\theta^2)} - \theta_2 \right)^{-1}$.

For A a closed set of M, denote by $B_r(A)$ the closed neighborhood of A with radius $r > 0$.

CLAIM 2. *Let Λ_i and Λ_j be the basic sets. Suppose that $2 \geq \operatorname{Ind} \Lambda_j \geq \operatorname{Ind} \Lambda_j \geq 1$ where $\operatorname{Ind} \Lambda$ denotes the dimension of the stable subbundle E^s of a basic set Λ. Then there are $r_1 > 0$ ($B_{r_1}(\Lambda_i) \subset B(\Lambda_i)$) and $\theta > 0$ such that if $x \in \Lambda_j$ and $y \in W^s(x) \cap B_{r_1}(\Lambda_i)$, then $\tan \sphericalangle (T_y W^s(x), \tilde{E}_i^u(y)) > \theta$.*

Proof. If this is false, for every n > 0 there are $x_n \in \Lambda_j$ ($\operatorname{Ind} \Lambda_j \geq \operatorname{Ind} \Lambda_i$) and $y_n \in W^s(x_n) \cap B_{1/n}(\Lambda_i)$ such that $\tan \sphericalangle (T_{y_n} W^s(x_n), \tilde{E}_i^u(y_n)) < 1/n$. Then, by (5) and (6) there are $z_n \in \Lambda_i$ and $w_n = \tilde{W}^s_{\varepsilon_1}(y_n) \cap \tilde{W}^u_{\varepsilon_1}(z_n)$. Since $y_n \to \Lambda_i$ as $n \to \infty$, there is a strictly increasing sequence $J_n > 0$ such that $f^k(y_n) \in B(\Lambda_i)$ for $0 \leq k \leq J_n$ and $f^{J_n+1}(y_n) \notin B(\Lambda_i)$. Put $\tau = \inf\{d(x, \Lambda_i) : x \in B(\Lambda_i)$ and $f^{-1}(x) \notin B(\Lambda_i)\} > 0$. Since

(7) $$d(f^j(y_n), f^j(w_n)) \leq \lambda^j d(y_n, w_n) \text{ for } 0 \leq j \leq J_n,$$

there is $N > 0$ such that for every $n \geq N$, $f^{J_n}(y_n) \in B(\Lambda_i) \setminus B_\tau(\Lambda_i)$ and $f^{J_n}(w_n) \notin B_{\tau/2}(\Lambda_i)$. Thus it is checked that there exists $c_1 > 0$ such that for every $n > 0$,

(8) $$f^j(B_{c_1}(f^{J_n}(y_n))) \cap B_{c_1}(f^{J_n}(y_n)) = \phi \text{ for all } j > 0.$$

Indeed, if for every $m > 0$ there are $n_m > 0$, $j_m > 0$ and $x'_m \in B_{\frac{1}{m}}(f^{J_{n_m}}(y_{n_m}))$ such that $f^{j_m}(x'_m) \in B_{\frac{1}{m}}(f^{J_{n_m}}(y_{n_m}))$, then $y = \lim_{m\to\infty} f^{J_{n_m}}(y_{n_m}) \in R(f)$, which is a contradiction since $y \notin \Omega(f) = R(f)$.

Similarly we can find $c_2 > 0$ such that for every $n > 0$

(9) $$f^j(B_{c_2}(f^{J^n}(w_n))) \cap B_{c_2}(f^{J^n}(w_n)) = \phi \text{ for all } j > 0.$$

Take and fix $0 < \theta' < \theta_2^{-1}(2 + \theta_2)^{-1}$. Then there is $N' > N$ such that for every $n \geq N'$, $\tan \measuredangle (T_{y_n}W^s(x_n), \tilde{E}_i^u(y_n)) \leq \theta'$.
By Claim 1, $\tan \measuredangle (D_{y_n}f^{J^n}(T_{y_n}W^s(x_n)), \tilde{E}_i^u(f^{J^n}(y_n))) \to 0$ as $n \to \infty$, and so

$$\tan \measuredangle (D_{y_n}f^{J^n}(T_{y_n}W^s(x_n)), \mathscr{E}_{f^{J^n}(y_n), f^{J^n}(w_n)}(\tilde{E}_i^u(f^{J^n}(w_n)))) \to 0$$

as $n \to \infty$ (by (7)). Here $\mathscr{E}_{x,y}$ denotes the parallel transform from T_xM to T_yM. Then, from (7), (8) and (9) there are $n \geq N'$ and $g \in \mathrm{Diff}^1(M)$ arbitrarily near to f such that $W^s(x_n, g) \cap W^u(z_n, g) \neq \phi$ and $W^s(x_n, g)$ does not meet transversely to $W^u(z_n, g)$, thus contradiction since $g \in \mathscr{P}^1(M)$.

CLAIM 3 (Lemma IV. 8 of Mañé [6]). *Let E^1 and E^2 be Banach spaces with norm $\| \cdot \|$, and denote by $B_r^i(p)$ the ball of radius r in E^i centered at p. Let $C > 0$ and $\varepsilon' > 0$ be constants such that ε' is so small that $\varepsilon' C < 1$. For $\rho_0 > 0$ take $0 < r \leq \rho_0$ and $0 < \varepsilon \leq \varepsilon'$ satisfying*

(10) $$\frac{\varepsilon(1+\varepsilon')}{1-\varepsilon' C} < \frac{r-\varepsilon}{C} \text{ and } \frac{\varepsilon(1+\varepsilon')}{1-\varepsilon' C} < r.$$

Suppose that $\psi : B_{\rho_0}^1(0) \to E^2$ and $\varphi : B_r^2(p) \to E^1$ are maps satisfying

(a) $\psi(0) = 0$, $\| \psi(w_1) - \psi(w_2) \| \leq \varepsilon' \| w_1 - w_2 \|$

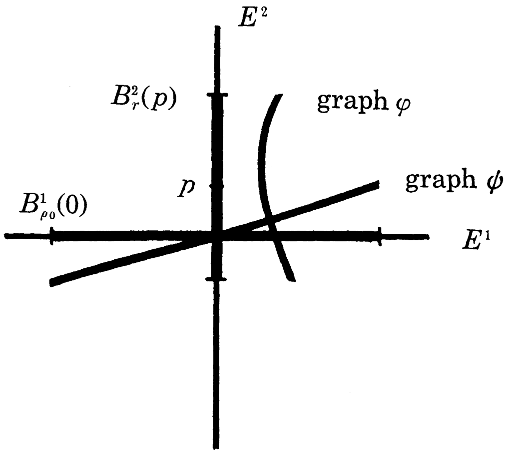
for $w_1, w_2 \in B_{\rho_0}^1(0)$,

(b) $\| \varphi(p) \| < \varepsilon$, $\| \varphi(w_1) - \varphi(w_2) \| \leq C \| w_1 - w_2 \|$

for $w_1, w_2 \in B_r^2(p)$, *and*

(c) $\| p \| < \varepsilon$.

Then graph $\varphi \cap$ graph $\psi \neq \phi$, where graph $\psi = \{(w, \psi(w)) : w \in B_{\rho_0}^1(0)\}$ and graph $\varphi = \{(\varphi(w), w) : w \in B_r^2(p)\}$.



Firstly we show for every $0 < \rho_1 < \min\{r, \frac{1-\varepsilon}{C}\}$, $\varphi(B^2_{\rho_1}(p)) \subset B^1_r(0)$. Take and fix $y \in B^2_{\rho_1}(p)$. Since $\| y - p \| < \rho_1 < \frac{r-1}{C}$, we have $C\| y - p \| + \varepsilon < r$. Thus

$$\| \varphi(y) \| \leq \| \varphi(\rho) \| + C\| y - p \| \leq \varepsilon + C\| y - p \| < r.$$

From this a map $\psi \circ \varphi : B^2_{\rho_1}(p) \to E^2$ is well defined. Since $\| \psi(\varphi(p)) - \psi(0) \| \leq \varepsilon' \| \varphi(0) - 0 \| = \varepsilon' \| \varphi(0) \| < \varepsilon'\varepsilon$, we have $\| \psi(\varphi(p)) \| < \varepsilon'\varepsilon$ (by (a)). Therefore, for $w_1, w_2 \in B^2_{\rho_1}(p)$

$$\| \psi\varphi(w_1) - \psi\varphi(w_2) \| \leq \varepsilon' \| \varphi(w_1) - \varphi(w_2) \| \leq \varepsilon' C \| w_1 - w_2 \|;$$

i.e. $\psi \circ \varphi : B^2_{\rho_1}(p) \to E^2$ is contracting. If we choose $\frac{\varepsilon(1+\varepsilon')}{1+\varepsilon' C} < \rho_2 < \min\{r, \frac{r-\varepsilon}{C}\}$, then for every $y \in B^2_{\rho_2}(p)$,

$$\begin{aligned} \| \psi\varphi(y) - p \| &\leq \| \psi\varphi(y) - \psi\varphi(p) \| + \| \psi\varphi(p) - p \| \leq \\ &\leq \varepsilon' C \| y - p \| + \| \psi\varphi(p) \| + \| p \| \leq \\ &\leq \varepsilon' C\rho_2 + \varepsilon'\varepsilon + \varepsilon < \rho_2. \end{aligned}$$

Thus $\psi \circ \varphi : B^2_{\rho_2}(p) \to B^2_{\rho_2}(p)$ is a contraction. Thus there exists $z \in B^2_{\rho_2}(p)$ such that $\psi \cdot \varphi(z) = z$.

Theorem B will be proved under the above claims. The technique of the proof is to derive a contradiction in proving the existence of a cycle among basic sets $\Lambda_1, \cdots, \Lambda_\ell$ under the assumption that f does not satisfy strong transversality. Remark that the dimension of M is 3. To prove Theorem B it is enough to see that

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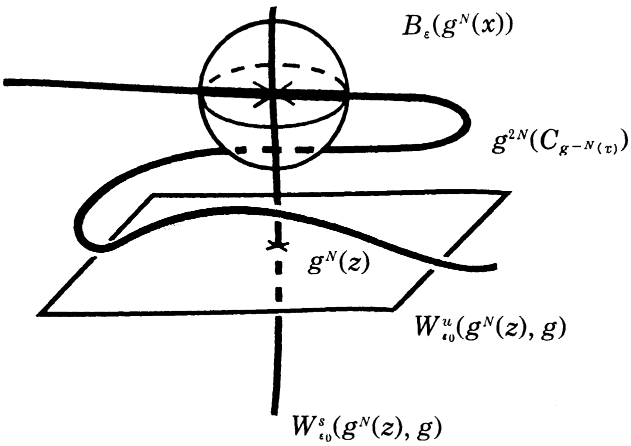
for $x \in M \setminus \Omega(f)$, dim $W^s(x)$ + dim $W^u(x) \geq$ dim M as explained before.

Suppose that there is $x \in M \setminus \Omega(f)$ such that dim $W^s(x)$ + dim $W^u(x) <$ dim $M = 3$ (i.e. dim $W^s(x)$ = dim $W^u(x) = 1$). Since $f \in \mathscr{P}^1(M)$, there are $g \in \mathscr{P}^1(M)$ and $\alpha > 0$ such that $f = g$ on a neighborhood of $\Omega(f)$, dim $W^\sigma(x, g)$ = dim $W^\sigma(x)$ for $\sigma = s, u$ and for the components $C^\sigma(x)$ of x in $W^\sigma(x, g) \cap B_\alpha(x)$ $(\sigma = s, u)$, $C^s(x) \cap C^u(x) = \{x\}$.

Let $\Omega(g) = \Lambda_1(g) \cup \cdots \cup \Lambda_\ell(g)$ be a spectral decomposition for g. Then there are $1 \leq i \neq j \leq \ell$, $y \in \Lambda_i(g)$ and $z \in \Lambda_j(g)$ such that $W^s(x, g) = W^s(z, g)$. and $W^u(x, g) = W^u(y, g)$. For simplicity suppose $y \in \Lambda_1(g)$ and $z \in \Lambda_2(g)$. *Let* $0 < \varepsilon_0 < r_1/2$ be a number as in (2) and fix $N > 0$ such that $g^{-N}(x) \in W^u_{\varepsilon_0/2}(g^{-N}(y), g)$ and $g^N(x) \in W^s_{\varepsilon_0/2}(g^N(z), g)$. Given the connected component $C_{g^{-N}_{(x)}}$ of $g^{-N}(x)$ in $B_{\varepsilon_0/4}(g^{-N}(x)) \cap W^u_{\varepsilon_0}(g^{-N}(y), g)$, we have $C_{g^{-N}_{(x)}} = B_{\varepsilon_0/4}(g^{-N}(x)) \cap W^u_{\varepsilon_0}(g^{-N}(y), g)$. Thus there is $0 < \varepsilon \leq \varepsilon_0/8$ such that $B_\varepsilon(g^N(x)) \cap g^{2N}(C_{g^{-N}_{(x)}})$ is the connected component of $g^N(x)$ in $B_\varepsilon(g^N(x)) \cap g^{2N}(C_{g^{-N}(x)})$.

Denote by $C_{g^N(x)}$ the connected component of $g^N(x)$ in $B_\varepsilon(g^N(x)) \cap g^{2N}(C_{g^{-N}(x)})$, and take and fix $0 < \varepsilon_2 \leq \varepsilon/2$ such that $d(v, w) < \varepsilon_2$ $(v, w \in M)$ implies $d(g^{-2N}(v), g^{-N}(w)) < \varepsilon_0/8$.



CLAIM 4. *Fix any* $w \in C_{g^N(x)} \cap B_{\varepsilon_2}(g^N(x)) \setminus \{g^N(x)\}$. *If there exists* $0 < r' < \varepsilon_2$ *such that* $B_{r'}(w) \cap C_{g^N(x)} \subset C_{g^N(x)} \setminus \{g^N(x)\}$, *and if for every* $w' \in B_{r'}(w) \cap C_{g^N(x)}$, dim $W^s(w', g) = 1$, *then* dim $W^s(v, g) = 1$ *for every* $v \in B_{\delta'}(w) \setminus C_{g^N(x)}$. *Here* $0 < \delta' = \delta'(r', g) < r'$ *is a number satisfying the property* (1) .

Proof. Note that if there is $v \in B_{\delta'}(w) \setminus C_{g^N_{(x)}}$ such that dim $W^s(v, g) = 2$, then $W^s(v, g) \cap (B_{r'}(w) \cap C_{g^N_{(x)}} = \phi$. Since $w \in C_{g^N_{(x)}} \cap B_{\varepsilon_2}(g^N(x))$, we have $g^{-2N}(w) \in B_{\varepsilon_0/8}(g^{-N}(x)) \cap W^u_{\varepsilon_0}(g^{-N}(y), g)$ and hence

(11) $$d(g^{-2N-n}(w),g^{-N-n}(x)) \leq \varepsilon_0/8 \text{ for all } n \geq 0.$$

Since $d(v, w) < \delta'$, by (1) there is $v' \in M$ such that $d(g^n(v'), g^n(v)) < r' < \varepsilon_0$ for all $n \geq 0$ and $d(g^{-n}(v'), g^{-n}(w)) < r' < \varepsilon/2 < \varepsilon_0/8$ for all $n \geq 0$. Thus

(12) $$v' \in W^s(v, g).$$

By using (11) it is checked that $d(g^{-2N-n}(v'), g^{-N-n}(x)) < \varepsilon_0/4$ for all $n \geq 0$ (i.e. $g^{-2N}(v') \in C_{g^{-N}(x)}$) Thus $v' \in g^{2N}(C_{g^{-N}(x)}) \cap B_\varepsilon(g^N(x)) = Cg^N_{(x)}$ (since $d(v', g^N(x)) \leq d(v', w) + d(w, g^N(x)) < \varepsilon/2 + \varepsilon_2 < \varepsilon$). Since $d(v', w) < r'$, we have $v' \in B_{r'}(w) \cap C_{g^N_{(x)}}$. By (12)

$$W^2(v, g) \cap (B_{r'}(w) \cap Cg^N_{(x)}) \neq \phi$$

which is a contradiction.

For $n \geq 1$ denote as $C_{g^{N+n}(x)}$ the connected component of $g^{N+n}(x)$ in $B_\varepsilon(g^{N+n}(z)) \cap g(C_{g^{N+n-1}(x)})$. Note that $C_{g^{N+n}(x)} \subset g(C_{g^{N+n-1}(x)})$ for all $n \geq 1$.

CLAIM 5. *For every $n > 0$ and $0 < \delta \leq \varepsilon_0$, there is $N' > n$ such that for every $w \in B_{1/N'}(g^{N+N'}(x))$,*

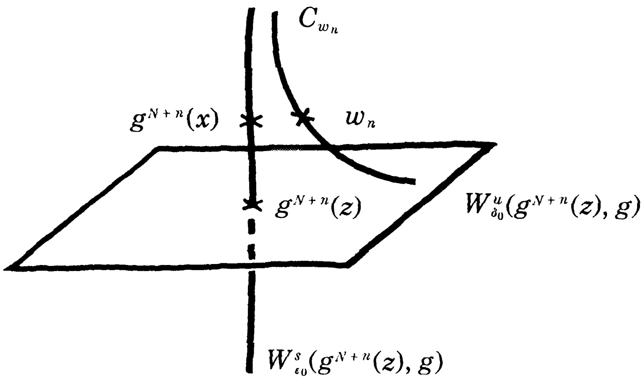
$$C_w \cap W^u_{\delta/2}(g^{N+N'}(z), g) \neq \phi,$$

where C_w is the connected component of w in $W^s(w, g) \cap B_{\varepsilon_0}(w)$.

Proof. If this is false, then there are $n_0 > 0$, $\delta_0 > 0$ and $w_n \in B_{1/n}(g^{N+n}(x))$ for all $n \geq n_0$ such that $C_{w_n} \cap W^u_{\zeta_0}(g^{N+n}(z), g) = \phi$. Let $r_1 > 0$ and $\theta > 0$ be numbers given in Claim 2 for $g \in \mathcal{P}^1(M)$. Clearly

$$\begin{aligned} d(w_n, g^{N+n}(z)) &\leq d(w_n, g^{N+n}(x)) + d(g^{N+n}(x), g^{N+n}(z)) \\ &\leq 1/n + \lambda^n d(g^N(x), g^N(z)) \to 0 \end{aligned}$$

as $n \to \infty$ by (2).



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For a moment we treat a neighborhood of $g^{N+n}(z)$ as it were $\mathbf{R}^3$. Let $E_n^1 = T_{g^{N+n}(x)} W_{\delta_0}^u(g^{N+n}(z), g)$, $E_n^2 = T_{g^{N+n}(x)} W_{\varepsilon_0}^s(g^{N+n}(z), g)$ and fix $n_1 > 0$ such that $d(g^{N+n}(x), g^{N+n}(z)) < r_1$ for $n \geq n_1$. Remark that $g^{N+n}(z) \in \Lambda_2(g)$ and put $p_n = \tilde{E}_2^u(g^{N+n}(x)) \cap E_n^2$ for $n \geq n_1$. Then

$$(*) \qquad d(p_n, g^{N+n}(z)) \to 0 \text{ as } n \to \infty.$$

Thus, by Claim 2 there are constants $C = C(\theta) > 0$, $r_2 = r_2(\theta) > 0$ and $n_2 \geq n_1$ such that for every $n \geq n_2$ there is a map $\varphi_n : B_{r_2}^2(p_n) \to E_n^1$ satisfying

$$(**) \qquad \| \varphi_n(p_n) \| \to 0 \text{ as } n \to \infty \text{ and}$$

$$\begin{cases} \| \varphi_n(w_1') - \varphi_n(w_2') \| \leq C \| w_1' - w_2' \| \text{ for } w_1', w_2' \in B_{r_2}^2(p_n), \\ \text{graph } \varphi_n \subset C_{w_n}. \end{cases}$$

Fix $0 < \varepsilon' < 1$ such that $0 < C\varepsilon' < 1$. Then there are $0 < \rho_0 < \delta_0$ and maps $\psi_n : B_{\rho_0}^1(0) \to E_n^2$ for $n > 0$ such that

$$\begin{cases} \psi_n(0) = 0, \\ \| \psi_n(w_1') - \psi_n(w_2') \| \leq \varepsilon \| w_1' - w_2' \| \text{ for } w_1', w^{\mathfrak{e}} \in B_{\rho_0}^1(0) \text{ and} \\ \text{graph } \psi_n \subset W_{\delta_0}^u(g^{N+n}(z), g) \text{ for } n > 0. \end{cases}$$

Put $r = \min\{\rho_0, r_2\}$ and fix $0 < \varepsilon \leq \varepsilon'$ such that satisfies (10). Then, from $(*)$ and $(**)$ we can take an integer $n_3 \geq n_2$ such that for every $n \geq n_3$, ψ_n and φ_n satisfy the assumptions of Claim 3. Thus

$$C_{w_n} \cap W_{\delta_0}^u(g^{N+n}(z), g) \neq \phi.$$

This is a contradiction.

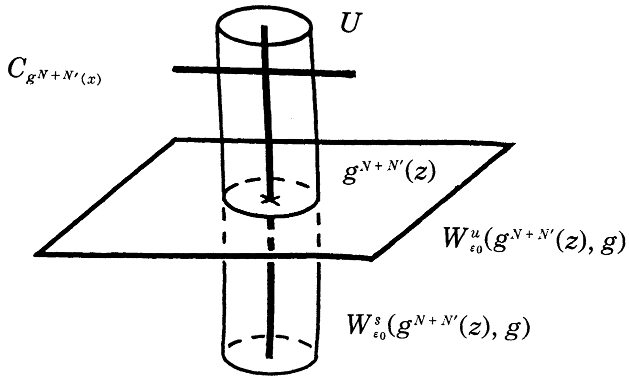
Take and fix $n > 0$ such that $d(g^{N+n}(x)), g^{N+n}(z)) < \delta/2$ where $0 < \delta = \delta(\varepsilon, g) < \varepsilon$ is a number given in (1). Let $N' = N'(n, \delta) \geq n$ be as in Claim 5 and put

$$B_{\delta/2}^u(g^{N+N'}(z)) = B_{\delta/2}(g^{N+N'}(z)) \cap W_{\varepsilon_0}^u(g^{N+N'}(z), g).$$

Claim 6. *There exists* $w \in B_{\delta/2}^u(g^{N+N'}(z)) \setminus \{g^{N+N'}(z)\}$ *such that* dim $W^s(w, g) = 1$ *and* $C_w \cap C_{g^{N+n'}(x)} = \phi$ *where* C_w *is the connected component of* w *in* $W^s(w, g) \cap B_{\varepsilon_0}(g^{N+N'}(z))$.

Before beginning with the proof of the Claim 6 we remark the following properties.

Remarks. (i) For every tubular neighborhood U of $W^s_{\varepsilon_0}(g^{N+N'}(z), g)$, there is no sink periodic point p of g such that $U \setminus W^s_{\varepsilon_0}(g^{N+N'}(z), g) \subset W^s(p, g)$.



To prove this, we suppose that there is a sink p satisfying $U \setminus W^s_{\varepsilon_0}(g^{N+N'}(z), g) \subset W^s(p, g)$. Then there are $N'' > N'$ and $0 < \varepsilon'' \leq \varepsilon$ such that for every $n \geq N''$ and $0 < \hat{\varepsilon} \leq \varepsilon''$, $C_{g^{N+n}(x)} \cap B_{\hat{\varepsilon}}(g^{N+n}(z)) \subset U$. For $n \geq N''$ we put

$$S^u(g^{N+n}(z)) = \partial B^u_{\varepsilon_0}(g^{N+n}(z)),$$

and for $0 < \hat{\varepsilon} \leq \varepsilon$ let $\hat{\delta} > 0$ be a number as in the definition of POTP of g. Then for every $\hat{\delta}$ there are $n_1(\hat{\delta}), n_2(\hat{\delta}) \geq N''$ such that

$$d(g^{N+n_1(\hat{\delta})}(x), g^{N+n_1(\hat{\delta})}(z)) < \frac{\hat{\delta}}{2}$$

and

$$d(g^{N+n_1(\hat{\delta})}(z), w) < \frac{\hat{\delta}}{2}$$

for every $w \in g^{-n_2(\hat{\delta})}(S^u(g^{N+n_1(\hat{\delta})+n_2(\hat{\delta})}(z)))$. Thus for every $w \in g^{-n_2(\hat{\delta})}(S^u(g^{N+n_1(\hat{\delta})+n_2(\hat{\delta})}(z)))$,

$$\{\cdots, g^{N+n_1(\hat{\delta})-2}(x), g^{N+n_1(\hat{\delta})-1}(x), w, g(w), \cdots\}$$

is a $\hat{\delta}$-pseudo orbit for g. However it is easy to see that if we fix $\hat{\varepsilon}$ small enough, then there exists a $\hat{\delta}$-pseudo orbit among them which can not be g–$\hat{\varepsilon}$-traced since

$$C_{g^{N+n_1(\hat{\delta})}(x)} \cap B_{\hat{\varepsilon}}(g^{N+n_2(\hat{\delta})}(x)) \subset U.$$

(ii) There is no stable manifold $W^s(w, g)$ with $\dim W^s(w, g) = 2$ and $\overline{W^s(w, g)} \supset W^s(g^{N+N'}(z), g)$ such that there is a sequence of points $\{w_n\}$ in $W^s(w, g) \cap C_{g^{N+N'}(x)} \setminus \{g^{N+N'}(x)\}$ satisfying $w_n \to g^{N+N'}(x)$ and $d_s(w_n, w_{n+1}) \to$

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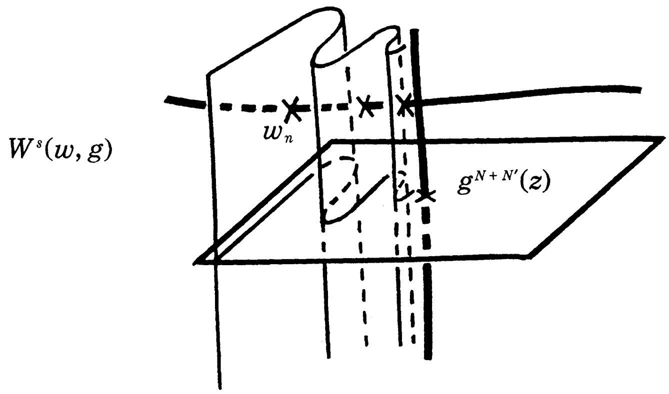
0 as $n \to \infty$. Here d_s is a metric on $W^s(w, g)$ induced from $\| \cdot \|$.

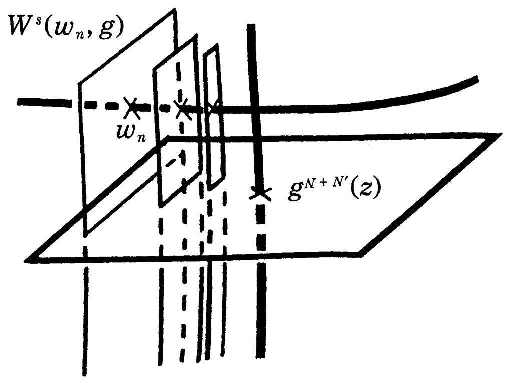


In fact, if there exists such a stable manifold then we can find $v \in g^{-2N-N'}(W^s(w, g))$ such that

$$\tan \measuredangle\ (T_v W^s(g^{-2N+N'}(w), g), \tilde{E}_1^u(v)) < \theta,$$

where $\theta > 0$ is a number given in Claim 2 for g. This is absurd since $W^s(w, g) \not\subset W^s(\Lambda_1(g), g)$.

(iii) If there is a sequence of points $\{w_n\}$ in $C_{g^{N+N'}(x)}$ such that $w_n \to g^{N+N'}(x)$, $r(\overline{W^s(w_n, g)}) \to 0$ as $n \to \infty$ and $\dim W^s(w_n, g) = 2$ for all $n \geq 0$, then Claim 6 is true. Here $r(\overline{W^s(w, g)})$ denotes the maximal radius of a closed ball in $(\overline{W^s(w, g)})$ centered at w with respect to d_s.



Indeed, fix $n > 0$ and let $\theta > 0$ be a number given in Claim 2 for g. Suppose that there is $v \notin W^s(w_n, g)$ such that $\dim W^s(v, g) = 2$ and $\partial W^s(w_n, g) \cap W^s(v, g) \neq \phi$. If we pick a point $v' \in \partial W^s(w_n, g) \cap W^s(v, g)$, then there are sufficiently large $m > 0$ and $v'' \in W^s(g^m(w_n), g)$ arbitrarily near to $g^m(v')$

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such that

$$\tan \measuredangle (T_{v''}W^s(g^m(w_n), g), \tilde{E}_1^u(v'')) < \theta.$$

This is a contradiction. Thus $\partial W^s(w_n, g)$ consists of two 1-dimensional stable manifolds (since $\bigcup \{W^s(p, g) : p$ is a sink periodic point of $g\}$ is open in M) .

Proof of Claim 6. We divides the proof into two cases.

Case 1. For every

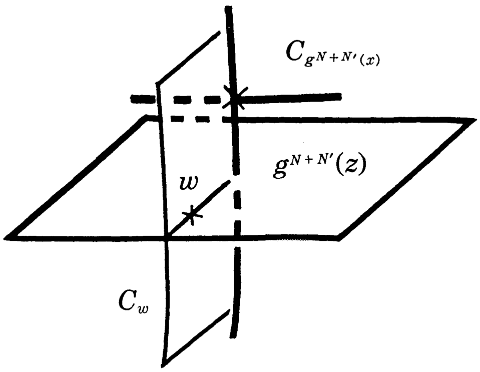
$$w \in C_{g^{N+N'}(x)} \cap B_{1/2N'}(g^{N+N'}(z)\} \setminus \{g^{N+N'}(z)\}$$

and $0 < r' < 1/N'$ such that

$$B_{r'}(w) \cap C_{g^{N+N'}(x)} \subset C_{g^{N+N'}(x)} \setminus \{g^{N+N'}(x)\}.$$

there is $w' \in B_{r'}(w) \cap C_{g^{N+N'}(x)}$ such that $\dim W^s(w', g) \geq 2$.

Note that $\bigcup \{W^s(\Lambda_k(g), g) : \Lambda_k(g)$ is an attractor$\}$ is an open set of M. By Remarks (i) , (ii) and (iii) we may assume that there is $w \in W^u_{\delta/2}(g^{N+N'}(z), g) \setminus \{g^{N+N'}(z)\}$ such that $\dim W^s(w, g) = 2$, $\overline{C_w} \cap W^s(g^{N+N'}(z), g) \neq \phi$ and $C_w \cap C_{g^{N+N'}(x)} = \phi$. Here C_w denotes the connected component of w in $W^s(w, g) \cap B_{\varepsilon_0}(g^{N+N'}(z))$.



For every $0 < \beta \leq \varepsilon$, let $0 < \gamma(\beta) \leq \beta$ be a number as in the definition of POTP of g. Take $v \in B_{\gamma(\beta)}(g^{N+N'}(x)) \cap C_w$. Then

$$\{\cdots, g^{N+N'-2}(x), g^{N+N'-1}(x), v, g(v), \cdots\}$$

is a $\gamma(\beta)$-pseudo orbit of g. Thus there exists $\hat{v} \in C_{g^{N+N'}(x)}$ such that $d(g^n(v), g^n(\hat{v})) < \beta$ for all $n \in \mathbf{Z}$, On the other hand, since $v \notin W^s(\Lambda_2(g), g)$, there exists $n_{v,\beta} > 0$, such that $g^i(v) \in B_{r_1/2}(\Lambda_2(g))$ for $0 \leq \mathrm{i} \leq n_{v,\beta}$ and $g^{n_{v,\beta}+1}(v) \notin \mathrm{B}_{r_1/2}(\Lambda_2(\mathrm{g}))$. Thus we have $B_\varepsilon(g^i(v)) \subset B_{r_1}(\Lambda_2(g))$ for $0 \leq i \leq n_{v,\beta}$. Let be $C_{\hat{v}}$ be the connected component of $\hat{v}$ in $C_{g^{N+N'}(x)} \cap B_\varepsilon(\hat{v})$. Then, by the hyperbolicity

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of $B_{r_1}(\Lambda_2(g))$ there is $0 < \varepsilon_3 \leq \varepsilon$ such that

$$\inf_{v,\beta} d_u(\partial(g^{n_{v,\beta}}(C_{\hat{v}})).\ g^{n_{v,\beta}}(\hat{v})) \geq \varepsilon_3,$$

where d_u denotes a metric on unstable manifolds induced from $\|\cdot\|$. Since $g \in \mathscr{P}^1(M)$ and $g^i(C_{\hat{v}}) \cap g^i(C_\omega) = \phi$ for $0 \leq i \leq n_{v,\beta}$, by using the methods stated in the proofs of Claims 2 and 4 we can find $g' \in \mathrm{Diff}^1(M)$ arbitrarily near to g such that $g'|_{\Omega(g)} = g|_{\Omega(g)}$, $W^s(g^{n_{v,\beta}}(v), g') \cap W^u(g^{n_{v,\beta}}(\hat{v}), g') \neq \phi$, and $W^2(g^{n_{v,\beta}}(v), g')$ does not meet transversely to $W^u(g^{n_{v,\beta}}(\hat{v}), g')$. This is a contradiction.

Case 2. There exists

$$w \in C_{g^{N+N'}(x)} \cap B_{1/2N'}(g^{N+N'}(\mathrm{x}))\backslash\{g^{N+N'}(x)\}$$

and $0 < r' < 1/2N'$ such that

$$B_{r'}(w) \cap C_{g^{N+N'}(x)} \subset C_{g^{N+N'}(x)}\backslash\{g^{N+N'}(x)\}$$

and for every $w' \in B_{r'}(w) \cap C_{g^{N+N'}(x)}$, $\dim W^s(w', g) = 1$.

By Claim 4, there is $0 < \delta' = \delta'(r', g) < 1/2N'$. such that for every $v \in B_{\delta'}(w)\backslash C_{g^{N+N'}(x)}$, $\dim W^s(v, g) = 1$. Denote by C_v the connected component of v in $W^s(v, g) \cap B_{\varepsilon_0}(v)$ for $v \in B_{\delta'}(w) \backslash C_{g^{N+N'}(x)}$. Take and fix $v \in B_{\delta'}(w) \backslash C_{g^{N+N'}(x)}$ such that $C_v \cap C_{g^{N+N'}(x)} = \phi$. Then there is $v' = C_v \cap W^u_{\delta/2}(g^{N+N'}(z), g) \neq \phi$ by Claim 5 (since $v \in B_{1/2N'}(g^{N+N'}(x))$). This completes the proof of Claim 6.

It is checked that for every $w \in B^u_{\delta/2}(g^{N+N'}(z))$,

$$W^s(w, g) \cap C_{g^{N+N'}(z)} \neq \phi.$$

Indeed, since $d(w, g^{N+N'}(x)) < \delta$ for $w \in B^u_{\delta/2}(g^{N+N'}(z))$, there is $w' \in M$ such that

$$d(g^n(w'), g^n(w)) < \varepsilon \text{ for all } n \geq 0$$

and

$$(13) \qquad d(g^{-n}(w'), g^{N+N'-n}(x)) < \varepsilon \text{ for all } n \geq 0$$

Thus $g^{-2N+N'}(w') \in C_{g^{-N}_{(x)}}$ and so $g^{-N'}(w') \in g^{2N}(C_{g^{-N}_{(x)}})$. Since $g^{-N'}(w') \in B_\varepsilon(g^N(x))$ (by (13)), we have $g^{-N'}(w') \in C_{g^N_{(x)}}$ and hence $w' \in C_{g^{N+N'}(x)}$. Thus $W^s(w, g) \cap C_{g^{N+N'}(x)} \neq \phi$ since $w' \in W^s(w,g)$.

Let $W \in B^u_{\delta/2}(g^{N+N'}(z)) \backslash \{g^{N+N'}(z)\}$ be as in Claim 6. Then $w \in W^u(\Lambda_i(g), g)$ and $\dim W^s(w, g) = 1$. Since $M = \bigcup_{i=1}^{\ell} W^s(\Lambda_i(g), g)$, we may suppose that $w \in W^s(\Lambda_3(g),g)$. Clearly $\mathrm{Ind}\,\Lambda_3(g) = 1$ and

$$w \in W^u(\Lambda_2(g), g) \cap W^s(\Lambda_3(g), g) \neq \phi.$$

It is easy to see that $\Lambda_2(g) \neq \Lambda_3(g)$. For, if $\Lambda_2(g) = \Lambda_3(g)$ then $w \in W^u(\Lambda_2(g), g) \cap W^s(\Lambda_2(g), g) = \Lambda_2(g)$. Thus $C_{g^{N+N'}_{(x)}} \cap W^s_\varepsilon(w, g) = \phi$. However, since

$$\{\cdots, g^{-1}(x), x, g(x), \cdots, g^{N+N'-1}(z), w, g(w), \cdots\}$$

is a δ-pseudo orbit of g, we have $C_{g^{N+N'}(x)} \cap W^s_\varepsilon(w, g) = \phi$. This is a contradiction. Hence $\Lambda_2(g) \neq \Lambda_3(g)$.

Since $w \in B^u_{\delta/2}(g^{N+N'}(z))$, we have $W^s(w, g) \cap C_{g^{N+N'}_{(x)}} \neq \phi$. Thus $W^s(\Lambda_1(g),g) \cap W^u(\Lambda_3(g), g) \neq \phi$.

The conclusions obtained above is summarized as follows

$$(14)\quad \begin{cases} \mathrm{Ind}\,\Lambda_3(g) = 1 \\ \Lambda_2(g) \neq \Lambda_3(g). \\ W^u(\Lambda_2(g), g) \cap W^s(\Lambda_3(g), g) \neq \phi \text{ and} \\ W^u(\Lambda_1(g), g) \cap W^s(\Lambda_3(g), g) \neq \phi. \end{cases}$$

By (14) there exists a cycle among basic sets of g. Indeed, since there are $z_1 \in \Lambda_1(g)$ and $z_2 \in \Lambda_2(g)$ such that $W^u(z_1, g) \cap W^s(z_2, g) \neq \phi$ and $\dim W^u(z_1, g) = \dim W^s(z_2, g) = 1$, by (14) we can find $z_3 \in \Lambda_3(g) \neq \Lambda_2(g)$ such that $W^u(z_1, g) \cap W^s(z_3, g) \neq \phi$, $\dim W^s(z_3, g) = 1$ and $W^u(\Lambda_2(g), g) \cap W^s(\Lambda_3(g), g) \neq \phi$. Since $W^u(z_1, g) \cap W^s(z_3, g) \neq \phi$ and $\dim W^u(z_1, g) = \dim W^s(z_3, g) = 1$, by the same manner we can find $z_4 \in \Lambda_4(g) \neq \Lambda_3(g)$ such that $W^u(z_1, g) \cap W^s(z_4, g) \neq \phi$, $\dim W^s(z_4, g) = 1$ and $W^u(\Lambda_3(g), g) \cap W^s(\Lambda_4(g), g) \neq \phi$. In this repetition we have a cycle among basic sets $\Lambda_1(g), \cdots, \Lambda_\ell(g)$ and reach a contradiction. We finish the proof of Theorem B.

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Kisarazu National College of Technology
Kisarazu, Chiba 292
Japan