Canad. J. Math. Vol. **65** (3), 2013 pp. 634–654
http://dx.doi.org/10.4153/CJM-2012-033-x

Laplace Equations and the Weak Lefschetz Property

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Abstract. We prove that r independent homogeneous polynomials of the same degree d become dependent when restricted to any hyperplane if and only if their inverse system parameterizes a variety whose $(d-1)$-osculating spaces have dimension smaller than expected. This gives an equivalence between an algebraic notion (called the Weak Lefschetz Property) and a differential geometric notion, concerning varieties that satisfy certain Laplace equations. In the toric case, some relevant examples are classified, and as a byproduct we provide counterexamples to Ilardi's conjecture.

1 Introduction

The goal of this note is to establish a close relationship between two a priori unrelated problems: the existence of homogeneous artinian ideals $I \subset k[x_0,\dots,x_n]$ that fail the Weak Lefschetz Property and the existence of (smooth) projective varieties $X \subset \mathbb{P}^N$ satisfying at least one Laplace equation of order $s \geq 2$. These are two longstanding problems that, as we will see, lie at the crossroads between Commutative Algebra, Algebraic Geometry, Differential Geometry, and Combinatorics.

An n-dimensional projective variety $X \subset \mathbb{P}^N$ is said to satisfy δ independent Laplace equations of order s if its s-osculating space at a general point $p \in X$ has dimension $\binom{n+s}{s} - 1 - \delta$. A homogeneous artinian ideal $I \subset k[x_0,x_1,\dots,x_n]$ is said to have the *Weak Lefschetz Property (WLP)* if there is a linear form $L \in k[x_0,x_1,\dots,x_n]$ such that, for all integers j, the multiplication map

$$\times L\colon \big(k[x_0,x_1,\dots,x_n]/I\big)_j \longrightarrow \big(k[x_0,x_1,\dots,x_n]/I\big)_{j+1}$$

has maximal rank, *i.e.*, is injective or surjective. One would naively expect this property to hold, and so it is interesting to find classes of artinian ideals failing WLP, and to understand what it is from a geometric point of view that prevents this property from holding.

The starting point of this paper was [1, Example 3.1] and the classical articles of Togliatti [19, 20]. In [1], Brenner and Kaid show that, over an algebraically closed

Received by the editors December 13, 2011; revised July 2, 2012.
Published electronically September 21, 2012.
The first author was supported by MIUR funds, PRIN project "Geometria delle varietà algebriche e dei loro spazi di moduli" (cofin 2008). The second author was partially supported by MTM2010-15256. The third author was supported by MIUR funds, PRIN project "Proprietà geometriche delle varietà reali e complesse" (cofin 2009). The first and third authors are members of italian GNSAGA.
AMS subject classification: **13E10**, 14M25, 14N05, 14N15, 53A20.
Keywords: osculating space, weak Lefschetz property, Laplace equations, toric threefold.

field of characteristic zero, any ideal of the form $(x^3, y^3, z^3, f(x, y, z))$ with $\deg f = 3$, fails to have the WLP if and only if $f \in (x^3, y^3, z^3, xyz)$. Moreover, they prove that the latter ideal is the only such monomial ideal that fails to have the WLP. A famous result of Togliatti (see [20]; or [5]) proves that there is only one nontrivial (in the sense to be defined in Section 4) example of a surface $X \subset \mathbb{P}^5$ obtained by projecting the Veronese surface $V(2,3) \subset \mathbb{P}^9$ and satisfying a single Laplace equation of order 2; X is projectively equivalent to the image of $\mathbb{P}^2$ via the linear system $\langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2\rangle \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$. Note that the linear system of cubics given by Brenner and Kaid's example $\langle x^3, y^3, z^3, xyz\rangle$ is apolar to the linear system of cubics given in Togliatti's example. A careful analysis of this example suggested us that there is relationship between artinian ideals $I \subset k[x_0, \dots, x_n]$ generated by r homogeneous forms of degree d that fail Weak Lefschetz Property and projections of the Veronese variety $V(n,d) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ to $X \subset \mathbb{P}^{\binom{n+d}{d}-r-1}$ satisfying at least a Laplace equation of order $d-1$. Our goal will be to exhibit such relationship with the hope of shedding more light on these fascinating and perhaps intractable problems of classifying the artinian ideals that fail the Weak Lefschetz Property and of classifying n-dimensional projective varieties satisfying at least one Laplace equation of order s. Our main theorem, Theorem 3.2, says that an ideal I generated by homogeneous forms of degree d, satisfying some reasonable assumptions, fails the WLP in degree $d-1$ if and only if its apolar linear system, corresponding to the homogeneous component of degree d of I^{-1}, parameterizes a variety that satisfies a Laplace equation of degree $d-1$.

We then give two examples of applications of Theorem 3.2, to linear systems of cubics in four variables and to ideals of polynomials in three variables respectively.

Next we outline the structure of this note. In Section 2 we fix the notation and collect the basic results on Laplace equations and the Weak Lefschetz Property needed in the sequel. Section 3 is the heart of the paper. In this section, we state and prove our main result (Theorem 3.2). In Section 4, we restrict our attention to the monomial case and give a complete classification in the case of smooth and quasi-smooth cubic linear systems on $\mathbb{P}^n$ for $n \leq 3$. In Section 5 we concentrate on the case $n = 2$ and specifically on ideals with 4 generators. We end the paper in Section 6 with some natural problems coming up from our work and a family of counterexamples to Ilardi's conjecture that work for any $n \geq 3$.

2 Definitions and Preliminary Results

In this section we recall some standard terminology and notation from commutative algebra and algebraic geometry, as well as some results needed in the sequel.

Set $R = k[x_0, x_1, \dots, x_n]$, where k is an algebraically closed field of characteristic zero and let $\mathfrak{m} = (x_0, x_1, \dots, x_n)$ be its maximal homogeneous ideal. We consider a homogeneous ideal I of R. The Hilbert function $h_{R/I}$ of R/I is defined by $h_{R/I}(t) := \dim_k(R/I)_t$. Note that the Hilbert function of an artinian k-algebra R/I has finite support and is captured in its *h-vector* $\underline{h} = (h_0, h_1, \dots, h_e)$, where $h_0 = 1$, $h_i = h_{R/I}(i) > 0$, and e is the last index with this property.

In the case of three variables, we will often use x, y, z instead of x_0, x_1, x_2.

2.1 The Weak Lefschetz Property

Definition 2.1 Let $I \subset R$ be a homogeneous artinian ideal. We will say that the standard graded artinian algebra R/I has the *Weak Lefschetz Property (WLP)* if there is a linear form $L \in (R/I)_1$ such that, for all integers j, the multiplication map

$$\times L \colon (R/I)_j \longrightarrow (R/I)_{j+1}$$

has maximal rank, *i.e.,* is injective or surjective. (We will often abuse notation and say that the ideal I has the WLP.) In this case, the linear form L is called a *Lefschetz element* of R/I. If for the general form $L \in (R/I)_1$ and for an integer number j the map $\times L$ does not have maximal rank, we will say that the ideal I fails the WLP in degree j.

The Lefschetz elements of R/I form a Zariski open, possibly empty, subset of $(R/I)_1$. Part of the great interest in the WLP stems from the fact that its presence puts severe constraints on the possible Hilbert functions, which can appear in various disguises (see, *e.g.,* [17]). Though many algebras are expected to have the WLP, establishing this property is often rather difficult. For example, it was shown by R. Stanley [18] and J. Watanabe [22] that a monomial artinian complete intersection ideal $I \subset R$ has the WLP. By semicontinuity, it follows that a *general* artinian complete intersection ideal $I \subset R$ has the WLP, but it is open whether *every* artinian complete intersection of height ≥ 4 over a field of characteristic zero has the WLP. It is worthwhile to point out that in positive characteristic there are examples of artinian complete intersection ideals $I \subset k[x, y, z]$ failing the WLP (see, *e.g.,* [13, Remark 7.10]).

Notation If $F_1, \dots, F_r$ are polynomials, $(F_1, \dots, F_r)$ denotes the ideal they generate, while $\langle F_1, \dots, F_r \rangle$ denotes the vector subspace generated by them.

Example 2.2 (1) The ideal $I = (x^3, y^3, z^3, xyz) \subset k[x, y, z]$ fails to have the WLP, because for any linear form $L = ax + by + cz$ the multiplication map

$$\times L \colon (k[x, y, z]/I)_2 \longrightarrow (k[x, y, z]/I)_3$$

is neither injective nor surjective. Indeed, since it is a map between two k-vector spaces of dimension 6, to show the latter assertion it is enough to exhibit a nontrivial element in its kernel. If $f = a^2x^2 + b^2y^2 + c^2z^2 - abxy - acxz - bcyz$, then f is not in I and we easily check that $L \cdot f$ is in I.

(2) The ideal $I = (x^3, y^3, z^3, x^2y) \subset k[x, y, z]$ has the WLP. Since the h-vector of R/I is (1,3,6,6,4,1), we only need to check that the map $\times L \colon (R/I)_i \to (R/I)_{i+1}$ induced by $L = x + y + z$ is surjective for $i = 2, 3, 4$. This is equivalent to checking that $(R/(I, L))_i = 0$ for $i = 3, 4, 5$. Obviously, it is enough to check the case $i = 3$. We have

$$\begin{aligned}(R/(I, L))_3 &\cong (k[x, y, z]/(x^3, y^3, z^3, x^2y, x + y + z))_3 \\ &\cong (k[x, y]/(x^3, y^3, x^3 + 3x^2y + 3xy^2 + y^3, x^2y))_3 \\ &\cong k[x, y]/(x^3, y^3, x^2y, xy^2))_3 = 0\end{aligned}$$

which proves what we want.

In this note we are mainly interested in artinian ideals I generated by homogeneous forms of fixed degree d. In this case we have the following easy but useful lemma.

Lemma 2.3 *Let $I \subset R = k[x_0, x_1, \dots, x_n]$ be an artinian ideal generated by $r \le \binom{n+d-1}{d}$ homogeneous forms $F_1, \dots, F_r$ of degree d. Let L be a linear form, let $\bar{R} = R/(L)$ and let $\bar{I}$ (resp. $\bar{F}_i$) be the image of I (resp. F_i) in $\bar{R}$. Consider the homomorphism $\phi_{d-1} \colon (R/I)_{d-1} \to (R/I)_d$ defined by multiplication by L. Then ϕ_{d-1} does not have maximal rank if and only if $\bar{F}_1, \dots, \bar{F}_r$ are k-linearly dependent.*

Proof First note that $(R/I)_{d-1} \cong R_{d-1}$,

$$\dim R_{d-1} = \binom{n+d-1}{d-1}, \quad \dim(R/L)_d = \binom{n+d-1}{n-1},$$
$$\dim(R/I)_d = \binom{n+d}{d} - r.$$

Consider the exact sequence

$$0 \longrightarrow \frac{[I:L]}{I} \longrightarrow R/I \xrightarrow{\times L} (R/I)(1) \longrightarrow (R/(I,L))(1) \longrightarrow 0,$$

where $\times L$ in degree $d-1$ is just ϕ_{d-1}. This shows that the cokernel of ϕ_{d-1} is just $(R/(I,L))_d$.

Since $r \le \binom{n+d-1}{d}$, we have $\dim(R/I)_{d-1} \le \dim(R/I)_d$. Hence, ϕ_{d-1} does not have maximal rank if and only if ϕ_{d-1} is not injective, if and only if $rk(\phi_{d-1}) < \binom{n+d-1}{d-1}$, if and only if

$$\begin{aligned}\dim(R/(I,L))_d &= \dim(\bar{R})_d - \dim \bar{I}_d = \binom{n+d-1}{n-1} - \dim\langle \bar{F}_1, \dots, \bar{F}_r\rangle_d \\ &\gneqq \dim(R/I)_d - \binom{n+d-1}{d-1} = \binom{n+d}{d} - \binom{n+d-1}{d-1} - r \\ &= \binom{n+d-1}{n-1} - r.\end{aligned}$$

Therefore, ϕ_{d-1} is not injective if and only if $\dim\langle \bar{F}_1, \dots, \bar{F}_r\rangle \lneqq r$, if and only if $\bar{F}_1, \dots, \bar{F}_r$ are k-linearly dependent. ∎

As an easy consequence we have the following useful corollary.

Corollary 2.4 *Let $F_1, \dots, F_r \in R = k[x_0, x_1, \dots, x_n]$ be a set of $\mathfrak{m}$-primary homogeneous forms of degree d. Let L be a linear form, let $\bar{R} = R/(L)$, and let $\bar{F}_i$ be the image of F_i in $\bar{R}$. If $r \le \binom{n-1+d}{d}$ and $\bar{F}_1, \dots, \bar{F}_r$ are k-linearly dependent, then the ideal $I = (F_1, \dots, F_r)$ fails the WLP and the same is true for any enlarged ideal $J = (F_1, \dots, F_r, F_{r+1}, \dots, F_t) \subsetneq R_d$ with $r \le t \le \binom{n-1+d}{d}$.*

Closing this subsection, we reformulate the Weak Lefschetz Property in the case $n = 2$ by using the theory of vector bundles on the projective space, and we refer to [1] for more information.

With any subspace $\langle F_1, \dots, F_r\rangle$ generated by r $\mathfrak{m}$-primary homogeneous forms of degree d is associated a kernel vector bundle K as in the following exact sequence on $\mathbb{P}^2$:

$$0 \longrightarrow K \longrightarrow \mathcal{O}^r \xrightarrow{F_1,\dots,F_r} \mathcal{O}(d) \longrightarrow 0.$$

The fact that K is locally free follows from the fact that $(F_1, \dots, F_r)$ is $\mathfrak{m}$-primary. It is well known that the bundle K splits on any line L as the sum of line bundles. On the general line L we have a splitting $K_{|L} \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_L(a_i)$, where $a_i \le 0$ for $1 \le i \le r-1$ and, moreover, we may assume that $a_1 \le \cdots \le a_{r-1}$. The $(r-1)$-ple $(a_1, \dots a_{r-1})$ is called the generic splitting type of K.

Theorem 2.5 *Assume $n = 2$. Let $I = (F_1, \dots, F_r)$ be an $\mathfrak{m}$-primary ideal generated by r homogeneous forms and let $(a_1, \dots a_{r-1})$ be the generic splitting type of the kernel bundle K. The following properties are equivalent:*

(i) *I has the WLP;*
(ii) $a_{r-1} < 0$.

Proof The forms $F_1, \dots F_r$ restricted to a general line L are dependent if and only if the restricted map

$$H^0(\mathcal{O}_L^r) \xrightarrow{F_1,\dots,F_r} H^0(\mathcal{O}_L(d))$$

has a nonzero kernel. The result follows because the kernel is $\oplus H^0(\mathcal{O}_L(a_i))$. ■

In the Brenner–Kaid example quoted in the introduction, we get as kernel a rank three vector bundle on $\mathbb{P}^2$ with generic splitting type $(-2, -1, 0)$.

Notation Let $V(n,d)$ denote the image of the projective space $\mathbb{P}^n$ in the d-tuple Veronese embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$.

2.2 Laplace Equations

In this section we adopt the point of view of differential geometry, as in [10].

Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. Let $x \in X$ be a smooth point. We can choose a system of N affine coordinates around x and a local parametrization of X of the form $\phi(t_1, \dots, t_n)$, where $x = \phi(0, \dots, 0)$ and the N components of ϕ are formal power series.

The tangent space to X at x is the k-vector space generated by the n vectors that are the partial derivatives of ϕ at x. Since x is a smooth point of X, these n vectors are k-linearly independent. Note that this is not the tangent space in the Zariski sense, but in differential-geometric language.

Similarly one defines the s-th osculating (vector) space $T_x^{(s)}X$ to be the span of all partial derivatives of ϕ of order $\le s$ (see, for instance, [10]). The expected dimension of $T_x^{(s)}X$ is $\binom{n+s}{s} - 1$, but in general $\dim T_x^{(s)}X \le \binom{n+s}{s} - 1$; if strict inequality holds for all smooth points of X, and $\dim T_x^{(s)}X = \binom{n+s}{s} - 1 - \delta$ for general x, then X is said to satisfy δ Laplace equations of order s. Indeed, in this case the partials of order s of ϕ are linearly dependent, which gives δ linear partial differential equations of order s, which are satisfied by the components of ϕ. We will also consider the projective s-th osculating space $\mathbb{T}_x^{(s)}X$, embedded in $\mathbb{P}^N$.

Remark 2.6 It is easy to prove (see, for instance, [10, §1(d)]) that, if X is a non-degenerate curve in $\mathbb{P}^N$, *i.e.,* X is not contained in any proper linear subspace of $\mathbb{P}^N$, then $\dim T_x^{(s)}X = s$ for a general point x. Hence X does not satisfy any Laplace equation.

Remark 2.7 It is clear that if $N < \binom{n+s}{s} - 1$, then X satisfies at least one Laplace equation of order s, but this case is not interesting and will not be considered in the following.

Remark 2.8 If X is uniruled by lines, *i.e.,* through any general point of X passes a line contained in X, then X satisfies a Laplace equation. Roughly speaking in this case it is possible to find a local parametrization of X in which one of the parameters appears at most at degree one. Hence the corresponding second derivative vanishes identically. The case of surfaces from which the general case follows immediately, is treated in detail in [10, §2].

If $X \subset \mathbb{P}^N$ is a rational variety, then there exists a birational map $\mathbb{P}^n \dashrightarrow X$ given by $N+1$ forms $F_0, \dots, F_N$ of some degree d of $k[x_0, x_1, \dots, x_n]$. From Euler's formula for homogeneous polynomials it follows that, for $s \le d$, the projective s-th osculating space $\mathbb{T}_x^{(s)}X$, for x general, is generated by the s-th partial derivatives of $F_0, \dots, F_N$ at the point x.

Assume that X is not a linear space. In the case $s = 2$, $n = 2$, the dimension of $\mathbb{T}_x^{(2)}X$ varies between 3 and 5. Moreover, $\dim \mathbb{T}_x^{(2)}X = 3$ for general $x \in X$ if and only if X is either a hypersurface or a ruled developable surface, *i.e.,* a cone or the developable tangent of a curve. The surfaces with $\dim \mathbb{T}_x^{(2)}X = 4$ for general $x \in X$ are not yet well understood in spite of the literature devoted to this topic (see [16], where they are called "superfici Φ", [10, p. 377], [5] [11], [19], [20], [21], [12]). As for surfaces in $\mathbb{P}^N$, $N \ge 5$ satisfying Laplace equations, only Del Pezzo surfaces, *i.e.,* projections of $V(2, 3)$, have been systematically studied. Besides the ruled surfaces, there are only a few smooth examples; in particular, the Togliatti surface introduced above (see the Introduction), a special complete intersection of quadrics in $\mathbb{P}^5$ that is a desingularization of the Kummer surface (see [3] and [4]), and some toric surfaces (see the examples given by Perkinson in [15], where the classification is given of toric surfaces and threefolds whose osculating spaces up to order $d - 1$ all have maximal dimension and all have dimension less than maximal for order d).

3 The Main Theorem

The goal of this section is to highlight the existence of a surprising relationship between a pure algebraic problem, the existence of artinian ideals $I \subset R$ generated by homogeneous forms of degree d and failing the WLP, and a pure geometric problem, the existence of projections of the Veronese variety $V(n,d) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ to $X \subset \mathbb{P}^N$ satisfying at least one Laplace equation of order $d-1$. Moreover, we will also discuss the geometry of some surfaces "apolar" to those satisfying the Laplace equation.

We start this section recalling the basic facts on Macaulay–Matlis duality that will allow us to relate the above mentioned problems. Let V be an $(n+1)$-dimensional k-vector space and set $R = \bigoplus_{i\geq 0} \operatorname{Sym}^i V^*$ and $\mathcal{D} = \bigoplus_{i\geq 0} \operatorname{Sym}^i V$. Let $\{x_0, x_1, \dots, x_n\}$, $\{y_0, y_1, \dots, y_n\}$ be dual bases of V^* and V respectively. So, we have the identifications $R = k[x_0, x_1, \dots, x_n]$ and $\mathcal{D} = k[y_0, y_1, \dots, y_n]$. There are products (see [7, p. 476])

$$\operatorname{Sym}^j V^* \otimes \operatorname{Sym}^i V \longrightarrow \operatorname{Sym}^{i-j} V$$
$$F \otimes D \longmapsto F \cdot D$$

making $\mathcal{D}$ into a graded R-module. We can see this action as partial differentiation. If $F(x_0, x_1, \dots, x_n) \in R$ and $D(y_0, y_1, \dots, y_n) \in \mathcal{D}$, then

$$F \cdot D = F(\partial/\partial y_0, \partial/\partial y_1, \dots, \partial/\partial y_n)D.$$

If $I \subset R$ is a homogeneous ideal, we define the *Macaulay's inverse system* I^{-1} for I as

$$I^{-1} := \{D \in \mathcal{D}, F \cdot D = 0 \text{ for all } F \in I\}.$$

Then I^{-1} is an R-submodule of $\mathcal{D}$ that inherits a grading of $\mathcal{D}$. Conversely, if $M \subset \mathcal{D}$ is a graded R-submodule, then $\operatorname{Ann}(M) := \{F \in R, F \cdot D = 0 \text{ for all } D \in M\}$ is a homogeneous ideal in R. In classical terminology, if $F \cdot D = 0$ and $\deg(F) = \deg(D)$, then F and D are said to be *apolar* to each other. In fact, the pairing

$$R_i \times \mathcal{D}_i \longrightarrow k \qquad (F, D) \mapsto F \cdot D$$

is exact; it is called the apolarity or Macaulay–Matlis duality action of R on $\mathcal{D}$.

For any integer i, we have $h_{R/I}(i) = \dim_k(R/I)_i = \dim_k(I^{-1})_i$. The following theorem is fundamental.

Theorem 3.1 *We have a bijective correspondence*

$$\begin{array}{ccc} \{\textit{Homogeneous ideals } I \subset R\} & \rightleftharpoons & \{\textit{Graded } R-\textit{submodules of } \mathcal{D}\} \\ I & \to & I^{-1} \\ \operatorname{Ann}(M) & \leftarrow & M \end{array}.$$

Moreover, I^{-1} is a finitely generated R-module if and only if R/I is an artinian ring.

When considering only monomial ideals, we can simplify by regarding the inverse system in the same polynomial ring R, and in any degree, d, the inverse system I_d^{-1} is spanned by the monomials in R_d not in I_d. Using the language of inverse systems, we will still call the elements obtained by the action *derivatives*.

Let I be an artinian ideal generated by r homogeneous polynomials $F_1, \dots, F_r \in R = k[x_0, x_1, \dots, x_n]$ of degree d. Let $I^{-1} \subset \mathcal{D}$ be its Macaulay inverse system. Associated with $(I^{-1})_d$ there is a rational map

$$\varphi_{(I^{-1})_d} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}.$$

The closure of its image $\overline{\operatorname{Im}(\varphi_{(I^{-1})_d})} \subset \mathbb{P}^{\binom{n+d}{d}-r-1}$ is the projection of the n-dimensional Veronese variety $V(n,d)$ from the linear system $\langle F_1, \dots, F_r\rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$. Let us call it $X_{n,(I^{-1})_d}$. Analogously, associated with I_d there is a morphism

$$\varphi_{I_d} \colon \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}.$$

Note that φ_{I_d} is regular (*i.e.,* defined everywhere), because I is artinian. Its image $\operatorname{Im}(\varphi_{I_d}) \subset \mathbb{P}^{r-1}$ is closed and is the projection of the n-dimensional Veronese variety $V(n,d)$ from the linear system $\langle (I^{-1})_d\rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$. Let us call it X_{n,I_d}. The varieties X_{n,I_d} and $X_{n,(I^{-1})_d}$ are usually called apolar.

We are now ready to state the main result of this section.

Theorem 3.2 (The Tea Theorem[1]) *Let $I \subset R$ be an artinian ideal generated by r homogeneous polynomials $F_1, \dots, F_r$ of degree d. If $r \le \binom{n+d-1}{n-1}$, then the following conditions are equivalent:*

(i) *the ideal I fails the WLP in degree $d-1$;*
(ii) *the homogeneous forms $F_1, \dots, F_r$ become k-linearly dependent on a general hyperplane H of $\mathbb{P}^n$;*
(iii) *the n-dimensional variety $X_{n,(I^{-1})_d}$ satisfies at least one Laplace equation of order $d-1$.*

Remark 3.3 Note that, in view of Remark 2.7, the assumption $r \le \binom{n+d-1}{n-1}$ ensures that the Laplace equations obtained in (iii) are not obvious in the sense of Remark 2.7. In the particular case $n = 2$, this assumption gives $r \le d+1$.

Proof The equivalence between (i) and (ii) follows immediately from Lemma 2.3. Let us see that (i) is equivalent to (iii). Since $(R/I)_{d-1} = R_{d-1}$ and

$$\dim R_{d-1} = \binom{n+d-1}{n} = \binom{n+d}{n} - \binom{n+d-1}{n-1} \le \binom{n+d}{n} - r = \dim(R/I)_d,$$

we have that the ideal I fails the WLP in degree $d-1$ if and only if for a linear form $L \in R_1$ the multiplication map

$$\times L \colon (R/I)_{d-1} \to (R/I)_d$$

is not injective. Via the Macaulay–Matlis duality, the latter is equivalent to saying that the rank of the dual map $(I^{-1})_d \longrightarrow (I^{-1})_{d-1}$ is $\le \binom{d+n-1}{n} - 1$; which is equivalent to saying that the $(d-1)$-th osculating space $\mathbb{T}_x^{(d-1)} X_{n,(I^{-1})_d}$ spanned by all partial derivatives of order $\le d-1$ of the given parametrization of $X_{n,(I^{-1})_d}$ has dimension $\le \binom{n+d-1}{n} - 2$; *i.e.,* $X_{n,(I^{-1})_d}$ satisfies a Laplace equation of order $d-1$. ■

[1] Since the first guess about the statement of the theorem emerged during a Tea discussion in Berkeley, we have always labeled the result in our discussions as the "Tea Theorem".

Remark 3.4 Note that for $n = 2$, $d = 3$, and $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset k[x_0, x_1, x_2]$, we recover Togliatti's example (see [19], [20], and [5]).

Definition 3.5 With notation as above, we will say that I^{-1} (or I) defines a *Togliatti system* if it satisfies the three equivalent conditions in Theorem 3.2.

Example 3.6 ([21]) Let $d = 2k + 1$ be an odd number and $n = 2$. Let $l_1, \ldots, l_d$ be general linear forms in 3 variables. Then the ideal $(l_1^d, \ldots, l_d^d, l_1l_2 \ldots l_d)$ is generated by $d + 1$ polynomials of degree d, and it fails the WLP in degree $d - 1$, because by [21, Théorème 3.1], $l_1^d, \ldots, l_d^d, l_1l_2 \cdots l_d$ become dependent on a general line $L \subset \mathbb{P}^2$. For $d = 3$ we recover Togliatti's example once more; for $d > 3$ we get nontoric examples. It is interesting to observe that a similar construction in even degree produces ideals that do satisfy the WLP.

Example 3.7 Let $n \geq 3$ and $d \geq 3$. Let $I = (LF_1, \ldots, LF_t, G_1, \ldots, G_n)$, where L is a linear form, $F_1, \ldots, F_t$ are general forms of degree $d - 1$, and $G_1, \ldots, G_n$ are general forms of degree d. If $\binom{n+d-2}{n-1} + 1 \leq t \leq \binom{n+d-1}{n-1} - n$, then I is artinian and fails the WLP in degree $d - 1$. Indeed the number of conditions imposed on the forms of degree $d-1$ to contain a linear form is equal to $\binom{n+d-2}{n-1}$. With the assumptions made on t, the number of generators $r = t + n$ is in the range of Theorem 3.2.

We will end this section by studying the geometry of some rational surfaces satisfying at least one Laplace equation of order 2 and the geometry of their apolar surfaces.

Example 3.8 In the case of the Togliatti surface the morphism $\varphi_{I_3} \colon \mathbb{P}^2 \to \mathbb{P}^3$ with $I_3 = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ is not birational. In fact, it is a triple cover of the cubic surface of equation $xyz = t^3$, which is singular at the three fundamental points of the plane $t = 0$.

Similarly in the case $n = 2$, $d = 4$, and $I_4 = (x_0^4, x_1^4, x_2^4, x_0^2x_1^2, x_0x_1x_2^2)$, the surface $X_{2,(I^{-1})_4} \subset \mathbb{P}^9$ has second osculating space of dimension 8 at a general point. Also the morphism $\varphi_{I_4} \colon \mathbb{P}^2 \to \mathbb{P}^4$ is not birational; it is a degree 4 cover of a singular Del Pezzo quartic, the complete intersection of two quadrics in $\mathbb{P}^4$.

Similar considerations can be made in the following example, where $n = 2$, $d = 5$, and $I_5 = (x_0^5, x_1^5, x_2^5, x_0^3x_1^2, x_0^2x_1^3, x_1^2x_2^3)$, but in this case we get a birational map $\varphi_{I_5} \colon \mathbb{P}^2 \to \mathbb{P}^5$.

4 The Toric Case

In this section, we will restrict our attention to the monomial case. First of all, we want to point out that for monomial ideals (*i.e.,* the ideals invariants for the natural toric action of $(k^*)^n$) on $k[x_0, \ldots, x_n]$) to test the WLP there is no need to consider a general linear form. In fact, we have

Proposition 4.1 *Let $I \subset R := k[x_0, x_1, \ldots, x_n]$ be an artinian monomial ideal. Then R/I has the WLP if and only if $x_0 + x_1 + \cdots + x_n$ is a Lefschetz element for R/I.*

Proof See [13, Proposition 2.2]. ∎

Fix $\mathbb{P}^n = \operatorname{Proj}(k[x_0, x_1, \dots, x_n])$. Denote by $\mathcal{L}_{n,d} := |\mathcal{O}_{\mathbb{P}^n}(d)|$ the complete linear system of hypersurfaces of degree d in $\mathbb{P}^n$ and set $n_d := \dim(\mathcal{L}_{n,d}) = \binom{n+d}{n} - 1$ its projective dimension. As usual denote by $V(n,d) \subset \mathbb{P}^{n_d}$ the Veronese variety.

Definition 4.2 A linear subspace $\mathcal{L} \subset \mathcal{L}_{n,d}$ is called a *monomial linear subspace* if it can be generated by monomials.

The example of the truncated simplex Consider the linear system of cubics

$$\mathcal{L} = |\{x_i^2 x_j\}_{0 \le i \ne j \le n}| \subset \mathcal{L}_{n,3}.$$

Note that $\dim \mathcal{L} = n(n+1) - 1$. Let $\varphi_{\mathcal{L}} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n(n+1)-1}$ be the rational map associated with $\mathcal{L}$. It has $n+1$ fundamental points (where it is not defined). The closure of its image $X := \overline{\operatorname{Im}(\varphi_{\mathcal{L}})} \subset \mathbb{P}^{n(n+1)-1}$ is (projectively equivalent to) the projection of the Veronese variety $V(n,3)$ from the linear subspace

$$\mathcal{L}' := \left| \left\langle x_0^3, x_1^3, \dots, x_n^3, \{x_i x_j x_k\}_{0 \le i < j < k \le n} \right\rangle \right|$$

of $\mathbb{P}^{\binom{n+3}{3}-1}$. Let us check that X satisfies a Laplace equation of order 2 and that it is smooth.

Since $\mathcal{L}$ and $\mathcal{L}'$ are apolar, we can apply Theorem 3.2 and we get that X satisfies a Laplace equation of order 2 if and only if the ideal

$$I = \left(x_0^3, x_1^3, \dots, x_n^3, \{x_i x_j x_k\}_{0 \le i < j < k \le n}\right) \subset R = k[x_0, x_1, \dots, x_n]$$

fails the WLP in degree 2; *i.e.,* for a general linear form $L \in R_1$ the map $\times L \colon (R/I)_2 \to (R/I)_3$ does not have maximal rank. By Lemma 2.3, it is enough to see that the restriction of the cubics $x_0^3, x_1^3, \dots, x_n^3, \{x_i x_j x_k\}_{0 \le i < j < k \le n}$ to a general hyperplane become k-linearly dependent and, by Proposition 4.1, it is enough to check that they become k-linearly dependent when we restrict to the hyperplane $x_0 + x_1 + \cdots + x_n = 0$, which follows after a straightforward computation. An alternative argument, due to [15, Proposition 1.1], is that all the vertex points in $\mathbb{Z}^{n+1}$, corresponding to the monomial basis of $\mathcal{L}$, are contained in the quadric with equation

$$2\left(\sum_{i=0}^{n} x_i^2\right) - 5\left(\sum_{0 \le i < j \le n} x_i x_j\right) = 0.$$

Then X is a projection of the blow-up of $\mathbb{P}^n$ at the $n+1$ fundamental points, embedded via the linear system of cubics passing through the blown-up points. Using the language of [8], it is the projective toric variety X_A, associated with the set A of vertices of the lattice polytope P_n defined as follows: let Δ_n be the standard simplex in $\mathbb{R}^n$, consider $3\Delta_n$, then P_n is obtained by removing all vertices so that the new edges have all length one: P_n is a "truncated simplex". By the smoothness criterium [8, Corollary 3.2, Ch. 5] (see also [15]), it follows that X is smooth. For instance, in the case $n = 2$, P_2 is the punctured hexagon of Figure 2.

In [11, p. 12], G. Ilardi formulated a conjecture, stating that the above example is the only smooth (meaning that the variety X is smooth) monomial Togliatti system of cubics of dimension $n(n+1)-1$. We will show that conjecture is incorrect, but we underline that it was useful to us because it pointed in the right direction.

We start by producing a class of examples of monomial Togliatti systems of cubics, holding for any $n \geq 3$. We will then give the classification of smooth and quasi-smooth monomial Togliatti systems for $n=3$ in Theorem 4.11. As a consequence, the conjecture in [11, p. 12] cannot hold, in the sense that the list in [11] is too short and we have to enlarge it. Correspondingly, in Remark 6.2, we propose a larger list for any n, which reduces to the list of the Theorem 4.11 for $n=3$.

A second example Consider the linear system of cubics

$$\mathcal{M} = \Big| \{x_i^2 x_j\}_{\substack{0 \leq i \neq j \leq n, \\ \{i,j\} \neq \{0,1\}}} \cup \{x_0 x_1 x_i\}_{2 \leq i \leq n} \Big| \subset \mathcal{L}_{n,3}.$$

Note that $\dim \mathcal{M} = n^2 + 2n - 4$. Let

$$\varphi_{\mathcal{M}} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n^2+2n-4}$$

be the rational map associated with $\mathcal{M}$. The closure of its image $X := \overline{\operatorname{Im}(\varphi_{\mathcal{M}})} \subset \mathbb{P}^{n^2+2n-4}$ is (projectively equivalent to) the projection of the Veronese variety $V(n,3)$ from the linear subspace

$$\mathcal{M}' := \Big| \big\langle x_0^3, x_1^3, \dots, x_n^3, x_0^2 x_1, x_0 x_1^2, \{x_i x_j x_k\}_{0 \leq i < j < k \leq n, (i,j) \neq (0,1)} \big\rangle \Big|$$

of $\mathcal{M}_{n,3} = \mathbb{P}^{\binom{n+3}{3}-1}$. Arguing as in the previous example, we can check that X satisfies a Laplace equation of order 2 and that it is smooth. The quadric containing all the vertex points in $\mathbb{Z}^{n+1}$ has equation

$$2\Big(\sum_{i=0}^{n} x_i^2\Big) - 5\Big(\sum_{0 \leq i < j \leq n} x_i x_j\Big) + 9 x_0 x_1 = 0.$$

Notice that $n^2 + 2n - 4 = n^2 + n - 1$ if and only if $n = 3$. Hence for $n = 3$ we have a counterexample to Ilardi's conjecture. Nevertheless X cannot be further projected without acquiring singularities; hence, for $n > 3$ this example does not give a counterexample to Ilardi's conjecture. See Section 6 for counterexamples to Ilardi's conjecture for any $n \geq 3$.

Now X is a projection of the blow-up of $\mathbb{P}^n$ at $n-1$ fundamental points plus the line through the remaining two fundamental points, embedded via a linear system of cubics. Also in this case, as in the previous one, X is a projective toric variety of the form X_A. Now there is a lattice polytope P obtained from $3\Delta_n$, removing $n-1$ vertices and the opposite edge, and A is the set of the vertices of P together with the $n-1$ central points of the 2-faces adjacent to the removed edge. By the above smoothness criterium, X cannot be further projected without acquiring singularities.

4.1 Geometric Point of View and Trivial Linear Systems

With notation as in Section 3, we now consider a monomial artinian ideal I, generated by a subspace $I_d \subset \operatorname{Sym}^d V^*$ (where $V \simeq \mathbb{C}^{n+1}$). Since we are in the monomial case, we will also assume $I_d^{-1} \subset \operatorname{Sym}^d V^*$.

Remark 4.3 Note that the assumption that I is artinian is equivalent to $I^{-1} \cap V(n,d) = \varnothing$. Indeed, if I is not artinian, then there exists a point $z \in \mathbb{P}^n$ that is a common zero of all polynomials in I. Then its Veronese image $v_d(z)$ belongs to $V(n,d) \cap I^{-1}$. Here $v_d(z)$ must be interpreted as $\sum z_\alpha \partial_\alpha$, where α denotes a multiindex of degree d. Conversely, if $v_d(z) \in I^{-1}$, then $(\sum z_\alpha \partial_\alpha)(F) = 0$ for all $F \in I_d$; therefore, being I generated by I_d, z is a common zero of the polynomials of I.

Let X be the closure of the image of $\varphi_{I_d^{-1}}$, it can be seen geometrically as the projection of $V(n,d)$ from I_d. The exceptional locus of this projection is $I \cap V(n,d)$ and corresponds via v_d to the base locus of the linear system $\langle I_d^{-1} \rangle$. Also, X can be interpreted as (a projection of) the blow up of $V(n,d)$ along $I \cap V(n,d)$. Since I is artinian, in the toric case $\varphi_{I_d^{-1}}$ is never regular, because I has to contain the d-th powers of the variables. On the contrary, the map φ_{I_d} is regular.

In this situation we assume that all 2-osculating spaces of X have dimension strictly less than $\binom{n+2}{2}$; *i.e.,* X satisfies a Laplace equation of order 2. Since the 2-osculating spaces of $V(n,d)$ have the expected dimension, this means that I meets the 2-osculating space $\mathbb{T}_x^{(2)} V(n,d)$ for all $x \in V(n,d)$.

Let $d = 3$. $V(n,3) \subset \mathbb{P}(\operatorname{Sym}^3(V^*))$ represents the homogeneous polynomials of degree 3 that are cubes of a linear form. Let $\sigma_2 V(n,3)$ denote its secant variety; its general element can be interpreted both as a sum of two cubes of linear forms and as a product of three linearly dependent linear forms. Let $\pi_{I_3} \colon V(n,3) \dashrightarrow X$ denote the projection with center I_3. We connect the singularities of X to the reciprocal position of I_3 and $\sigma_2 V(n,3)$.

Proposition 4.4 *If $I \cap \sigma_2 V(n,3)$ strictly contains $\sigma_2(I \cap V(n,3))$, then X is singular.*

Proof The points of $I \cap \sigma_2 V(n,3)$ give rise to nodes of X, except those of $\sigma_2(I \cap V(n,3))$, because $I \cap V(n,3)$ is the indeterminacy locus of π_{I_3}. Note that $\sigma_2(I \cap V(n,3)) \subset I$, because I is an ideal. ■

Among Togliatti systems, not necessarily monomial, we detect two types that we call trivial.

Definition 4.5 A Togliatti system of forms of degree d is trivial of type A if there exists a form Q of degree $d-1$ such that, for every $L \in V^*$, $QL \in I$; that is, Q belongs to the saturation of I.

Note that the ideal generated by a quadratic form Q defines a trivial Togliatti system of cubics of type A that is not artinian, but adding $s \geq n$ suitable forms $F_1, \dots, F_s \in \operatorname{Sym}^3 V^*$ we get a linear system $Q\langle x_0, \dots, x_n \rangle + \langle F_1, \dots, F_s \rangle$, which is an artinian trivial Togliatti system of type A.

In the toric case, if Q is a quadratic monomial, then Q has rank ≤ 2; therefore, $I = (Q) + (F_1, \ldots, F_s)$ meets $\sigma_2 V(n,3)$ in infinitely many points outside I. In particular, by Proposition 4.4, a toric trivial Togliatti system of cubics of type A cannot parameterize a smooth variety.

Example 4.6 Consider the 12-dimensional linear system of cubics

$$\mathcal{L} =$$
$$\langle x_0^2x_1, x_0^2x_2, x_0^2x_3, x_1^2x_0, x_1^2x_2, x_1^2x_3, x_2^2x_0, x_2^2x_1, x_2^2x_3, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3\rangle \subset \mathcal{L}_{3,3}.$$

Let $\varphi_{\mathcal{L}} : \mathbb{P}^3 \to \mathbb{P}^{11}$ be the rational map associated with $\mathcal{L}$. The closure of its image $X := \overline{\mathrm{Im}(\varphi_{\mathcal{L}})} \subset \mathbb{P}^{11}$ is (projectively equivalent to) the projection from the linear subspace

$$\mathcal{L}' := \langle x_0^3, x_1^3, x_2^3, x_3^3, x_0x_1x_2, x_0x_3^2, x_1x_3^2, x_2x_3^2\rangle$$

of the Veronese variety $V(3,3) \subset \mathbb{P}(\mathcal{L}_{3,3}) = \mathbb{P}^{19}$. We easily check that X is not smooth. In fact $\mathrm{Sing}(X) = \{(0,0,0,1)\}$. Finally, let us check that X satisfies a Laplace equation of order 2. Since $x_0^3, x_1^3, x_2^3, (x_0+x_1+x_2)^3, x_0x_1x_2, x_0(x_0+x_1+x_2)^2, x_1(x_0+x_1+x_2)^2, x_2(x_0+x_1+x_2)^2$ are k-linearly dependent, applying Lemma 2.3 and Proposition 4.1 we get that the ideal

$$I = (x_0^3, x_1^3, x_2^3, x_3^3, x_0x_1x_2, x_0x_3^2, x_1x_3^2, x_2x_3^2) \subset R = k[x_o, x_1, x_2, x_3]$$

fails the WLP in degree 2. Therefore, using that $\mathcal{L}$ and $\mathcal{L}'$ are apolar and Theorem 3.2, we conclude that X satisfies a Laplace equation of order 2. Alternatively, we could observe that X is ruled, because the variable x_3 appears in the polynomials of the linear system $\mathcal{L}$ only up to degree 1, or, alternatively, the polynomials of $\mathcal{L}'$ contain all monomials of degree ≥ 2 in x_3.

Definition 4.7 A Togliatti system of forms of degree d is trivial of type B if there is a point $p \in V(n,d)$ such that the intersection of I with the $(d-1)$-osculating space at p meets all the other $(d-1)$-osculating spaces.

A trivial Togliatti system of type B is given in Example 3.7. To explain this, let us recall that, if $p \in V(n,d)$ is identified with L^d, where $L \in V^*$, then $\mathbb{T}_p^{(1)}V(n,d)$ is formed by the multiples of L^{d-1}, $\mathbb{T}_p^{(2)}V(n,d)$ by the multiples of L^{d-2}, and so on. From this description it follows that a sufficient condition to have a Togliatti system of cubics of type B is

$$\dim_k(I \cap \mathbb{T}_p^{(2)}) > \binom{n+2}{2} - n - 1 = \binom{n+1}{2},$$

because this number is the codimension of the intersection of two osculating spaces inside one of them. We found several cases when this happens even if $\dim I \cap \mathbb{T}_p^{(2)} = \binom{n+2}{2} - n - 1$.

Remark 4.8 G. Ilardi has a different notion of trivial Laplace equations in [11, Remark 1.2], which corresponds to varieties embedded in a space of dimension smaller than the expected dimension of the osculating spaces; see Remark 2.7. Still another definition can be found in [5].

Proposition 4.9 *Let I be a monomial artinian ideal I, generated in degree 3. Assume that I is trivial of type B of the form $I = (LF_1, \dots, LF_t, G_1, \dots, G_n)$, where L, F_i, G_j are monomials of degrees $1, 2, 3$ respectively, and $t > \binom{n+1}{2}$. Then the variety X is singular.*

Proof Since I is monomial, we can assume that $L = x_0$ and $G_i = x_i^3$, for all $i \geq 1$. We want to prove that I meets the tangent space at $p = L^3$ outside $I \cap V(n, 3)$, giving a singularity of X. We are done if among the polynomials $F_1, \dots, F_t$ there is a multiple of x_0 different from x_0^2. In view of the assumption on t, the unique case to check separately is when $t = \binom{n+1}{2} + 1$ and $\{F_1, \dots, F_t\}$ contains x_0^2 and all monomials of degree 2 in $x_1, \dots, x_n$. But in this case, looking at the corresponding polytope P, we see that the vertex $x_0^2 x_1$ has edges in P connecting to the $2n-2$ vertices $x_0^2 x_2, \dots x_0^2 x_n, x_1^2 x_2, \dots, x_1^2 x_n$, so that for $n \geq 3$ we get $2n - 2 > n$, hence the polytope P is not simple and the variety X is not smooth by [8, Proposition 4.12, Chap. 5]. ∎

Remark 4.10 According to [8, Chap. 5], a toric variety is called *quasi-smooth* if every cone of the fan $\mathcal{F}(X)$ associated with X is simplicial and the normalization morphism $\widetilde{X} \to X$ is bijective. We note that the variety X of Proposition 4.9 is even not quasi-smooth.

Note that a monomial artinian ideal I generated in degree three contains the monomials x_i^3 for $i = 0, \dots, n$.

We are now ready to give a complete classification of monomial Togliatti systems of cubics in the cases $n = 2$ and 3.

In the case $n = 2$, let $k[a, b, c]$ be the base ring; we recall that the only nontrivial monomial Togliatti system is a^3, b^3, c^3, abc (see [5, 21]). In view of the next classification theorem for $n = 3$, we recall also that all toric surfaces are quasi-smooth according to [8] chap. 5, §2.

Theorem 4.11 *Let $I \subset k[a, b, c, d]$ be a monomial artinian ideal of degree 3. Let X be the closure of the image of its apolar linear system. Assume that X is a smooth threefold and does satisfy a Laplace equation of degree 2. Then, up to a permutation of the coordinates, I^{-1} is one of the following three examples:*

(i) $(a^2b, a^2c, a^2d, ab^2, ac^2, ad^2, b^2c, b^2d, bc^2, bd^2, c^2d, cd^2)$, *$X$ is of degree 23, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the 4 coordinate points;*

(ii) $(abc, abd, a^2c, a^2d, ac^2, ad^2, b^2c, b^2d, bc^2, bd^2, c^2d, cd^2)$, *$X$ is of degree 18, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the line $\{c = d = 0\}$ and in the two points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$;*

(iii) $(abc, abd, acd, bcd, a^2c, ac^2, a^2d, ad^2, b^2c, bc^2, b^2d, bd^2)$, *$X$ is of degree 13, in $\mathbb{P}^{11}$, it is isomorphic to $\mathbb{P}^3$ blown up in the two lines $\{a = b = 0\}$ and $\{c = d = 0\}$.*

Moreover, if we substitute "smooth" with "quasi-smooth" (see Remark 4.10) we have the further cases:

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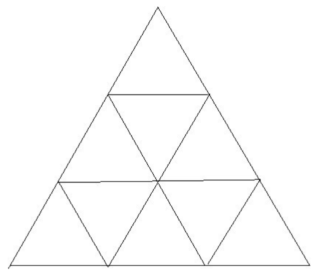

Figure 1: full triangle

(iv) $(acd, bcd, a^2c, a^2d, ac^2, ad^2, b^2c, b^2d, bc^2, bd^2, c^2d, cd^2)$*, this example is trivial of type A (indeed the apolar ideal contains* $ab * (a, b, c, d)$*); X is of degree* 18*, in* $\mathbb{P}^{11}$ *and its normalization is isomorphic to* $\mathbb{P}^3$ *blown up in the line* $\{c = d = 0\}$ *and in the two points* $(0, 0, 1, 0)$ *and* $(0, 0, 0, 1)$*;*

(iv$'$) *a projection of case* (ii) *removing one or both of the monomials* abc, abd*, or a projection of case* (iii) *removing a subset of the monomials* (abc, abd, acd, bcd)*, or a projection of case* (iv) *removing one or both of the monomials* (acd, bcd)*.*

Proof Consider the apolar ideal I. Since it is monomial and artinian, I contains (a^3, b^3, c^3, d^3) and j generators more, with $1 \leq j \leq 6$.

Due to [15, Proposition 1.1], in order to check that the four cases satisfy a Laplace equation of degree 2, it is enough to check that the vertex points in $\mathbb{Z}^4$ are contained in a quadric. This is $Q := 2(a^2 + b^2 + c^2 + d^2) - 5(ab + ac + ad + bc + bd + cd)$ in case (i) (it corresponds to a sphere with the same center of the tetrahedron); it is $Q + 9ab$ (a quadric of rank three) in case (ii); it is

$$Q + 9ab + 9cd = (-2a - 2b + c + d)(-a - b + 2c + 2d)$$

in case (iii), and it is ab in case (iv). An alternative approach for proving that the four cases satisfy a Laplace equation of degree 2 would be to apply Theorem 3.2(ii) directly.

Every case corresponds to a convex polytope contained in the full tetrahedron with vertices the powers a^3, b^3, c^3, d^3. This tetrahedron has four faces, as in Figure 1.

The convex polytope corresponding to case (i) is the truncated tetrahedron already described. It is instructive to describe its faces, which are four "punctured" hexagons as in Figure 2 and four smaller regular triangles. It is [15, Theorem 3.5(4)]. It has degree $3^3 - 4 = 23$ in $\mathbb{P}^{11}$. Note that the projection of this example is not quasi-smooth, because, when we remove a vertex, the resulting polytope has four faces meeting in a vertex (see [8, chap. 5, Proposition 4.12]).

The case (ii) corresponds to [15, Theorem 3.5(5)]. It has degree 18 in $\mathbb{P}^{11}$ The degree computation follows from the fact that the equivalence of a line in the (excess) intersection of three cubics in $\mathbb{P}^3$ counts seven, according to the [6, Example 9.1.4(a)]. So $3^3 - 7 - 2 = 18$.

The convex hull has the following faces: one rectangle, two full trapezoids, two punctured hexagons, and two triangles.

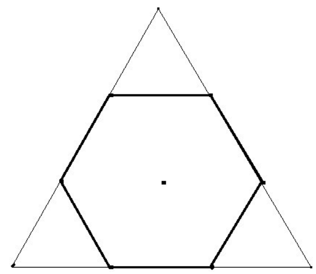

Figure 2: punctured hexagon

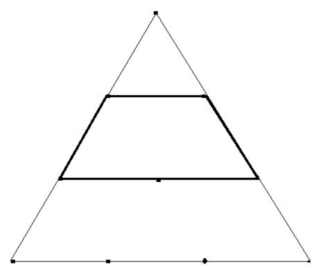

Figure 3: full trapezoid

The picture of the full trapezoid is like Figure 3, and it is important to remark that all three vertices of the longer side are included.

The projection of this case is never smooth, but when removing the mid vertices of the long sides of the trapezoids, we get a quasi smooth variety, appearing in (iv′) of our statement. To understand why these cases are not smooth, note that [8, Corollary 3.2(a), Chap. 5] is not satisfied when Γ is one of the vertices of the long side of the trapezoid.

Case (iii) can be seen both as [15, Theorem 3.5(2) or (3)]. Our variety X is $\mathbb{P}^3$ blown up on two skew lines L_1 and L_2. To see it as a particular case of case (2) of [15], consider that there are two natural maps from X to $\mathbb{P}^1$, with fiber given by the Hirzebruch surface isomorphic to $\mathbb{P}^2$ blown up in one point.

Fix a line L_i. The map takes a point p to the plane spanned by L_i and p. These planes through L_i make the target $\mathbb{P}^1$.

To see it as a particular case of case (3) of [15], consider that through a general point p there is a unique line meeting L_1 in p_1 and L_2 in p_2. The map from X to the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ takes p to the pair (p_1, p_2).

The convex polyhedron has six faces, four full trapezoids and two full (long) rectangles. The argument regarding the projection is analogous to the previous case and we omit it.

Case (iv) does not appear in [15, Theorem 3.5], because it is not smooth.

The convex hull has the following faces: one rectangle, two punctured trapezoids,

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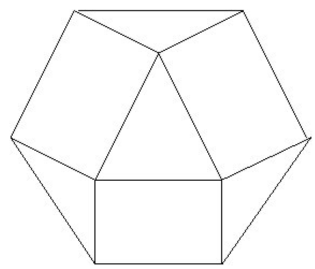

Figure 4: nonquasi smooth cases with 13 vertices

two full hexagons and two triangles. The presence of the punctured trapezoids is crucial for the nonsmoothness, exactly as we saw in the projection of case (ii).

A computer check shows that this list is complete; in all the remaining cases the convex polytope has at least four faces meeting in some vertex.

Let us just underline that there are exactly four monomial Togliatti (cubic) systems with 13 generators, their apolar ideals are obtained by adding to (a^3, b^3, c^3, d^3) the monomials $a^2 * (b, c, d)$ and their cyclic permutations. They are trivial of type A.

The faces are three full trapezoids, one full hexagon. The convex hull is topologically equivalent to the Figure 4, where the four meeting faces are evident, so it is not quasi smooth. ■

Remark 4.12 The computations have been performed using Macaulay2 [9].

5 Bounds on the Number of Generators

In this section we concentrate on the case $n = 2$. We will see how, using Theorem 3.2, it is possible to translate into geometric terms a result expressed in purely algebraic terms involving WLP.

Let $\mathcal{L}$ be a linear system of curves of degree d and (projective) dimension $N \leq \binom{d+1}{2} - 1$, defining a map $\phi_{\mathcal{L}} : \mathbb{P}^2 \to \mathbb{P}^N$ having as image a surface X which satisfies exactly one Laplace equation of order $d - 1$.

With notations as in Theorem 3.2, let I^{-1} be the ideal generated by the equations of the curves in $\mathcal{L}$ and I its apolar system, generated by r polynomials.

Note that if $\mathcal{L}$ is a Togliatti system with $r = 3$, then $\mathcal{L}$ is trivial of type A and I is not artinian. The Togliatti example described in Remark 3.4 is a nontrivial example with $r = 4$ and I artinian. It is a classical result that this is the only nontrivial example with $d = 3$ (see [20] and [5]).

We consider now the case $r = 4$ with $d \geq 4$.

Theorem 5.1 *Let $I \subset R := k[x, y, z]$ be an artinian ideal generated by 4 homogeneous polynomials of degree $d \geq 4$. Then*

(i) *I satisfies the WLP in degree $d - 1$;*
(ii) *if d is not multiple of 3, then I satisfies the WLP everywhere;*
(iii) *if d is multiple of 3 but not of 6, then there exists I that fails the WLP.*

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Proof Let $I = (F_1, \dots, F_4)$ and denote by $\mathcal{E}$ the syzygy bundle of $F_1, \dots, F_4$; *i.e.*, $\mathcal{E}$ is the rank three bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = -4d$, which enters in the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(-d)^4 \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

From [1, Theorem 3.3], if $\mathcal{E}$ is not semistable, then I has the WLP. So we assume that $\mathcal{E}$ is semistable and consider the normalized bundle $\mathcal{E}_{\text{norm}} = \mathcal{E}(k)$ with $k = [4d/3]$. We distinguish three cases, according to the congruence class of d modulo 3. If $d \equiv 1 \bmod 3$, then $c_1(\mathcal{E}_{\text{norm}}) = -1$, hence by the Theorem of Grauert-Mülich [14] it follows that the restriction of $\mathcal{E}_{\text{norm}}$ to a general line L is $\mathcal{E}_{\text{norm}} \mid_L \simeq \mathcal{O}_L^2 \oplus \mathcal{O}_L(-1)$. Then by [1, Theorem 2.2] I has the WLP. Similarly, if $d \equiv 2 \bmod 3$, then $c_1(\mathcal{E}_{\text{norm}}) = -2$, and on a general line $\mathcal{E}_{\text{norm}} \mid_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(-1)^2$. Finally, assume that $d = 3\lambda$, $\lambda \geq 2$. There are two possibilities for $\mathcal{E}_{\text{norm}} \mid_L$: it is isomorphic either to $\mathcal{O}_L^3$, and we conclude as in the two previous cases, or to $\mathcal{O}_L(-1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(1)$. Hence $\mathcal{E} \simeq \mathcal{E}_{\text{norm}}(-4\lambda)$ and $\mathcal{E} \mid_L \simeq \mathcal{O}_L(-1-4\lambda) \oplus \mathcal{O}_L(-4\lambda) \oplus \mathcal{O}_L(1-4\lambda)$. Consider the exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(1) \to \mathcal{E} \mid_L (1) \to 0 \tag{5.1}$$

and its twists. The only critical situation is obtained twisting by $4\lambda - 2$; it is then isomorphic to

$$0 \to \mathcal{E}(4\lambda-2) \to \mathcal{E}(4\lambda-1) \to \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1) \oplus \mathcal{O}_L \to 0,$$

where the second arrow is the multiplication by L. By the semistability of $\mathcal{E}$ we get $H^0(\mathcal{E}(4\lambda-2)) = H^0(\mathcal{E}(4\lambda-1)) = (0)$. Also $H^2(\mathcal{E}(4\lambda-2)) = (0)$: indeed, by Serre's duality, $H^2(\mathcal{E}(4\lambda-2)) \simeq H^0(\mathcal{E}^*(-4\lambda-1))$, and this is zero by the semistability of $\mathcal{E}^*$, because $c_1(\mathcal{E}^*(-4\lambda-1) = -3$. Therefore the cohomology exact sequence of (5.1) becomes

$$0 \to k \to H^1\big(\mathcal{E}(4\lambda-2)\big) \to H^1\big(\mathcal{E}(4\lambda-1)\big) \to k \to 0,$$

where k is the base field. But $H^1(\mathcal{E}(4\lambda-2)) \simeq (R/I)_{4\lambda-2}$ and $H^1(\mathcal{E}(4\lambda-1)) \simeq (R/I)_{4\lambda-1}$, so I fails the WLP in degree $4\lambda - 2 = d + (\lambda - 2)$. With similar arguments we get that this is the only degree in which I fails the WLP, so in particular WLP always holds in degree $d-1$. Finally, [13, Corollary 7.4] shows that the ideal $(x^d, y^d, z^d, x^\lambda y^\lambda z^\lambda)$, with $d = 3\lambda$ odd, fails the WLP. ∎

Remark 5.2 (1) Theorem 5.1(ii) was stated for the monomial case in [13, Theorem 6.1]. An analogous proof holds for homogeneous polynomials that are not necessarily monomials, and we include it here for the sake of completeness.

(2) U. Nagel has pointed out to us that if d is a multiple of 6 and I is a monomial ideal then I does have the WLP. This follows from [2, Theorem 6.3].

(3) Theorem 5.1 is optimal, *i.e.*, for all $d \geq 4$ and $5 \leq r \leq d+1$ there exist examples of ideals I generated by r polynomials of degree d that fail the WLP in degree $d-1$.

Let $I = (xF, yF, zF, G_1, \dots, G_{r-3})$, where F is a homogeneous polynomial with $\deg F = d-1$ and $G_1, \dots, G_{r-3}$ are general forms of degree d. I is an artinian ideal because $r \geq 5$, and I^{-1} defines a surface satisfying a Laplace equation of order $d-1$.

Putting everything together we get the following corollary.

Corollary 5.3 *Let $I \subset k[x, y, z]$ be an artinian ideal generated by r forms $F_1, \dots, F_r$ of degree $d \geq 4$ and let $S \subset \mathbb{P}^{\binom{d+2}{2}-r-1}$ be the projection of the Veronese surface $V(2, d)$ from the linear system $\langle F_1, \dots, F_r \rangle$. The following hold:*

(i) *if $r = 4$, then all the $(d-1)$-th osculating spaces of $S \subset \mathbb{P}^{\binom{d+2}{2}-5}$ have the expected dimension;*

(ii) *for all $5 \leq r \leq d+1$, there exists a surface $S \subset \mathbb{P}^{\binom{d+2}{2}-r-1}$ with a $(d-1)$-th osculating space of dimension less than the expected one.*

Proof (i) It follows from Theorems 3.2 and 5.1.

(ii) It follows from Theorem 3.2 and Remark 5.2(3). We observe that it is possible to find for all $d \geq 4$ and $5 \leq r \leq d+1$ examples of *smooth* surfaces $S \subset \mathbb{P}^{\binom{d+2}{2}-r-1}$ with a $(d-1)$-osculating space of dimension smaller than the expected one. For instance, take $r = 5$, $F = x^{d-1} + y^{d-1} + z^{d-1}$ and $G_1 = x^d$, $G_2 = y^d$, $G_3 = z^d$ and apply Remark 5.2(3). ∎

6 Final Comments

A further interesting project is the classification of all Togliatti linear systems of cubics on $\mathbb{P}^n$, in the monomial case, accomplished here for $n \leq 3$ (see Theorem 4.11). It is possible to generalize the three examples in Theorem 4.11 constructing suitable projections of blow ups of $\mathbb{P}^n$ along unions of linear spaces of codimension ≥ 2 corresponding to partitions of the $n+1$ fundamental points.

Among the three examples in Theorem 4.11, the third one is a ruled threefold, while the first two are not. How can we distinguish the ruled examples from the nonruled ones? Since all ruled varieties satisfy Laplace equations of all orders (see Remark 2.8), the nonruled ones are much more interesting to find.

The second case of Theorem 4.11 generalizes to $n \geq 4$ and gives for any $n \geq 3$ a counterexample to Ilardi's conjecture in [11, p. 12]. In fact, we have the following example.

Example 6.1 We consider the monomial artinian ideal

$$I = (x_0, x_1, \dots, x_{n-2})^3 + (x_{n-1}^3, x_n^3, x_0x_{n-1}x_n, x_1x_{n-1}x_n, \dots, x_{n-2}x_{n-1}x_n)$$
$$\subset k[x_0, \dots, x_n].$$

Since $\dim I_3 = \binom{n+1}{3} + n + 1$, we get that $\dim(I_3^{-1}) = n(n+1)$. Let X be the closure of the image of $\varphi_{I_3^{-1}}$, which can be seen as the projection of $V(n, 3)$ from I_3. X is a smooth n-fold in $\mathbb{P}^{n(n+1)-1}$ isomorphic to $\mathbb{P}^n$ blown up at the linear space $x_{n-1} = x_n = 0$ and in the two points $(0, \dots, 0, 1, 0)$ and $(0, \dots, 0, 1)$. Moreover, it easily follows from Theorem 3.2 that X satisfies a Laplace equation of degree 2. A quadric in $\mathbb{Z}^{n+1}$ containing the vertices of the corresponding polytope, analogous to the one in the proof of Theorem 4.11(ii), has equation:

$$2(x_0^2 + \cdots + x_n^2) - 5\Big(\sum_{i,j=0,i<j}^{n} x_ix_j\Big) + 9\Big(\sum_{i,j=0,i<j}^{n-2} x_ix_j\Big) = 0.$$

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Remark 6.2 The examples of Theorem 4.11 and Example 6.1 can be seen as special cases of a class of smooth monomial Togliatti systems of cubics. Let $E_0, \dots, E_n$ be the fundamental points in $\mathbb{P}^n$, and let Π be a partition of the set $\{E_0, \dots, E_n\}$ such that each part contains at most $n-1$ points. Let us consider the blow up of $\mathbb{P}^n$ along the linear subspaces generated by the parts of Π and its embedding with the cubics. Since we are performing a blow up along a torus invariant subscheme, we get a toric variety, which corresponds to a polytope P. It is the n-dimensional simplex truncated along the faces associated with the blown up spaces. Finally let us consider the projection from the points corresponding to the centers of the full hexagons in P. The toric variety X obtained in this way is smooth. We conjecture that all smooth monomial Togliatti systems of cubics are obtained in this way.

Acknowledgment This work began in the Winter of 2009 during our visit to the MSRI in Berkeley. We thank the organizers of the Algebraic Geometry program for their kind hospitality, and Rita Pardini, with whom we shared tea and discussed the Tea Theorem.

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