ANSELM'S ONTOLOGICAL ARGUMENT AND GRADES OF BEING

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Abstract. Anselm described god as "something than which nothing greater can be thought" [1, p. 93], and Descartes viewed him as "a supreme being" [7, p. 122]. I first capture those characterizations formally in a simple language for monadic predicate logic. Next, I construct a model class inspired by Stoic and medieval doctrines of *grades of being* [8, 20]. Third, I prove the models sufficient for recovering, as internal mathematics, the famous ontological argument of Anselm, and show that argument to be, on this formalization, valid. Fourth, I extend the models to incorporate a modality fit for proving that any item than which necessarily no greater can be thought is also necessarily real. Lastly, with the present approach, I blunt the sharp edges of notable objections to ontological arguments by Gaunilo and by Grant. A trigger warning: every page of this writing flouts the old saw "Existence is not a predicate" and flagrantly.

David Charles McCarty, Professor of Philosophy at Indiana University, passed away on November 25, 2020. At the time of his death, his paper "Anselm's Ontological Argument" was close to publication in Review of Symbolic Logic*, though additional steps remained to be completed. We are posthumously publishing the paper with minimal changes. Thanks to Stewart Shapiro and Paul Spade for their contributions to bringing the manuscript to a publishable form. This paper is an illustration of Prof. McCarty's wide ranging knowledge and interests, using Boolean-valued models to reconstruct a medieval argument for the existence of God, drawing on the idea of grades of being.*

–Jamie Tappenden

§1. The Anselm and the Descartes formulae. As formalisms go, the object language $\mathcal{L}$ (inspired by Scott [18]) could not be simpler: first-order and monadic with 'R' as sole predicate. Read informally, 'Rx' is to convey that x is real. The existential quantifier '$\exists x$' is glossed, "There is an item x." Conditionals '$Rx \to Ry$' are construed with reference to grades of being: '$Rx \to Ry$' will assert, "Item y is at least as real as item x." (I elaborate on and justify these readings later.) Herein and throughout, 'item' is my count noun of choice for individuals in general, be they existent, imaginary, nonexistent, or even contradictory. Particular numbers and sets are items, as are particular people—living or dead, medium-sized dry goods, individual unicorns, and the round square cupola on Berkeley College. (If you have trouble with this last idea, please consult [11].) Within the bounds of $\mathcal{L}$ and in such terms, I identify formulae capturing Anselm's and Descartes's characterizations of god.

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In his *Fifth Meditation*, Descartes supposed an item possessing an absolutely highest grade of being, "a first and supreme being" [7, p. 122]. What I call '*the Descartes formula*' in $\mathcal{L}$ expresses that characterization:

$$\forall y(Ry \rightarrow Rx).$$

Its informal reading is "x is a supreme reality," alternatively "x has a grade or extent of being at least that of any item y." Shortly put, "x is at least as real as any y."

In *Proslogion*, Anselm posited an item with a maximal grade of being, specifically, "something than which nothing greater can be thought" [1, p. 93]. In $\mathcal{L}$, there is a natural rendition of this idea:

$$\forall y((Rx \rightarrow Ry) \rightarrow (Ry \rightarrow Rx)).$$

I call this '*the Anselm formula*.' It is, in predicate logic, equivalent to

$$\neg\exists y((Rx \rightarrow Ry) \wedge \neg(Ry \rightarrow Rx)).$$

In plainspeak, the former asserts that, for any item y at least as real as x, x and y have the same grade of being, or are equally real. The latter asserts that there is no item y strictly more real than x. Proofs to come will show that the Anselm formula, when true in the favored models, selects out, like the Descartes formula, an item of maximum grade, or highest reality. As promised, I justify these readings later. A subsequent section will direct needed attention to Anselm's phrase 'can be thought.'

It is a theorem of pure, monadic predicate logic that any item specified by either the Anselm or the Descartes formula is real, provided that there is any item whatsoever. To prove that, assume that

$$\exists z.Rz,$$

or "there is an item," and that the Descartes formula governs x:

$$\forall y(Ry \rightarrow Rx).$$

Alphabetic change of bound variable reveals the foregoing as logically equivalent to

$$\exists z.Rz \rightarrow Rx.$$

Given the assumption $\exists z.Rz$,

$$Rx$$

is an immediate consequence. Therefore, the sequent

$$\exists z.Rz \wedge \forall y(Ry \rightarrow Rx) \vdash Rx$$

is valid. Read informally, the sequent says, "If there is any item at all, an item of maximum grade of being is truly real." Were one to ignore, for the moment, the first conjunct in the sequent's premise, the remaining sequent would tell us, "Any item of the highest grade of being really exists." Later, I demonstrate that one can, without loss, ignore at times the first conjunct, $\exists z.Rz$, since it holds in all relevant models.

As in Anselm's own argument [1, pp. 93–94], an analogous deduction from the Anselm formula starts with a *reductio*. Assume both that there is an item, or

$$\exists z.Rz,$$

and that x possesses a maximal grade of being, which is the Anselm formula,

$$\forall y((Rx \to Ry) \to (Ry \to Rx)).$$

Then, for *reductio*, suppose that x is not real:

$$\neg Rx.$$

By predicate logic, it follows immediately that

$$\forall y(Rx \to Ry).$$

From the Anselm formula, one now concludes that

$$\forall y(Ry \to Rx),$$

which is the Descartes formula. From this point, follow the derivation above from the Descartes argument to the conclusion "x is real:"

$$Rx.$$

Putting that result together with the assumptions $\exists z.Rz$ and the Anselm formula, it is plain that the sequent

$$\exists z.Rz \wedge \forall y((Rx \to Ry) \to (Ry \to Rx)) \vdash Rx$$

is also a theorem pure logic. (Its converse is equally a theorem.) Informally, it asserts that, if there is an item, then any item of maximal being has also to exist in reality. Again, the initial conjunct of the premise can sometimes be put aside, and the sequent then inform us that any item of a maximal grade of being really exists.

At this stage, there is no presumption that I have already proven, as Descartes and Anselm hoped to do, that—in Descartes's case—any greatest being exists in reality or, à la Anselm, that a being no greater than which can be thought also exists. Such 'proofs,' were they attempted here and now, would rest entirely upon the informal readings I postulated for the symbols 'Rx,' '$\exists x$,' and '$Rx \to Ry$,' readings that may well turn out fantastical, even inappropriate. To give proofs that attend the original ontological arguments closely, I need, at least, to construct a rigorous model theory on which the Descartes and Anselm formulae are not only satisfiable, and possess the required definitory properties, but also take on their informal readings *demonstrably*. This I now do.

§2. Grades of being, extensions, and Boolean algebras. Anselm thought there to be a rising hierarchy among beings articulated by accretions in reality: items holding a higher grade of being are more real than those of relatively lower grade, as he explained in Chapter 31 of *Monologion*:

> [I]t is clear that a living substance exists more than a nonliving one, that a sentient substance exists more than a nonsentient one, and that a rational substance exists more than a nonrational one [1, p. 47].

Some of my contemporaries may sneer at the proposal that a particular leaf has a higher grade of being or really exists to a larger extent than a particular stone—although leaves are self-sustaining and self-regulating in ways stones are not. Illuminati of the twenty-first century would, I think, grasp the point of, if not concur with, the contention that even relatively impermanent stones are more fully real than evanescent rainbows, mirages, and afterimages. Would they not grant that Abraham Lincoln, as portrayed in a reliable, scholarly biography, has a higher grade of reality than he would have showing up in a Hollywood film where he, together with Grant and Sherman, figure as homicide detectives or vampire slayers? Among imaginary entities, increases in reality are commonly recognized; most would allow that a man with skin thick as a rhino's and naturally Kelly green in color is more real than any loquacious Euclidean triangle.

Even those of a generous philosophic cast of mind may object, "The above quotation introduces a notion of grades for existent beings, but does not itself issue much precedent for extending the scale from the real into the nonreal, from stones into a realm populated by chimeras and round, square cupolas." In reply, I point out, first, that Anselm ordered the grades not only by accretion, as above, but also by diminution. In *Monologion*, he wrote,

> I think that this same point [the order on grades of being] can also be readily seen by means of the following consideration. From some substance which lives, perceives, and reasons, let us imaginatively remove first what is rational, next what is sentient, then what is vital, and finally the remaining bare existence. Now, who would not understand that this substance, thus destroyed step by step, is gradually reduced to less and less existence—and, in the end, to non-existence? [1, p. 46]

The scale of grades thus extends from real items downward to include ones non-existent. (A parallel grading by decreasing actuality—or increasing potentiality—features in Aristotle's *Metaphysics*. "And, as we have seen, all individual things in the world may be graded to the extent to which they are infected with potentiality" [16, p. 174].) Furthermore, Anselm's order by accretion is meant to be identical to that by diminution:

> Yet those characteristics which, when removed one at a time, reduce a being to less and less existence increase its existence more and more when added to it again in reverse order [1, pp. 46–47].

Second, from the same section in *Monologion*, it is equally plain that the scale was not meant to govern real items *qua real items* principally or exclusively, but to govern *created things*, and not as they really exist but in so far as they are likenesses of the highest being:

> [E]very created thing both exists and is excellent in proportion to its likeness to what exists supremely and is supremely great [1, p. 46],

and

> [E]very created nature consists of a higher degree of existence and excellence to the extent that it is seen to approximate this Word [1, p. 47].

Loads of imaginary items—think of Sherlock Holmes or the chimera—stand in obvious likeness to real items and were intentionally created, one supposes, as such.

Third, in Anselm's work, grades of being attach as well to mental images and formulated plans prior to their execution, as revealed in his treatment, in the course of the Chapter 2 argument from *Proslogion*, of the artist and his painting [1, p. 94]:

> For when an artist envisions what he is about to paint, he has it in his understanding, but he does not yet understand [judge] that what he has painted exists.

A crux of that famous ontological argument requires the ordering, in grade of being, of an item that holds a maximal position on the scale relative to one that would hold a maximal position—except that it is unreal:

> But surely that than which a greater cannot be thought cannot be only in the understanding, it could be thought to exist also in reality—which is greater [1, p. 94].

Accordingly, an x existing in reality is to have a higher grade of being than such an x when in the understanding merely; what the artist only envisions (or the fool imagines dimly) has a strictly lower grade of being than what the artist envisions and produces.

Adopting Anselm's grades of created being with full metamathematical seriousness, I assume that the grades constitute (or can be treated as constituting) a Zermelo–Fraenkel (ZF) set S. I define a natural ordering $\leq$ over grades, rather than over individual items, so that, for G_1 and G_2 grades,

$$G_1 \leq G_2$$

just in case any item of grade G_2 is at least as real as any item of grade G_1. At the cost of a relabeling, one can allow that, when items of G_1 have just as much reality as those of G_2 and conversely, G_1 and G_2 are identical as grades. It is then easy to prove that the relation $\leq$ is reflexive, antisymmetric, and transitive. In other words, $\leq$ is a partial order on S:

PROPOSITION. *The order $\leq$ over the grades of being in S, as just defined, is a partial order.*

N.B. One can distinguish between grades of being, the order on which may be linear, and the medieval image of the *Tree of Porphyry*, which naturally carries, on its nodes, a nonlinear order [12].

It is a simple result that any partial order $\preceq$ on a set X can be embedded faithfully into the subset order $\subseteq$ on a field of subsets of X: just map the $\preceq$-ordered elements X into the $\preceq$-downward-closed subsets $\hat{X}$ that they determine. The field of $\preceq$-downward-closed sets is in its turn embedded into the full powerset of X, the collection of all subsets of X. Hence, every partial order—such as the order $\leq$ on grades of being in S—can be recovered as a natural, $\subseteq$-suborder on a complete powerset Boolean algebra $\mathfrak{B}$ on S, a structure with all intersections, unions, and relative complements, behaving in the familiar Boolean or 'classical logic' fashion.

N.B. No one is presuming that either Anselm or Descartes had a grasp, implicit or explicit, of the technicalities of twentieth century metamathematics featuring so largely in this essay. Without question, it would be a noble and worthwhile task to reconstruct Anselm's ontological argument using exclusively the logical tools of the late eleventh and early twelfth centuries of our era. But not all symphonic performances, to be great, have to be played using period instruments. I want here to use the strongest foundational magnifying glass at my disposal to reveal the fine structures of the ontological arguments—but carefully, so that distortion is held to a minimum.

That this representation—of grades by $\subseteq$-ordered sets—is *faithful* means that the function $\lambda X.\hat{X}$ from grades into Boolean values is injective, and that, if $G_1 \leq G_2$, then representative $\hat{G}_2$ is a superset of representative $\hat{G}_1$, and conversely:

PROPOSITION. *The mapping $\lambda X.\hat{X}$ on the grades is faithful.*

Proof. Assume $G_1 \leq G_2$ and $H \,\varepsilon\, \hat{G}_1$. Then, by definition of $\hat{G}_1$, $H \leq G_1$. Because $\leq$ is transitive, $H \leq G_2$. Therefore, $H \,\varepsilon\, \hat{G}_2$. Consequently, $\hat{G}_1 \subseteq \hat{G}_2$. Conversely, it follows from $\hat{G}_1 \subseteq \hat{G}_2$ and $G_1 \,\varepsilon\, \hat{G}_1$ that $G_1 \,\varepsilon\, \hat{G}_2$ and $G_1 \leq G_2$. So, if $\hat{G}_2 \subseteq \hat{G}_1$, then $G_2 \leq G_1$. Now, from the foregoing and anti-symmetry for $\leq$, one concludes that the $\lambda X.\hat{X}$ mapping is injective. $\square$

Such algebras $\mathfrak{B}$ treat particular grades G as (sub)sets $\hat{G}$ that share with all sets the characteristic of being *extensions*. On this representation, a $\leq$-higher grade of being means a $\subseteq$-greater extension. Ergo, it makes perfect sense to speak interchangeably of grades of being or extents of being. With all the ingredients assembled, the following theorem is proven.

THEOREM. *The grades G of being under their canonical order $\leq$ are faithfully represented by a range of extensions $\hat{G}$ ordered by $\subseteq$ within the complete powerset Boolean algebra $\mathfrak{B}$ on the set of all grades S.*

Thanks to the reflexivity of $\leq$, each G is a member of $\hat{G}$. Consequently, one says that the Boolean algebra $\mathfrak{B}$ on the set S is *covered* by the grades:

PROPOSITION. *For each $H \in S$, there is a grade G of being such that $H \,\varepsilon\, \hat{G}$.*

In $\mathfrak{B}$, there is always a $\subseteq$-greatest element, the set S of all grades, called '*True*,' plus a least element, the empty set $\emptyset$ or *False*. If, in addition, there is a $\leq$-greatest grade of being G, then $\hat{G}$ is that $\subseteq$-greatest element, the set S itself. (Not all the models I construct from grades feature a $\leq$-greatest.)

PROPOSITION. *When G is the greatest grade under the order $\leq$,*

$$\hat{G} = S = \text{True}.$$

Proof. When G is the $\leq$-greatest grade, then every grade H is $\leq G$. Therefore, the $\leq$-downward-closed set $\hat{G}$ coincides with the whole set S. $\square$

As remarked, on ancient and medieval conceptions of grades of being, grades respect reality by preserving it under $\leq$. That is, if $G_1 \leq G_2$, then items of grade G_2 are at least as real as those of grade G_1. If there is a highest grade, items with that grade exist with the greatest reality; items of markedly lower grades are, relatively speaking, much less real. As Anselm put it,

> So without doubt every being exists more and is more excellent to the extent that it is more like that Being which exists supremely and is supremely excellent. Thus, it is quite obvious that in the Word, through which all things were created, there is no likeness of created things but is, rather, true and simple Existence—whereas in created things there is not simple and absolute existence but a meagre imitation of this true Existence [1, p. 47].

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Finally, I assume throughout that there is at minimum one item a and one grade of being G for that item. Therefore, the resultant Boolean algebras $\mathfrak{B}$ must contain at least the two distinct elements *True* and *False*.

What then have I assumed so far about Anselm's grades of being? In truth, not all that much. I supposed that there are some items and that they have grades of being of such natures that they (or their set-theoretic alter egos) can be collected into a set recognized under the ZF axioms. Also, I assumed that the grades admit a partial order that respects the real existence of items that have those grades.

§3. Boolean-valued models and grades of being. Once one makes sense of items having varied grades of being as Anselm understood them, especially when he glosses them in terms of likeness or imitation, one can also make sense of grades of truth: for Anselm, the assertion "x is real" has a higher grade of truth when x is a human being than it does when x is a stone. A sophisticated notion of grades of truth capturing this idea is that embodied in Boolean-valued models. There, truth-values are not confined to the two-member set consisting of *True* and *False* only, but can, at times, be drawn from an infinite collection. Moreover, the Boolean-valued idea yields a notion of model and of internal mathematics that validates all the standard, classical inferences.

As Dana Scott [17] made clear, Tarski, Mostowski, and others explored Boolean-valued interpretations for first-order formulae prior to [15]. For instance, [4] featured a Boolean-valued model of type theory. Later, [21] and the unpublished [19] extended the idea to set theory, thereby obtaining independence results like those of [5] and [6]. Among other things, these developments revealed that the standard Tarskian conception of model—over the two ordinary truth-values T and F—admits generalization in a scientifically fruitful way to a conception of model over an arbitrary or complete Boolean algebra of extended truth-values, without in any way curtailing the logic of the interpreted language. Familiar propositional and first-order classical logics remain sound and complete with respect to their respective classes of Boolean-valued models.

In general, to interpret the language $\mathcal{L}$, a Boolean-valued model is no more than a function $[\![R]\!]$ mapping a nonempty domain or set D into a complete Boolean algebra $\mathfrak{B}$. The base model $[\![R]\!]$ then extends uniquely to an interpretation function

$$\lambda\phi.[\![\phi]\!]$$

assigning to sentences ϕ, perhaps with parameters a and b from D, elements of $\mathfrak{B}$. The interpretation $\lambda\phi.[\![\phi]\!]$ obeys the recursion clauses displayed below. Once again, the designated operations on $\mathfrak{B}$ are $\cup$, $\cap$, $\sim$, $\bigcup$, and $\bigcap$: join, meet, relative complement, least upper bound, and greatest lower bound, respectively.

$$\begin{aligned}
[\![Ra]\!] &= [\![R]\!](a),\\
[\![\phi \vee \psi]\!] &= [\![\phi]\!] \cup [\![\psi]\!],\\
[\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!],\\
[\![\phi \rightarrow \psi]\!] &= \sim [\![\phi]\!] \cup [\![\psi]\!],\\
[\![\neg\phi]\!] &= \sim [\![\phi]\!],\\
[\![\exists x\phi]\!] &= \textstyle\bigcup_{a\,\varepsilon\,D}[\![\phi(a)]\!], \text{ and}\\
[\![\forall x\phi]\!] &= \textstyle\bigcap_{a\,\varepsilon\,D}[\![\phi(a)]\!].
\end{aligned}$$

DEFINITIONS.

1. *For* Γ *a set of* $\mathcal{L}$*-formulae, the Boolean value* $[\![\Gamma]\!]$ *of* Γ *in a model* $[\![R]\!]$ *is just the Boolean greatest lower bound of the values* $[\![\psi]\!]$ *of the formulae* ψ *belonging to* Γ:
$$[\![\Gamma]\!] \;=\; \bigcap_{\psi\,\varepsilon\,\Gamma}[\![\psi]\!].$$
2. *For a formula* ϕ *and a set of formulae* Γ, *all from* $\mathcal{L}$, $\Gamma \vDash \phi$ (Γ *entails* ϕ) *just in case, for every complete Boolean algebra* $\mathfrak{B}$ *and Boolean-valued model* $[\![R]\!]$ *over* $\mathfrak{B}$, $[\![\Gamma]\!] \subseteq [\![\phi]\!]$.
3. *Given Boolean-valued model* $[\![R]\!]$ *and* $\mathcal{L}$*-formula* ϕ, ϕ *is* true *or* satisfied *in* $[\![R]\!]$ *just in case* $[\![\phi]\!]$ =True.
4. *Single turnstyle '*$\vdash$*' stands for classical first-order derivability over* $\mathcal{L}$.

PROPOSITION (*Soundness and Completeness*). *For a formula* ϕ *and set of formulae* Γ *from* $\mathcal{L}$, $\Gamma \vdash \phi$ *if and only if* $\Gamma \vDash \phi$.

Proof. See [15]. □

More specifically, I henceforth assume there to be grades of being G attached to all the items a and b in D, and that they have been represented as $\hat{G}$s within a complete powerset Boolean algebra $\mathfrak{B}$, as in Section 2. The idea is to let the function $[\![R]\!]$ mapping a into $[\![Ra]\!]$ and b into $[\![Rb]\!]$ and so on take a and b into their respective (represented) grades of being. Henceforth, for $a, b\,\varepsilon\, D$, I always assume that $[\![Ra]\!]$ and $[\![Rb]\!]$ are their grades of being. Until further notice, I assume that the items in the domains D and their grades of being in the models $[\![R]\!]$ over items in D need not be the very items and the very grades conceived by Anselm; they might be formal or 'pretend' items and abstract grades discovered within a universe of ZF and obeying the assumptions on items and grades listed above. I am working here in the same model-building spirit as that exemplified in nonstandard models of arithmetic: the elements of the domain in a nonstandard model of arithmetic need not themselves be natural numbers, or even comprise a countable collection. Once I return to Anselm's argument in Section 5, I will then insist that the items in the model constructed there be those Anselm presupposed and their grades of being be their proper Anselmian grades.

Just as every item is presumed to have a grade of being, so it is in perfect accord with traditional conceptions of grades to assume that our Boolean-valued models are *full*:

DEFINITION. *Model* $[\![R]\!]$ *is* full *whenever, for each representative* $\hat{G}$ *in S of a grade of being G, there is an item* a *in D such that*
$$[\![Ra]\!] \;=\; \hat{G}.$$

Simply, this means that all grades of being must be grades of (some manner of) *beings*. Every grade of being is the grade of some item or other in D.

When b has a grade of being $\geq$ that of a, then $[\![Ra]\!] \subseteq [\![Rb]\!]$ in $\mathfrak{B}$, because the representation preserves the ordering. This relationship in $\mathfrak{B}$ is readily captured by the conditional $\rightarrow$ of the formal language under interpretation $[\![\phi]\!]$—as already indicated in Section 1 *infra*.

PROPOSITION. *For* a *and* b ε D, b *has an extent of being that is at least that of* a*—or* $[\![Ra]\!] \subseteq [\![Rb]\!]$*—if and only if the formula* $Ra \rightarrow Rb$ *is true in* $[\![R]\!]$.

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Proof. By the relevant clause of the above display, for any x that belongs to the set S underlying $\mathfrak{B}$, $x \,\varepsilon\, [\![Ra \to Rb]\!]$ just in case

$$x \,\varepsilon\, (\sim [\![Ra]\!] \cup [\![Rb]\!]).$$

This last obtains whenever either

$$x \not\varepsilon [\![Ra]\!] \;\text{ or }\; x \,\varepsilon\, [\![Rb]\!].$$

That in turn holds if and only if $[\![Ra]\!] \subseteq [\![Rb]\!]$. Therefore,

$$[\![Ra \to Rb]\!] = \mathit{True\ just\ in\ case}\ [\![Ra]\!] \subseteq [\![Rb]\!]. \qquad \square$$

In other words, when $[\![Ra]\!]$ and $[\![Rb]\!]$ are grades or extents of being, b exists at least as much as a does whenever the statement $Ra \to Rb$ is true in $[\![R]\!]$.

From the foregoing proposition, it follows immediately that the Descartes and Anselm formulae capture the property of being a *maximum grade of being*. These results verify a main claim of Section 1.

PROPOSITION. *Item $a \,\varepsilon\, D$ has a maximum grade of being just in case the Descartes formula $[\![\forall y(Ry \to Ra)]\!]$ is true under interpretation $[\![R]\!]$.*

Proof. By fullness, a has a maximum grade of being if and only if

$$\forall b \,\varepsilon\, D.\ [\![Rb]\!] \subseteq [\![Ra]\!].$$

By the last proposition, this holds in turn if and only if

$$\forall b \,\varepsilon\, D\ [\![Rb \to Ra]\!] \text{ is true.}$$

This holds if and only if

$$\bigcap_{b \,\varepsilon\, D} [\![Rb \to Ra]\!] \;=\; \mathit{True},$$

and because $\forall$ is interpreted in terms of $\bigcap$, that is equivalent to

$$\forall y(Ry \to Ra) \text{ is true.} \qquad \square$$

Now, I turn to the Anselm formula and its interpretation over model $[\![R]\!]$.

PROPOSITION. *Item $a \,\varepsilon\, D$ has a maximum grade of being if and only if the formula $\forall y((Ra \to Ry) \to (Ry \to Ra))$ is true in $[\![R]\!]$.*

Proof. It is easy to prove, in first-order predicate logic, that the Descartes formula

$$\forall y(Ry \to Ra)$$

and the Anselm formula

$$\forall y((Rx \to Ry) \to (Ry \to Rx))$$

are logically equivalent. $\square$

N.B. There is no assumption that the grades of being in any model are ordered linearly.

Now, as promised, I show that the first conjunct $\exists z.Rz$ can be elided from each of the relevant sequents featuring the Descartes and Anselm formulae.

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Proposition. *Over any Boolean algebra* $\mathfrak{B}$ *formed as above from representatives* $\hat{G}$ *of grades of being* G, $[\![\exists z.Rz]\!]$ =True *under the model* $[\![R]\!]$.

Proof. According to the definition of $[\![\exists z.\phi]\!]$,

$$[\![\exists z.Rz]\!] = \bigcup_{a \,\varepsilon\, D} [\![Ra]\!].$$

Hence, to show that $[\![\exists z.Rz]\!]$ $=$*True*, it suffices to note that $\mathfrak{B}$ is covered by the represented grades $\hat{G}$, and that the models are always full. □

§4. Consistency. To show the Cartesian and Anselmian characterizations consistent logically, I assume, for the sake of this section only, that there is a single unique grade G of being and that a single item a has that grade. Once again, G is represented as an extension $\hat{G}$ within the complete Boolean algebra $\mathfrak{B}$ on the carrier set S which is, this time, the powerset of the singleton set $\{G\}$. That powerset has therefore exactly two elements: the empty set $\emptyset$, and the singleton $\{G\}$, with $\emptyset$ as the $\subseteq$-least element *False* in $\mathfrak{B}$, and $\{G\}$ as the $\subseteq$-greatest element *True*. Both Descartes and Anselm formulae, with parameter 'a' substituted for free variable x, interpret to *True* in the model $[\![R]\!]$ over this $\mathfrak{B}$ with $[\![Ra]\!] = \{G\}$ and $D = \{a\}$.

Theorem. *Descartes and Anselm formulae are both satisfied in the Boolean-valued model* $[\![R]\!]$ *just described.*

Proof. I begin with the Descartes formula, $\forall y(Ry \to Rx)$. Given the clauses defining $[\![R]\!]$, one sees that

$$[\![\forall y(Ry \to Ra)]\!]$$

is identical to

$$\bigcap_{b \,\varepsilon\, D} [\![Rb \to Ra]\!].$$

Because a is the sole member of D, the latter evaluates over $\mathfrak{B}$ to

$$[\![Ra \to Ra]\!],$$

which holds, as proved above, if and only if

$$[\![Ra]\!] \subseteq [\![Ra]\!].$$

Therefore,

$$\forall y(Ry \to Ra)$$

is true in $[\![R]\!]$.

The Anselm formula,

$$\forall y((Rx \to Ry) \to (Ry \to Rx)),$$

holds of item a under interpretation $[\![R]\!]$ because, as already noted, it is formally logically equivalent to the Descartes formulae, and the relevant formal logic is sound with respect to Boolean-valued interpretations $[\![R]\!]$. □

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N.B. Neither the Descartes nor the Anselm formula is logically true over the class of models circumscribed herein. For example, let $[\![R_1]\!]$ be a model constructed over a complete Boolean algebra on a set of grades that are (order isomorphic to) the natural numbers $n \varepsilon \mathcal{N}$ ordered canonically. In this set of grades, there is no greatest element. Assume that a is an item in domain D and that

$$[\![R_1 a]\!] = \hat{m},$$

where m is a fixed natural number. Then, a simple calculation shows that, over $[\![R_1]\!]$,

$$[\![\forall y(Ry \to Ra)]\!] = \hat{m}.$$

In the model, the top element *True* is (isomorphic to) the set of all natural numbers $\mathcal{N}$, which is obviously not the same as the collection $\hat{m}$ of $\leq$-predecessors of the individual natural number m. Hence, both Anselm and Descartes formulae fail to be true in the model $[\![R_1]\!]$.

§5. Proper classes and a model for Anselm's *Proslogion*. The chief contentions of Sections 5 and 6 are that (1) the underlying skeleton of Anselm's ontological argument is the sequent featuring the Anselm formula, derived earlier, a logical truth of pure first-order monadic predicate logic, and (2) the visible flesh of the argument is internal mathematics: a process of *interpreting* and employing the interpreted sequent over a particular Boolean-valued model $[\![\mathcal{R}]\!]$, the Boolean values of which capture the grades of actual, possible, and impossible items. Therefore, in this section, $[\![\mathcal{R}]\!]$ is constructed from "stuff in the world," rather than from merely formal or 'pretend' items and grades, if you allow that possible and imaginary items are "in the world," and that such items truly have grades of being. These items are Anselm's *thinkables*. (If you don't believe in them or their grades, these two sections may still be engaging, either as historical reconstruction or abstract machinery.) To be specific, the sequent in question is that with the Anselm formula as antecedent and the reality claim Rx consequent:

$$\forall y((Rx \to Ry) \to (Ry \to Rx)) \vdash Rx.$$

I show that a reasonable facsimile of the argument from *Proslogion*, Chapter 2, emerges from its interpretation over $[\![\mathcal{R}]\!]$.

Within conventional set theory, it is impossible to issue any guarantee for the existence of a standard set with members all and only the thinkable items. On the outlook of *ZF* set theory, the collection of all thinkables has to be a proper class, a collection unlimited, within the cumulative hierarchy, by any ordinal rank. To see this, assume for *reductio* that the collection of all thinkable ordinal numbers comprises a set. If so, that set has an ordinal α as its least upper bound. However, the successor $\alpha + 1$ of α then exists and is thinkable as well, since I (just now) thought of it. Hence, the collection of all thinkable ordinals cannot be bounded in rank by any ordinal. So, it cannot be a set. And, if the class of thinkable ordinals is not a set, then the class of all thinkable items is not a set either, according to the Separation principle.

I exploit one of the assumptions listed at the close of Section 2 above to avoid potential paradoxes by cutting down the collections of all thinkable items to a manageable set-size. At the end of Section 2, I supposed that the grades of being always comprise a set S which is the range of the canonical "grade of being of" function from items to grades. For our purposes, it will therefore suffice to use the Axiom of Choice, over that set of grades S, to select, for each grade, a single illustrative item having that grade. I have assumed that all our models are full, i.e., that every grade is the grade of some item or other, so this selection makes sense for every grade. For example, from Anselm's grade of sentient things, we may select a single plant, perhaps an individual rose bush. And likewise for all the grades. By the principle of Replacement applied to set S and the selection function, the items so selected comprise a set. Consequently, in the model construction to follow, I can take it, without loss, that both the needed thinkable items and their grades comprise sets suitable for treatment in ZF.

§6. Anselm's ontological argument. Working step-by-step, following Anselm's own exposition, I now recover the details of the Chapter 2 *Proslogion* argument as mathematics internal and external to $[\![\mathcal{R}]\!]$. The reader has been warned: I violate repeatedly the old taboo against treating existence as a predicate. Telling arguments against the taboo are collected at [2, pp. 39–66] and [14, pp. 32–38, 130–161]. (I confess my disappointment with the latter author; he seems to dismiss *ab initio* and without hesitation the very idea of grades of being [14, p. 152].)

Anselm draws the intriguing distinction between *existing in reality* and *existing in the understanding only*, a distinction on which the cogency of his reasoning seems to depend. Using contemporary notions, one can retrace the distinction simply, cleanly, and with accuracy. For any (chosen) thinkable, its 'existing in the understanding only' is syntactic, handled within the formal language $\mathcal{L}$ and its first-order deductive apparatus, conceived as mathematics internal to the structure $[\![\mathcal{R}]\!]$. This identification is not far-fetched, given Anselm's description of the fool's thinking in *Proslogion* Chapter 4:

> For in one sense an object is thought when the word signifying it is thought .... Thus in [this] first sense ...God can be thought not to exist [1, p. 95].

Here, Anselm is responding to the implicit question, "How does the fool (who *said* in his heart that god does not exist) think of god, all the while believing that god does not exist?" In effect, Anselm's answer is that the fool has the Anselm formula in his understanding only, and grasps it exclusively as a string of words, that is, syntactically. The other arm of the contrast, 'a existing in reality,' I take to be captured by the notion of obtaining in the model $[\![\mathcal{R}]\!]$—as defined and treated set-theoretically—so that, when

$$[\![Ra]\!] = True,$$

Ra holds in the structure $[\![\mathcal{R}]\!]$, which represented 'existing in reality,' and so a really exists. Once again, I remind the reader that I do not suppose Anselm to share with us a twenty-first century appreciation of either syntax, semantics, or set theory.

Needless to say, here and in Anselm's reasoning, a conceptual bridge links the understanding so construed with reality so construed, and I maintain that Anselm uses it implicitly. The bridge can be traversed in two directions. In the one direction, it is by applying the Soundness Theorem. In the other, it is via expressibility. First, heading from understanding to reality, the Soundness Theorem assures us that $[\![\mathcal{R}]\!]$ is a model of classical deduction and of its internal mathematics. For instance, with any $\mathcal{L}$-sentences ϕ and ψ with parameter a, when

$$\phi(a) \vdash \psi(a),$$

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if

$$[\![\phi(a)]\!] = True,$$

then

$$[\![\psi(a)]\!] = True,$$

as well. Consequently, if $\vdash Ra$, then

$$[\![Ra]\!] = True.$$

Second, going in the opposite direction—from reality to understanding—the 'existence in reality' condition

$$[\![Ra]\!] = True$$

is expressed in the understanding, in the internal mathematics, by the formula Ra. This implies that one can take the real existence of an item as a target for investigation in the internal logic and mathematics, just as Anselm does 'in his understanding.' Anselm thereby employs his understanding to investigate what obtains in reality.

Now, I inaugurate the ontological argument proper. Anselm calls upon the fool:

> *But surely when this very Fool hears the words "something than which nothing greater can be thought," he understands what he hears. and what he understands is in his understanding* [1, p. 93].

In our rendition, Anselm is directing attention to the Anselm formula,

$$\forall y((Rx \to Ry) \to (Ry \to Rx)),$$

asserting it to be within the fool's (as well as our) understanding. After a brief excursus, already noted, into the difference in grade between an artist's plan for a painting and its realization, Anselm states,

> So even the Fool is convinced that something than which nothing greater can be thought exists at least in his understanding [1, p. 94].

This means that, in the understanding, in the internal mathematics, an item a is governed by the Anselm formula, so that

$$\forall y((Ra \to Ry) \to (Ry \to Ra)).$$

This renders into the formalism Anselm's statement that

> *[W]e believe you to be something than which nothing greater can be thought* [1, p. 93].

After all, the watchword Anselm chose for *Proslogion* was, *Credo ut intelligam* [1, p. 93]: I believe in order to understand.

Next, Anselm opens the pivotal *reductio*, assuming

> *it were only in the understanding* [1, p. 94],

and not in reality. Accordingly, Anselm is now supposing—for the sake of argument—that a, a bearer of the maximality property, is unreal, that is,

$$\neg Ra.$$

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Until further notice, we continue to reason internally over the model, using the interpreted object language; in other words, we work 'in the understanding.' I have already shown that

$$\exists z.Rz \wedge \forall y((Ra \to Ry) \to (Ry \to Ra)) \vdash Ra$$

is a formal theorem of predicate logic and, hence, 'in the understanding,' too. I have explained that we can, without fear, drop the first conjunct

$$\exists z.Rz$$

from the antecedent, since it is a truth of the internal mathematics. It then follows, from the assumption of $\neg Ra$, that

$$\neg\forall y((Ra \to Ry) \to (Ry \to Ra)),$$

which is logically equivalent to

$$\exists y((Ra \to Ry) \wedge \neg(Ry \to Ra)).$$

On the given interpretation, this means that there is a thinkable item y with grade of being strictly greater than that of a. Consequently, a both satisfies the Anselm formula and, at the same time, fails to satisfy it. This contradicts what the fool (and we) understand a to be. In Anselm words,

> *Therefore, if that than which a greater cannot be thought existed only in the understanding, then that than which a greater cannot be thought would be that than which a greater can be thought. But surely this conclusion is impossible* [1, p. 94],

The *reductio* is thereby closed. Hence, the original assumption that

$$\neg Ra$$

fails and—still working in the understanding—we conclude that

$$Ra$$

holds.

At this point, we cross the bridge, provided by the Soundness Theorem, to 'existing in reality.' The logic of the internal mathematics is sound with respect to the model $[\![\mathcal{R}]\!]$, so we can conclude that

$$[\![Ra]\!] = \mathit{True},$$

and a possesses the highest grade of being. Thus, a is fully real. Anselm concludes,

> *Hence, without doubt, something than which a greater cannot be thought exists both in the understanding and in reality.*

In other words, he claims both that the Anselm maximality condition holds of a,

$$\forall y((Ra \to Ry) \to (Ry \to Ra)),$$

and that a really exists, i.e.,

$$Ra \text{ is true.}$$

In summary, then, if you accept a modicum of ordinary set theory together with the assumptions about items and grades above listed, namely, that there are items both real and imaginary, that there are grades of being for such items comprising a set, and that a natural order on those grades respects reality, you must to accept Anselm's conclusion: any item of maximal grade exists in reality.

Incidentally, one can, in similar fashion, reconstruct the ontological argument from Descartes's *Fifth Meditation* [7, pp. 120–123] over a suitable domain D and Boolean algebra $\mathfrak{B}$. I leave that reconstruction as an exercise for the reader.

§7. Necessary existence. Both Descartes [7, p. 122] and Anselm [1, p. 94] wished, in addition, their ontological arguments to prove that god's existence is in some sense necessary. To capture the modality within the Boolean-valued approach, I can extend the models by attaching a topology τ to the powerset Boolean algebra $\mathfrak{B}$.

DEFINITIONS.

1. *A* topology *on the set S underlying $\mathfrak{B}$ is a collection of subsets of S containing S itself, as well as $\emptyset$, and closed under finite intersections $\cap$ and arbitrary unions $\bigcup$.*
2. *Relative to a topology τ, the* interior *of a set $X \subseteq S$—in symbols 'Int(X)'—is the $\subseteq$-largest member Y of τ such that $Y \subseteq X$.*

Once a topology is attached to $\mathfrak{B}$, for any formula ϕ of $\mathcal{L}$, the formula $\Box\phi$ (read informally, "ϕ is necessary") is added, and is interpreted over the extended model $[\![\Box R]\!]$ so that

$$[\![\Box\phi]\!] = \mathrm{Int}([\![\phi]\!]).$$

As reported in his [13], Kazimierz Kuratowski discovered, in effect, that the properties making τ a topology are just those ensuring that, under this interpretation, the $\Box$ has at least the deductive properties codified in the modal system $\mathbf{S}_4$ [10, pp. 46–49].

This scheme for modeling modality is little more than conventional 'possible worlds' semantics in disguise. Say that a subset X of S is *open* whenever X is identical to its own interior or

$$X = \mathrm{Int}(X).$$

Then, define a binary accessibility relation $A(x, y)$ on elements x and y of $\mathfrak{B}$'s carrier set S to hold in case every open set having x as a member also contains y as a member. It is easy to show that the modality determined in the standard fashion from $A(x, y)$ is semantically identical to that described in the preceding paragraph.

In the case of Anselm's ontological argument, the following is a provable sequent in predicate $\mathbf{S}_4$:

$$\exists z.\Box Rz \wedge \forall y((\Box Rx \to \Box Ry) \to (\Box Ry \to \Box Rx)) \vdash \Box Rx.$$

In an extended model $[\![\Box R]\!]$ for that argument, one adopts a topology containing as open sets all the collections $\hat{G}$. This guarantees, plausibly enough, that every item has its grade of being necessarily. In this fashion, $\Box Rx$ can be interpreted in the model $[\![\Box R]\!]$—and in accord with Anselm's text [1, p. 94]—as "it cannot be thought that $\neg Rx$."

§8. Objections: Gaunilo's island and Grant's devil. The idea of grades of being blunts and turns the point of Gaunilo's 'greatest island' objection to Anselm's ontological argument. The monk Gaunilo of Marmoutiers attempted to parody Anselm's argument, maintaining

> You can no more doubt that this [lost] island which is more excellent than all other lands exists somewhere in reality than you can doubt that it is in your understanding. And since for it to exist in reality as well as in the understanding is more excellent [than for it to exist in the understanding alone], then, necessarily, it really exists [1, p. 119].

Let island-items i be thinkable, actual or nonactual, islands. Sufficient representatives of the island-items are already contained in the structure described in Section 5 *infra*. Add to the language $\mathcal{L}$ a single monadic I predicate. When a is any item, $[\![\mathfrak{R}]\!]$ assigns *True* to Ia just in case a is an island-item, and $\emptyset$ otherwise. $[\![\mathfrak{R}]\!]i$ then assigns to island-items i grades of being suitable to them. In analogy with the foregoing, the Gaunilo or 'greatest island-item' property is

> x is an island-item no greater island-item than which can be thought.

The Gaunilo formula expressing that property over the model is

$$\forall y(((Ix \wedge Rx) \rightarrow (Iy \wedge Ry)) \rightarrow ((Iy \wedge Ry) \rightarrow (Ix \wedge Rx))).$$

At this point, construction of a little semantical counterexample suffices to show that

$$[\![Ix \wedge Rx]\!] \;=\; \mathit{True}$$

does not follow from

$$[\![\forall y(((Rx \wedge Ix) \rightarrow (Ry \wedge Iy)) \rightarrow ((Ry \wedge Iy) \rightarrow (Rx \wedge Ix)))]\!] \;=\; \mathit{True}$$

alone. Some assumption that is at least as strong as

$$[\![\exists z(Iz \wedge Rz)]\!] \;=\; \mathit{True}$$

is required. Just consider a model with a two-element domain $\{i, j\}$ in which Ix holds only of i only, while j gets the highest grade of being and i some lower grade.

Therefore, a natural and proper response to the Gaunilo parody consists in showing that

$$[\![\exists z(Iz \wedge Rz)]\!] \;=\; \bigcup_{i\,\varepsilon\, D}[\![Ri]\!] \;=\; \mathit{True}$$

is unlikely to hold in the Anselmian model $[\![\mathfrak{R}]\!]$ of Section 5, extended to I as above. The value $\bigcup_{i\in D}[\![(Ii \wedge Ri)]\!]$ is, by definition, the least upper bound in the Boolean algebra $\mathfrak{B}$ generated from the various $[\![(Ii \wedge Ri)]\!]$, $i\,\varepsilon\, D$, $[\![Ri]\!]$ being the grade of being of island-item i. The island-items are things inanimate, nonsentient, and nonrational, therefore, their grades or extents of being must all be relatively low, below the grade G of, say, an individual fieldmouse. Taken together, the union of those low grades must be below G and, hence,

$$\bigcup_{i\in D}[\![(Ii \wedge Ri)]\!]$$

cannot have *True*, the highest grade, as its least upper bound.

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In just this spirit, Bonaventure responded to Gaunilo's argument. He pointed out that an island is, by its nature, a defective being or *ens defectivum* [3, 1.1 *ad* 6]. As Anselm wrote in reply to Gaunilo,

> [I]f anyone finds for me anything else (whether existing in reality or only in thought) to which he can apply the logic of my argument, then I will find and make him a present of that lost island—no longer to be lost.

That is, if islands or island-items could attain higher grades of being ("to which he can apply the logic of my argument,") then the most excellent island would indeed exist. However, islands and island-items are not the sorts of things to which the logic of the argument, as anatomized here using grades of being, will apply.

C.K. Grant [9, pp. 71–72] argued that, had Anselm succeeded in proving that the maximally perfect item indeed exists, then, by parity of reasoning, one ought to be able to demonstrate that the devil, the maximally imperfect item, does not exist. Presumably, in our case, this would mean that it ought to follow logically either from x having the least grade of being,

$$\forall y(Rx \to Ry),$$

or from x having a minimal grade of being,

$$\forall y((Ry \to Rx) \to (Rx \to Ry)),$$

that $\neg Rx$.

Neither is the case. First, nowhere was it shown *simpliciter* and logically that items having the maximum or a maximal grade have to be real. What was shown, in both cases, is that, if one adjoins the assumption "there is an item,"

$$\exists z.Rz,$$

then Rx will follow logically from the maximum or maximality condition. Second, it is worth asking, "What happens if, to the above 'devilish' minimality conditions, one adds the assumption that there is an item that doesn't exist?" The following sequent is logically true.

$$\exists z.\neg Rz \wedge \forall y(Rx \to Ry) \vdash \neg Rx.$$

Ergo, granting that there are nonexistents, one sees that, if x occupies a minimum grade of being or

$$[\![\forall y(Rx \to Ry)]\!] = \textit{True},$$

then

$$[\![Rx]\!] = \textit{False},$$

and

$$[\![\neg Rx]\!] = \textit{True}.$$

However, as Grant himself pointed out [9, p. 72], a conclusion such as the foregoing does not in fact demonstrate that the devil is unreal. One can respond that the devil, even if imaginary, should not possess an absolutely minimum or minimal grade of being, for the devil is presumably quite sentient and highly rational. Nonexistent rocks or round, square cupolas, it appears, will have even lower grades of being than a cunning, powerful devil.

§9. Coda. If there is a principal shortcoming in the numerous earlier formalizations of the ontological arguments, it is a tacit demand for strict uniformity. Each such formalization is laid out for viewing within some single object language, standardly that of first- or second-order predicate logic or a modal extension thereof, accompanied by a single informal (and, all too often, vaguely specified) interpretation. It is this blithe assumption of uniformity—that every step in the original argument must submit to reëxpression within one formalism—that I question. In the present essay, the arguments are anatomized using more varied logical instruments, among them, an object language and its formal logic with an explicitly specified informal reading, mathematics internal to a model or class of models (i.e., the object language under a formal semantical interpretation), as well as a full, semiformal, set-theoretic metatheory for constructing models and discerning their properties.

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