BULL. AUSTRAL. MATH. SOC. 35J85, 49J40
VOL. 64 (2001) [381–393]

HEMIVARIATIONAL INEQUALITIES WITH THE POTENTIAL CROSSING THE FIRST EIGENVALUE

SOPHIA TH. KYRITSI AND NIKOLAOS S. PAPAGEORGIOU

In this paper we study a nonlinear hemivariational inequality involving the p-Laplacian. Our approach is variational and uses a recent nonsmooth Linking Theorem, due to Kourogenis and Papageorgiou (2000). The use of the Linking Theorem instead of the Mountain Pass Theorem allows us to assume an asymptotic behaviour of the generalised potential function which goes beyond the principal eigenvalue of the negative p-Laplacian with Dirichlet boundary conditions.

1. INTRODUCTION

Let Z be a bounded domain with a $C^{1,\alpha}$, $0 < \alpha < 1$, boundary Γ. We consider the following nonlinear hemivariational inequality:

$$(1) \qquad \left\{ \begin{array}{l} -\operatorname{div}\Big(\|Dx(z)\|^{p-2}Dx(z)\Big) \in \partial j\big(z, x(z)\big) \text{ almost everywhere on } Z \\ x|_\Gamma = 0 \end{array} \right\}$$

where $2 \leqslant p < \infty$, $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function measurable in $z \in Z$ and locally Lipschitz in $x \in \mathbb{R}$ and $\partial j(z,x)$ denotes the subdifferential in the sense of Clarke (generalised subdifferential, see Section 2). Recently nonlinear hemivariational inequalities were studied by Gasinski and Papageorgiou **[9, 10, 11, 12, 13]** using nonsmooth critical point theory and producing mountain pass type critical points for the nonsmooth locally Lipschitz energy functional. The results involved asymptotic conditions at $\pm\infty$ restricted by the first eigenvalue of the negative p-Laplacian with Dirichlet boundary conditions, that is, of $\big(-\Delta_p, W_0^{1,p}(Z)\big)$ with $-\Delta_p x = -\operatorname{div}\big(\|Dx\|^{p-2}Dx\big)$. In this paper we consider asymptotic behaviour which goes beyond the first eigenvalue. This means that we have to abandon the use of the mountain pass theorem and look for critical points of the linking type. In this direction we use the recent general linking type theorem due to Kourogenis and Papageorgiou **[18]**. For semilinear hemivariational inequalities (that is, $p = 2$) going beyond the first eigenvalue to higher ones, presents no problem since we have a full knowledge of the spectrum of $\big(-\Delta, H_0^1(Z)\big)$. In fact results in this direction were obtained

Received 17th January, 2001

https://doi.org/10.1017/S0004972700019857 Published online by Cambridge University Press

by Goeleven, Motreanu and Panagiotopoulos [**14, 15**] and Gasinski and Papageorgiou [**12**]. However for $p > 2$ (quasilinear problems) we encounter serious difficulties whose sources are the lack of full-knowledge of the spectrum of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$ and the lack of Rayleigh quotients (that is, variational expressions) for the higher eigenvalues, which makes it difficult to construct links for the quasilinear case.

Hemivariational inequalities arise in physical problems when we deal with nonconvex, nonsmooth energy functionals. Such functions appear in mechanics and engineering if one wants to consider more realistic mechanical laws of nonmonotone and multivalued nature. For concrete applications we refer to the books of Naniewicz and Panagiotopoulos [**22**] and Panagiotopoulos [**23**].

In the next section for the convenience of the reader, we recall the basic definitions and facts from the nonsmooth critical point theory as this was originally formulated by Chang [**5**] and recently extended by Kourogenis and Papageorgiou [**18**] and outline the spectral properties of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$.

2. Preliminaries

Let X be a Banach space and X^* its topological dual. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood U of x and a constant k depending on U such that $|f(z) - f(y)| \leqslant k\|z - y\|$ for all $z, y \in U$. It is well-known from convex analysis that a proper convex and lower semicontinuous function $g : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom} g = \{x \in X : g(x) < +\infty\}$. By analogy with the directional derivative of a convex function, for a locally Lipschitz function $f : X \to \mathbb{R}$ we define the generalised directional derivative at $x \in X$ in the direction $h \in X$, by

$$f^0(x; h) = \limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{f(x' + th) - f(x')}{t}.$$

It is easy to check that $h \to f^0(x; h)$ is a sublinear, continuous function. So by the Hahn-Banach Theorem $f^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leqslant f^0(x; h) \text{ for all } h \in X\}.$$

This set $\partial f(x)$ is called the "generalised subdifferential" of f at $x \in X$. If $f, g : X \to \mathbb{R}$ are both locally Lipschitz functions, then $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $\lambda \in \mathbb{R}$. Moreover, if $f : X \to \mathbb{R}$ is convex, then as we already mentioned f is locally Lipschitz and the generalised subdifferential coincides with the subdifferential in the sense of convex analysis (see Hu and Papageorgiou [**16**]). Also, if f is strictly differentiable (in particular if f is continuously Gateaux differentiable at $x \in X$), then $\partial f(x) = \{f'(x)\}$.

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. A point $x \in X$ is said to be a "critical point" of f if $0 \in \partial f(x)$. It is easy to see that if $x \in X$ is a local extremum of f, then $0 \in \partial f(x)$. It is well-known that in smooth critical point theory, a compactness condition, known as the "Palais-Smale condition" plays a prominent role. A weaker form of this condition was proposed by Cerami [**4**], and Bartolo, Benci and Fortunato [**3**] showed that Cerami's weaker condition suffices to have a deformation lemma and through it obtain minimax theorems for locating critical points. Kourogenis and Papageorgiou [**18**] gave a nonsmooth version of Cerami's condition. Namely a locally Lipschitz functional $f : X \to \mathbb{R}$ satisfies the "nonsmooth C-condition", if every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{f(x_n)\}_{n\geqslant 1}$ is bounded and $(1 + \|x_n\|)m(x_n) \to 0$, has a strongly convergent subsequence. Here $m(x) = \inf\{\|x^*\| : x^* \in \partial f(x)\}$. Using this compactness type condition, Kourogenis and Papageorgiou [**18**] proved a nonsmooth deformation lemma and then with its help obtained among other things a "Linking Theorem". For easy reference we recall that theorem. First a definition:

Definition. Let $C_1 \subseteq C$ and D be subsets of X. We say that C_1 and D "link" in X, if

(a) $C_1 \cap D = \emptyset$ and

(b) for every $\vartheta \in C(C, X)$ with $\vartheta|_{C_1}$ =identity, we have $\vartheta(C) \cap D \neq \emptyset$.

Kourogenis and Papageorgiou [**18**], proved the following "Linking Theorem":

Theorem 1. *If X is a reflexive Banach space, $C_1 \subseteq C$ and D nonempty subsets of X with D closed, C_1 and D link in X, $\Gamma = \{\vartheta \in C(C, X) : \vartheta|_{C_1} = identity\}$, $\phi : X \to \mathbb{R}$ is locally Lipschitz and satisfies the nonsmooth C-condition, $c = \inf\limits_{\vartheta\in\Gamma} \sup\limits_{c\in C} \phi(\vartheta(c))$ and $\sup\limits_{C_1} \phi \leqslant \inf\limits_{D} \phi$, then $c \geqslant \inf\limits_{D} \phi$ and c is a critical value of ϕ, that is, there exists a critical point $x \in X$ of ϕ such that $\phi(x) = c$. Moreover, if $c = \inf\limits_{D} \phi$, then there exists $x \in D$ such that $x \in K_c = \{x \in X : 0 \in \partial\phi(x),\ \phi(x) = c\}$.*

About the spectrum of $(-\Delta_p, W_0^{1,p}(Z))$ we know the following (see Anane [**1**], Lindqvist [**21**]).

Consider the following nonlinear eigenvalue problem.

$$(2)\qquad \left\{\begin{array}{l} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = \lambda|x(z)|^{p-2}x(z) \text{ almost everywhere on } Z \\ x|_\Gamma = 0. \end{array}\right\}.$$

The least $\lambda \in \mathbb{R}$ for which (2) has a nontrivial solution is called the first (principal) eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(Z))$. The first eigenvalue λ_1 is positive, isolated and simple (that is, the associated eigenfunctions are constant multiples of each other). Moreover, for λ_1 we have a variational characterisation via the Rayleigh quotient, that is,

$$(3)\qquad \lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z),\ x \neq 0\right].$$

The minimum is realised at the normalised eigenfunction u_1. Note that if u_1 minimises the Rayleigh quotient, then so does $|u_1|$ and so it follows that the first eigenfunction u_1 does not change sign on Z. In fact since we have assumed Z to have a $C^{1,\alpha}$-boundary Γ, from the regularity theorem of Lieberman [**20**] we have that $u_1 \in C^{1,\beta}(\overline{Z})$, $0 < \beta < 1$ and $u_1(z) \neq 0$ for all $z \in Z$ and so we may assume that $u_1(z) > 0$ for all $z \in Z$. The Liusternik-Schnirelmann theory, gives in addition to λ_1, a whole strictly increasing sequence $\{\lambda_k\}_{k\geqslant 1} \subseteq \mathbb{R}_+$ for which problem (2) has a nontrivial solution. These numbers are defined as follows. Let $G = \left\{x \in W_0^{1,p}(Z) : \|Dx\|_p = 1\right\}$ and let $\psi : G \to \mathbb{R}_-$ be given by $\psi(x) = -\|x\|_p^p$. We set

$$c_n = \inf_{K\in A_n} \sup_{x\in K} \psi(x) \tag{4}$$

$$\left\{\begin{array}{l} -\operatorname{div}\Big(\big\|Dx(z)\big\|^{p-2}Dx(z)\Big) \in \partial j\big(z, x(z)\big) \text{ almost everywhere on } Z \\ x|_\Gamma = 0 \end{array}\right\} \tag{5}$$

where $A_n = \left\{K \subseteq G : K \text{ is symmetric, closed and } \gamma(K) \geqslant n\right\}$, with γ denoting the Krasnoselskii $\mathbf{Z}_2$-genus (see Struwe [**24**, p.86]). The sequence $\left\{\lambda_n = -(1/c_n)\right\}_{n\geqslant 1}$ is strictly increasing and tends to $+\infty$. These numbers are the so-called "Liusternik-Schnirelmann eigenvalues" (or "variational eigenvalues") of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$. For $p = 2$ we know that these are all the eigenvalues of $\left(-\Delta, H_0^1(Z)\right)$ but for $p > 2$ we cannot say this. Recently Anane and Tsouli [**2**] proved that if $\lambda_2^* = \inf\left[\lambda > \lambda_1 : \lambda \text{ is an eigenvalue of } \left(-\Delta_p, W_0^{1,p}(Z)\right)\right]$, then $\lambda_2^* = \lambda_2$, that is, the second eigenvalue and the second Liusternik-Schnirelmann eigenvalue of $\left(-\Delta_p, W_0^{1,p}(Z)\right)$ coincide. Set

$$V_k = \Big\{x \in W_0^{1,p}(Z) : -\operatorname{div}\Big(\big\|Dx(z)\big\|^{p-2}Dx(z)\Big) = \lambda_k\big|x(z)\big|^{p-2}x(z) \text{ almost everywhere on } Z\Big\},\ k \geqslant 1.$$

These are symmetric, closed cones, but in general are not subspaces of $W_0^{1,p}(Z)$, unless λ_k is simple. Also if $W_n = \bigcup_{k=1}^{n} V_k$ and $\widehat{W}_n = \bigcup_{k>n} V_n$, then in contrast to the linear case $(p = 2)$, for $p > 2$ in general we do not have the inequalities

$$\|Dx\|_p^p \leqslant \lambda_k\|x\|_p^p \text{ for } x \in W_k \text{ and } \|Dx\|_p^p \geqslant \lambda_{k+1}\|x\|_p^p \text{ for } x \in \widehat{W}_k.$$

Precisely this negative fact is the source of problems in constructing linkings in the quasilinear case.

3. Auxiliary results

In this section we prove some auxiliary results which will pave the way for the existence theorem in Section 4. Our hypotheses on the nonsmooth potential function $j(z, x)$ are the following:

$$H(j)\ j : Z \times \mathbb{R} \to \mathbb{R}$$

is a function such that

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leqslant \alpha_1(z) + c_1|x|^{r-1}$$

with $\alpha_1 \in L^{r'}(Z)$, $c_1 > 0$,

$$1 \leqslant r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geqslant N \end{cases}, \quad \frac{1}{r} + \frac{1}{r'} = 1$$

and $j(\cdot\,,0) \in L^{r'}(Z)$;

(iv) $\lim\limits_{|x|\to\infty} \big[u(z,x)x - pj(z,x)\big] = -\infty$ uniformly for almost all $z \in Z$ and all $u(z,x) \in \partial j(z,x)$;

(v) $\lambda_1 \leqslant \liminf\limits_{|x|\to\infty} \dfrac{pj(z,x)}{|x|^p} \leqslant \limsup\limits_{|x|\to\infty} \dfrac{pj(z,x)}{|x|^p} < \lambda_2$ uniformly for almost all $z \in Z$.

REMARK. A more restricted version of hypothesis H(j)(iv) was employed in the context of smooth problems by Costa and Magalhaes [**7**, Theorem 2], where it was also assumed that $\lim\limits_{|x|\to\infty} \big(pj(z,x)\big)/|x|^p = \lambda_1$ uniformly for almost all $z \in Z$. So our work here extends in many different directions of Costa and Magalhaes [**7**, Theorem 2].

Note that by virtue of hypothesis H(j)(iii) and the Lebourg mean value theorem (see Lebourg [**19**] or Clarke [**6**, p.41]) we have that for almost all $z \in Z$ and all $x \in \mathbb{R}$

$$\big|j(z,x)\big| \leqslant \alpha_1'(z) + c_1'|x|^r$$

with $\alpha_1' \in L^{r'}(Z)$ and $c_1' > 0$. We introduce the energy functional $\phi : W_0^{1,p}(Z) \to \mathbb{R}$ defined by $\phi(x) = \big(\|Dx\|_p^p\big)/p - \int_Z j\big(z,x(z)\big)dz$. From Hu and Papageorgiou [**17**, p.313], we know that $J : L^r(Z) \to \mathbb{R}$ defined by $J(x) = \int_Z j\big(z,x(z)\big)dz$ is locally Lipschitz, in particular then since $W_0^{1,p}(Z)$ is embedded continuously in $L^r(Z)$, $J|_{W_0^{1,p}}$ is locally Lipschitz. So ϕ is locally Lipschitz.

PROPOSITION 2. *If hypotheses H(j) hold, then ϕ satisfies the nonsmooth C-condition.*

PROOF: Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$\phi(x_n) \to \xi \text{ and } \big(1 + \|x_n\|\big)m(x_n) \to 0 \text{ as } n \to \infty.$$

We can find $x_n^* \in \partial\phi(x_n)$ such that $\|x_n^*\| = m(x_n)$, $n \geqslant 1$. The existence of such an element follows from the weak compactness of $\partial\phi(x_n)$ and the weak lower semicontinuity of the norm. Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z)\big)_{\mathbb{R}^N} dz \text{ for all } x, y \in W_0^{1,p}(Z).$$

Here and in what follows by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $\big(W_0^{1,p}(Z), W^{-1,q}(Z)\big)$. We know (see for example Gasinski and Papageorgiou [**13**]) that A is monotone, continuous, hence maximal monotone (see Hu and Papageorgiou [**16**], p.309). Also if $\widehat{J} = J|_{W_0^{1,p}(Z)}$, as we already mentioned $\widehat{J}$ is locally Lipschitz on $W_0^{1,p}(Z)$ and from Chang [**5**, Theorem 2.2] and Clarke [**6**, Theorem 2.7.5, p.83] we have that for all $x \in W_0^{1,p}(Z)$, $\partial\widehat{J}(x) \subseteq L^{r'}(Z)$ and if $u \in \partial\widehat{J}(x)$, then $u(z) \in \partial j\big(z, x(z)\big)$ almost everywhere on Z. Then for every $n \geqslant 1$ we have

$$x_n^* = A(x_n) - u_n \text{ with } u_n \in \partial\widehat{J}(x_n).$$

From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ we have

$$\begin{aligned}\Big|\big\langle x_n^*, x_n\big\rangle - p\phi(x_n) + p\xi\Big| &\leqslant \|x_n^*\|\|x_n\| + \big|p\phi(x_n) - p\xi\big| \\ &\leqslant m(x_n)\big(1 + \|x_n\|\big) + p\big|\phi(x_n) - \xi\big| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Note that if by $(\cdot,\cdot)_{rr'}$ we denote the duality brackets for the pair $\big(L^r(Z), L^{r'}(Z)\big)$, we have

$$\langle x_n^*, x_n\rangle = \big\langle A(x_n), x_n\big\rangle - (u_n, x_n)_{rr'} = \|Dx_n\|_p^p - \int_Z u_n(z)x_n(z)dz.$$

So we obtain that

$$p\phi(x_n) - \langle x_n^*, x_n\rangle = \int_Z \Big(u_n(z)x_n(z) - pj\big(z, x(z)\big)\Big)dz \to p\xi \text{ as } n \to \infty. \tag{6}$$

We shall show that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not the case. Then by passing to a subsequence if necessary we may assume that $\|x_n\| \to \infty$. Throughout this proof the norm $\|\cdot\|$ on the Sobolev space $W_0^{1,p}(Z)$ is defined by $\|x\| = \|Dx\|_p$ which by Poincare's inequality is equivalent to the usual Sobolev norm. Let $y_n = x_n/\|x_n\|$, $n \geqslant 1$. Since $\|y_n\| = 1$, $n \geqslant 1$, we may assume (at least for a subsequence) that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$, $y_n \to y$ in $L^p(Z)$ (recall that $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$), $y_n(z) \to y(z)$ almost everywhere on Z and $\big|y_n(z)\big| \leqslant k(z)$ almost everywhere on Z with $k \in L^p(Z)$. Because of hypothesis H(j)(v) we can find $\lambda_1 \leqslant \eta < \lambda_2$ and $M > 0$ such that for almost all $z \in Z$ and all $|x| \geqslant M$ we have

$$j(z, x) \leqslant (\eta/p)|x|^p.$$

Also from the growth of j established earlier, we have that for almost all $z \in Z$ and all $|x| < M$

$$\big|j(z, x)\big| \leqslant \beta(z) \text{ with } \beta \in L^{r'}(Z).$$

Therefore we can say that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$j(z,x) \leqslant (\eta/p)|x|^p + \beta(z).$$

For every $n \geqslant 1$, we have

$$\begin{aligned}
\frac{\phi(x_n)}{\|x_n\|^p} &= \frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{j\big(z,x_n(z)\big)}{\|x_n\|^p}dz \\
&= \frac{1}{p} - \int_Z \frac{j\big(z,x_n(z)\big)}{\|x_n\|^p}dz \quad (\text{since } \|Dy_n\|_p = \|y_n\| = 1) \\
&\geqslant \frac{1}{p} - \int_Z \frac{\eta\big|x_n(z)\big|^p}{p\|x_n\|^p}dz - \int_Z \frac{\beta(z)}{\|x_n\|^p}dz \\
&= \frac{1}{p} - \int_Z \frac{\eta}{p}\big|y_n(z)\big|^p dz - \int_Z \frac{\beta(z)}{\|x_n\|^p}dz.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$0 \geqslant \frac{1}{p} - \frac{\eta}{p}\|y\|_p^p = \frac{1}{p}\big(1 - \eta\|y\|_p^p\big).$$

From this inequality it follows that $y \neq 0$.

Because of hypothesis H(j)(iv), we can find $M_1 > 0$ such that for almost all $z \in Z$, all $|x| \geqslant M_1$ and all $u \in \partial j(z,x)$ we have

$$ux - pj(z,x) \leqslant -1.$$

On the other hand from hypothesis H(j)(iii) and the resulting from it growth of j, we infer that for almost all $z \in Z$, all $|x| < M_1$ and all $u \in \partial j(z,x)$, we have

$$ux - pj(z,x) \leqslant \eta_1 \quad \text{for some } \eta_1 > 0.$$

Let $C = \big\{z \in Z : y(z) \neq 0\big\}$. Evidently $|C|_N > 0$ ($|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$) and for almost all $z \in C$ we have $\big|x_n(z)\big| \to \infty$. We have

$$\begin{aligned}
&\int_Z \Big(u_n(z)x_n(z) - pj\big(z,x_n(z)\big)\Big)dz \\
&\quad = \int_C \Big(u_n(z)x_n(z) - pj\big(z,x_n(z)\big)\Big)dz + \int_{C^c} \Big(u_n(z)x_n(z) - pj\big(z,x_n(z)\big)\Big)dz \\
&\quad \leqslant \int_C \Big(u_n(z)x_n(z) - pj\big(z,x_n(z)\big)\Big)dz + \eta_1|C^c| \to -\infty \text{ as } n \to \infty
\end{aligned}$$

a contradiction to (5). This proves the boundedness of $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$. Thus we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^r(Z)$ (from the compact embedding of $W_0^{1,p}(Z)$ in $L^r(Z)$). We have

$$\langle x_n^*, x_n - x\rangle = \big\langle A(x_n), x_n - x\big\rangle - (u_n, x_n - x)_{rr'}.$$

https://doi.org/10.1017/S0004972700019857 Published online by Cambridge University Press

From the choice of the sequence $\{x_n\}_{n\geqslant 1}$ we have that $|\langle x_n^*, x_n - x\rangle| \leqslant m(x_n)\|x_n - x\| \to 0$ as $n \to \infty$. Also since $\{u_n\}_{n\geqslant 1} \subseteq L^{r'}(Z)$ is bounded (hypothesis H(j)(iii)), it follows that $(u_n, x_n - x)_{rr'} \to 0$ as $n \to \infty$. Therefore $\lim \langle A(x_n), x_n - x\rangle = 0$. But as we already said A is maximal monotone, hence it is generalised pseudomonotone and so

$$\|Dx_n\|_p^p = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_p^p$$

(see Hu and Papageorgiou [**16**, p.365]).

Remark that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and $\|Dx_n\|_p \to \|Dx\|_p$. The space $L^p(Z, \mathbb{R}^N)$ is uniformly convex, hence it has the Kadec-Klee property (see Hu and Papageorgiou [**16**, p.28]) and so $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$. Therefore we conclude that $x_n \to x$ in $W_0^{1,p}(Z)$ which proves that ϕ satisfies the nonsmooth C-condition. $\square$

In the next proposition we prove an anticoercivity property of ϕ on $\mathbb{R}u_1$, the eigenspace of the principal eigenvalue $\lambda_1 > 0$.

Proposition 3. *If hypotheses H(j) hold, then $\phi(tu_1) \to -\infty$ when $|t| \to +\infty$.*

Proof: Set $k(z,x) = j(z,x) - (\lambda_1/p)|x|^p$. Evidently for every $x \in \mathbb{R}$ $z \to k(z,x)$ is measurable, while for almost all $z \in Z$ $x \to k(z,x)$ is locally Lipschitz. From Clarke [**6**, Proposition 2.3.14, p.48], we know that for $x > 0$ the function $x \to k(z,x)/|x|^p$ is locally Lipschitz and for almost all $z \in Z$ and all $x > 0$ we have

$$\begin{aligned}
\partial\left(\frac{k(z,x)}{|x|^p}\right) &= \frac{|x|^p \partial k(z,x) - p|x|^{p-2}xk(z,x)}{|x|^{2p}} \\
&= |x|^{p-1}\frac{\left(x\partial k(z,x) - pk(z,x)\right)}{|x|^{2p}} \\
&= \frac{x\partial j(z,x) - \lambda_1|x|^p - pj(z,x) + \lambda_1|x|^p}{|x|^{p+1}} = \frac{\partial j(z,x)x - pj(z,x)}{|x|^{p+1}}.
\end{aligned}$$

By virtue of hypothesis $H(j)_2$(iv), given $\eta_2 > 0$ we can find $M_2 > 0$ such that for almost all $z \in Z$, all $x > M_2$ and all $u \in \partial j(z,x)$ we have

$$ux - pj(z,x) < -\eta_2,$$

hence

$$\partial\left(\frac{k(z,x)}{|x|^p}\right) < -\frac{\eta_2}{|x|^{p+1}}$$

for almost all $z \in Z$ and all $x > M_2$.

Since for $z \in Z\backslash E$ with $|E|_N = 0$, the function $x \to k(z,x)/|x|^p$ is locally Lipschitz on $(M_2, +\infty)$ it is differentiable at every $x \in (M_2, +\infty)\backslash L(z)$ with $|L(z)| = 0$ (here $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$). We define

$$\mu_0(z,x) = \begin{cases} \left(\dfrac{k(z,x)}{x^p}\right)'_x & \text{if } x \in (M_2, +\infty)\backslash L(z) \\ 0 & \text{if } x \in L(z) \end{cases}$$

https://doi.org/10.1017/S0004972700019857 Published online by Cambridge University Press

So for all $z \in Z \setminus E$ and all $x \in (M_2, +\infty) \setminus L(z)$ we have

$$\mu_0(z,x) \leqslant -\frac{\eta_2}{x^{p+1}} = \frac{d}{dx}\Big(\frac{1}{p}\frac{\eta_2}{x^p}\Big).$$

We integrate the above inequality on the interval $[v,y]$ with $v < y,\ v,y \in (M_2, +\infty)$ and obtain

$$\int_v^y \mu_0(z,x)dx = \int_v^y \Big(\frac{k(z,x)}{x^p}\Big)' dx \leqslant \eta_2 p \int_v^y \Big(\frac{1}{x^p}\Big)' dx,$$

hence

$$\frac{k(z,x)}{y^p} - \frac{k(z,v)}{v^p} \leqslant \frac{\eta_2}{p}\Big(\frac{1}{y^p} - \frac{1}{v^p}\Big).$$

Let $y \to +\infty$. Then by virtue of hypothesis H(j)(v) we have that $\liminf\limits_{y\to+\infty}\big(k(z,x)/y^p\big) \geqslant 0$. So we obtain that

$$\frac{k(z,v)}{v^p} \geqslant \frac{\eta_2}{p}\frac{1}{v^p} \quad \text{for all } v \in (M_2, +\infty),$$

hence

$$k(z,v) \geqslant \frac{\eta_2}{p} \quad \text{for all } v \in (M_2, +\infty).$$

For every $t > 0$ let $K_t = \{z \in Z : tu_1(z) > M_2\}$. Since $u_1(z) > 0$ for all $z \in Z$ (see Section 2), we see that $|K_t^c|_N \to 0$ as $t \to +\infty$. For $t > 0$ we have

$$\begin{aligned}
\phi(tu_1) &= \frac{t^p}{p}\|Du_1\|_p^p - \int_Z j\big(z, tu_1(z)\big)dz \\
&= \frac{t^p}{p}\|Du_1\|_p^p - \int_{K_t} j\big(z, tu_1(z)\big)dz - \int_{K_t^c} j\big(z, tu_1(z)\big)dz \\
&\leqslant \frac{t^p}{p}\|Du_1\|_p^p - \frac{\eta_2}{p}|K_t|_N - \frac{t^p\lambda_1}{p}\|u_1\|_p^p + \frac{\lambda_1}{p}\int_{K_t^c}\big|tu_1(z)\big|^p dz \\
&\qquad\qquad + \int_{K_t^c}\alpha_1'(z)dz + c_1'\int_{K_t^c}\big|tu_1(z)\big|^p dz \\
&\leqslant -\frac{\eta_2}{p}|K_t|_N + \frac{\lambda_1}{p}M_2^p|K_t^c|_N + \int_{K_t^c}\alpha_1'(z)dz + c_1 M_2^p|K_t^c|_N \to -\frac{\eta_2}{p}|Z|_N \text{ as } t \to +\infty.
\end{aligned}$$

Therefore $\limsup\limits_{t\to+\infty}\phi(tu_1) \leqslant -(\eta_2/p)|Z|_N$. But $\eta_2 > 0$ was arbitrary. So we deduce that $\phi(tu_1) \to -\infty$ as $t \to +\infty$. Similarly we can show that $\phi(tu_1) \to -\infty$ as $t \to -\infty$. Therefore we conclude that $\phi(tu_1) \to -\infty$ as $|t| \to \infty$. ∎

Let $V = \{x \in W_0^{1,p}(Z) : \|Dx\|_p^p = \lambda_2\|x\|_p^p\}$.

PROPOSITION 4. *If hypotheses H(j) hold, then $\phi(v) \to +\infty$ as $\|v\| \to \infty$, $v \in V$.*

PROOF: Recall that because of hypothesis H(j)(v) there exist $\lambda_1 \leqslant \eta < \lambda_2$ and $M > 0$ such that for almost all $z \in Z$ and all $|x| \geqslant M$ we have

$$pj(z,x) \leqslant \eta |x|^p.$$

So if $v \in V$, we have

$$\begin{aligned}
\phi(v) &= \frac{1}{p}\|Dv\|_p^p - \int_Z j\big(z, v(z)\big)dz \\
&= \frac{1}{p}\|Dv\|_p^p - \int_{\{|v| \geqslant M\}} j\big(z, v(z)\big)dz - \int_{\{|v| < M\}} j\big(z, v(z)\big)dz \\
&\geqslant \frac{1}{p}\|Dv\|_p^p - \frac{\eta}{p}\|v\|_p^p - \eta_3 \ \text{ for some } \eta_3 > 0, \\
&= \frac{1}{p}\Big(1 - \frac{\eta}{\lambda_2}\Big)\|Dv\|_p^p - \eta_3 \ \text{ from the definition of } V.
\end{aligned}$$

Since $\eta < \lambda_2$, it follows that $\phi(v) \to +\infty$ as $\|v\| \to \infty$, $v \in V$. ∎

4. EXISTENCE THEOREM

Using the three auxiliary results of Section 3 and theorem 1 (the nonsmooth linking theorem), in this section we prove the following existence theorem for problem (1):

THEOREM 5. *If hypotheses H(j) hold, then problem* (1) *has a solution $x \in W_0^{1,p}(Z)$.*

PROOF: By virtue of proposition 4, we can find $k_1 > -\infty$ such that

$$k_1 = \inf\big[\phi(v) : v \in V\big].$$

Also because of proposition 3 we can find $t^* > 0$ large enough such that if $y = t^* u_1$, then we have $\phi(\pm y) < k_1$. Let

$$G = \big\{x \in W_0^{1,p}(Z) : \|Dx\|_p = 1\big\}$$

and

$$U = \big\{x \in G : -\psi(x) = \|x\|_p^p > -c_2\big\},$$

where $c_2 < 0$ is given by (4). Evidently U is open in C and because

$$\|\pm u_1\|_p^p = (1/\lambda_1)\|\pm Du_1\|_p^p = 1/\lambda_1 = -c_1 > -c_2,$$

we see that $\pm u_1 \in U$. We shall show that in fact u_1 and $-u_1$ belong to different path-connected components of U. Suppose that this was not the case. Then we could find a continuous curve ϑ in U joining u_1 and $-u_1$ (that is $\vartheta(0) = u_1$ and $\vartheta(1) = -u_1$). Set $H = \{\vartheta\} \cup \{-\vartheta\}$. Evidently $H \subseteq U$ is compact, symmetric and so $\gamma(H) > 1$, that is $H \in A_2$. Note that from the definition of U and since $H \subseteq U$, we have that $\sup\left[\psi(x) = -\|x\|_p^p : x \in H\right] < c_2$, a contradiction to (4). Therefore u_1 and $-u_1$ belong to different path-connected components of U. Let W be the path-connected component of U with $u_1 \in W$. Hence $-W$ is the path-connected component of U with $-u_1 \in -W$. We set $E = t^* W$ and $F = E \cup (-E)$. Then since $\lambda_2 = -1/c_2$ (see Section 2), we have $\|Dw\|_p^p < \lambda_2 \|w\|_p^p$ for all $w \in F$ and $\|Dw\|_p^p = \lambda_2 \|w\|_p^p$ for all $w \in \partial F$ and so we infer that $\partial F \subseteq V$. Set $C = [-y, y] = \left\{r \in W_0^{1,p}(Z) : r = \lambda(-y) + (1-\lambda)y, \text{ for some } \lambda \in [0,1]\right\}$ $C_1 = \partial C = \{-y, y\}$ and $D = V$. We claim that C_1 and $D = V$ link in $W_0^{1,p}(Z)$. To see this first note that because $\phi(\pm y) < k = \inf_V \phi$, we have $C_1 \cap V = \emptyset$. Also if $\vartheta_1 \in C\left(C, W_0^{1,p}(Z)\right)$ such that $\vartheta_1|_{C_1} = \text{identity}$, then $\vartheta_1(C) \cap V \supseteq \vartheta_1(C) \cap \partial F \neq \emptyset$, which establishes the linking of C_1 and V is $W_0^{1,p}(Z)$. Now we are in a position to apply theorem 1 and obtain $x \in W_0^{1,p}(Z)$ such that $\phi(x) \geqslant k_1$ and $0 \in \partial\phi(x)$. Then we have $Ax - u = 0$ in $W^{-1,q}(Z)$ for some $u \in \partial \widehat{J}(x)$. So for all $\vartheta \in W_0^{1,p}(Z)$ we have

$$\langle A(x), \vartheta \rangle = (u, \vartheta)_{rr'}$$

hence

$$\Big\langle -\operatorname{div}\big(\|Dx\|^{p-2} Dx\big), \vartheta \Big\rangle = (u, \vartheta)_{rr'} \quad \text{(by Green's indentity)}$$

and so

$$-\operatorname{div}\Big(\big\|Dx(z)\big\|^{p-2} Dx(z)\Big) = u(z) \in \partial j\big(z, x(z)\big)$$

almost everywhere on Z with $x|_\Gamma = 0$, that is, $x \in W_0^{1,p}(Z)$ is a solution of (1). ∎

REMARK. Hemivariational inequalities incorporate as a special case quasilinear elliptic problems with discontinuities. This is the case when $j(z,x) = \int_0^x f(z,r)dr$ with f a measurable function, locally bounded in the second variable. Problems with discontinuities were examined by Chang [5] (semilinear problems) and by Kourogenis and Papageorgiou [18] (quasilinear problems).

A simple example of a function j satisfying hypotheses H(j) is the following (for simplicity we drop the z-dependence):

$$j(x) = \frac{\lambda_1}{p}|x|^p + k(x)$$

where

$$k(x) = \begin{cases} \pm x \ln x & \text{if } x > 0 \\ |x| & \text{if } x \leqslant 0 \end{cases}.$$

https://doi.org/10.1017/S0004972700019857 Published online by Cambridge University Press

Note that in this case the resulting hemivariational inequality is at resonance at $\lambda_1 > 0$. Such problems were studied under different hypotheses by Gasinski and Papageorgiou [**9, 10, 13**] (for the semilinear case see also Goeleven, Motreanu and Panagiotopoulos [**15**] and Gasinski Papageorgiou [**12**]).

One of the first classes of problems studied in the context of hemivariational inequalities, were semipermeability problems. They arise in heat conduction, flow through porous media and electrostatics. We consider an open bounded domain $Z \subseteq \mathbb{R}^3$ refered to a fixed orthogonal Cartesian coordinate system $Oz_1z_2z_3$ and we formulate the equation $-\Delta_p x = f$ for nonlinear stationary problems (see Diaz [**8**]). On the $C^{1,\alpha}$ boundary Γ of Z we assume that x equals zero. The function x represents the temperature in heat conduction problems, whereas in problems of hydraulics the pressure and in problems of electrostatics the electric potential. We assume that $f = f_1 + f_2$, where f_1 is given and f_2 is related to x with the relation $-f_1(z) \in \partial j\big(z, x(z)\big)$ on Z, where j is a locally Lipschitz (hence in general nonconvex and nonsmooth) energy function and ∂j denotes the generalised gradient in the sense of Clarke [**6**] of $x \to j(z,x)$. The relation $-f_1(z) \in \partial j\big(z, x(z)\big)$ describes, in the language of heat conduction the behaviour of a semipermeable membrane of finite thickness occupying Z or the behaviour of a temperature controller producing heat in order to regulate the temperature in Z. So, for example when the temperature x is less than h, then the heat per unit volume supplied is constant, say α. When $x = h$ heat is supplied for constant temperature until a desired value b is reached. The supplied heat-temperature relation follows a parabola until the temperature h_1 is reached. Then we have a change of heat from a value $-c$ to $-d$ with the temperature remaining constant at $x = h_1$ and then the heat supply remains constant whereas the temperature may increase. We can also think of another temperature-control problem, in which the temperature is regulated in order to deviate as little as possible from the interval $[h_1, h_2]$. In both cases the law of the system is nonmonotone and multivalued (we fill-in the gaps at the discontinuity points) and the resulting inequality is a hemivariational inequality.

References

[1] A. Anane, 'Simplicite et isolation de la premier valeur propre du p-Laplacien avec poids', *C.R. Acad. Sci. Paris Ser. I Math.* **305** (1987), 725–728.

[2] A. Anane and N. Tsouli, 'On the second eigenvalue of the p-Laplacian', in *Nonlinear Partial Differential equations*, Pitman Research Notes Math. Ser. **343** (Longman, Harlow, 1996), pp. 1–9.

[3] P. Bartolo, V. Benci and D. Fortunato, 'Abstract critical point theorems and applications to some nonlinear problems with strong resonance', *Nonlinear Anal.* **7** (1983), 981–1012.

[4] G. Cerami, 'An existence criterion for the critical points on unbounded manifolds', (in Italian), *Instit. Lombardo Accad. Sci. Lett. Rend. A* **112** (1978), 332–336.

[5] K.C. Chang, 'Variational methods for nondifferentiable functionals and their applications to partial differential equations', *J. Math. Anal. Appl.* **80** (1981), 102–129.

[6] F.H. Clarke, *Optimization and nonsmooth analysis* (Wiley, New York, 1983).

[7] D.G. Costa and C.A. Magalhaes, 'Existence results for perturbations of the p-Laplacian', *Nonlinear Anal.* **24** (1995), 409–418.

[8] J.I. Diaz, *Nonlinear partial differential equations and free boundaries Volume 1: Elliptic equations*, Research Notes in Math. **106** (Pitman, London, 1985).

[9] L. Gasinski and N.S. Papageorgiou, 'Nonlinear hemivariational inequalities at resonance', *Bull.Austral. Math. Soc.* **60** (1999), 353-364.

[10] L. Gasinski and N.S.Papageorgiou, 'An existence theorem for nonlinear hemivariational inequalities at resonance', *Bull.Austral. Math. Soc.* **63** (2001), 1–14.

[11] L. Gasinski and N.S. Papageorgiou, 'Multiple solutions for nonlinear hemivariational inequalities near resonance', *Funkcial. Ekvac.* **43** (2000), 271–284.

[12] L. Gasinski and N.S. Papageorgiou, 'Existence of solutions and of multiple solutions for eigenvalue problems of hemivariational inequalities', *Adv. Math. Sci. Appl* (to appear).

[13] L. Gasinski and N.S. Papageorgiou, 'Solutions and multiple solutions for quasilinear hemivariational inequalities at resonance', *Proc. Royal Soc. Edinburgh* **131A** (2001), 1–21.

[14] D. Goeleven, D. Motreanu and P. Panagiotopoulos, 'Multiple solutions for a class of eigenvalue problems in hemivariational inequalities', *Nonlinear Anal.* **29** (1997), 9–26.

[15] D. Goeleven, D. Motreanu and P. Panagiotopoulos, 'Eigenvalue problems for variational-hemivariational inequalities at resonance', *Nonlinear Anal.* **33** (1998), 161–180.

[16] S. Hu and N.S. Papageorgiou, *Handbook of multivalued analysis Volume I: Theory*, Mathematics and its Applications **419** (Kluwer, Drodrecht, 1997).

[17] S. Hu and N.S. Papageorgiou, *Handbook of multivalued analysis Volume II: Applications*, Mathematics and its Applications **500** (Kluwer, Drodrecht, 2000).

[18] N. Kourogenis and N.S. Papageorgiou, 'Nonsmooth critical point theory and nonlinear elliptic equations at resonance', *J. Austral. Math. Soc. Ser. A* **69** (2000), 245–271.

[19] G. Lebourg, 'Valeur moyenne pour gradient généralisé', *C.R. Acad. Sci. Paris Ser. A-B* **281** (1975), 795–797.

[20] G.M. Lieberman, 'Boundary regularity for solutions of degenerate elliptic equations', *Nonlinear Anal.* **12** (1998), 1203–1219.

[21] P. Lindqvist, 'On the equation $\mathrm{div}(|Dx|^{p-2}Dx) + \lambda|x|^{p-2}x = 0$', *Proc. Amer. Math. Soc.* **109** (1991), 157–164.

[22] Z. Naniewicz and P. Panagiotopoulos, *Mathematical theory of hemivariational inequalities and applications*, Monographs and Textbooks in Pure and Applied Mathematics **188** (Marcel-Dekker, New York, 1995).

[23] P. Panagiotopoulos, *Hemivariational inequalities. Applications to mechanics and engineering* (Springer-Verlag, New York, 1993).

[24] M. Struwe, *Variational methods. Applications to nolinear partial differential equations and Hamiltonian systems*, A series of modern surveys in Mathematics **34** (Springer-Verlag, Berlin, 1990).

National Technical University
Department of Mathematics
Zografou Campus
Athens 15780
Greece

National Technical University
Department of Mathematics
Zografou Campus
Athens 157 80
Greece
e-mail:npapg@math.ntua.gr